9.7 Day 1

Friday, February 21, 2020 9:14 AM

Taylor / Maclaurin Polynomials

* A polynomial can converge to a function.

$f(x) = \sin x$

$P_7(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$

$f(0) = \sin(0) = 0$ $\qquad f'''(0) = -\cos(0) = -1$

$f'(0) = \cos(0) = 1$ $\qquad f^{(4)}(0) = \sin(0) = 0$

$f''(0) = -\sin(0) = 0$ $\qquad f^{(5)}(0) = \cos(0) = 1$

$$p(x) = 0 + 1x + \frac{1}{2}(0)x^2 + \frac{1}{2}\cdot\frac{1}{3}(-1)x^3 + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}(0)x^4 + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{5}(1)x^5 + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{5}\cdot\frac{1}{6}(0)x^6 + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}(-1)x^7$$

$$P_7(x) + x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

☆ we can build poly. that shares 1st, 2nd, 3rd ..... derivs w/ $f(x)$ @ $x = c$

Build $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

so that $P(0) = 1$ $\qquad p(0) = a_0 = a_0 = 1$

$P'(0) = 2$ $\qquad P'(x) = a_1 + 2a_2 x + 3a_3 x^2$ $\qquad P'(0) = a_1 = 2$

$P''(0) = 3$ $\qquad P''(x) = 2a_2 + 6a_3 x$

$P'''(0) = 4$ $\qquad P''(0) = 3$ $\qquad 3 = 2a_2$ $\qquad a_2 = 3/2$

$P'''(0) = 6a_3$ $\qquad 6a_3 = 4$ $\qquad a_3 = \frac{4}{6} = \frac{4}{1\cdot 2\cdot 3}$

$$p(x) = 1 + \frac{2x}{1!} + \frac{3}{2!}x^2 + \frac{4}{3!}x^3$$

$$= \sum_{n=0}^{3} \frac{(x+1)x^n}{n!} \quad \text{or} \quad \sum_{n=1}^{4} \frac{n \cdot x^{n-1}}{(n-1)!}$$

Defn: Taylor Polynomial

- n^{th} Taylor Polynomial (Maclaurin Polynomial) by f @ $x = 0$

$$P(x) = \frac{f(0)}{\ } + \frac{f'(0)x}{\ } + \frac{f''(0)x^2}{\ } + \dots + \frac{f^{(n)}(0)x^n}{\ }$$

$$P_n(x) = \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \dots + \frac{f^{(n)}(0)\,x^n}{n!}$$

nth Taylor Polynomial for f centered @ x = c

$$P_n(x) = \frac{f(c)}{0!} + \frac{f'(c)\cdot(x-c)}{1!} + \frac{f''(c)\,(x-c)^2}{2!} + \dots + \frac{f^{(n)}(c)\,(x-c)^n}{n!}$$

* Notice the "Taylor Tripod"

- nth deriv @ x = c
- nth power of x = c
- n! in the denom.

Example: Write a 5th order MacLauren Polynomial for $f(x) = \cos x$ @ $x = 0$ (Don't need to give this info)

$f(x) = \cos x$

$f(0) = 1$

$f'(0) = -\sin(0) = 0$

$f''(0) = -\cos(0) = -1$

$f'''(0) = \sin(0) = 0$

$f^{(4)}(0) = \cos(0) = 1$

$$P_5(x) = 1 + \frac{0x}{1!} - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \frac{0x^5}{5!}$$

$$P_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

use to approx $\cos(0.5)$ $\quad P_5(0.5) = 1 - \frac{(0.5)^2}{2!} + \frac{(0.5)^4}{4!} \approx 0.877$

Ex: 3rd order Taylor Poly. for $y = \sin x$ @ $x = \frac{\pi}{4}$

$f(\pi/4) = \frac{\sqrt{2}}{2}$

$f'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$$P(x) = \sqrt{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^3$$

$$P_3(x) = \frac{\sqrt{2}}{2} + \frac{\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)}{1!} - \frac{\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^2}{2!} - \frac{\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^3}{3!}$$

$$\left\{\begin{array}{l} f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\ f''\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \\ f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \end{array}\right.$$

you try 3^{rd} order Taylor Poly. for $y = e^x$ centered @ $x = 2$

$$P_3(x) = e^2 + \frac{e^2(x-2)}{1!} + \frac{e^2(x-2)^2}{2!} + \frac{e^2(x-2)^3}{3!}$$

Ex: 3^{rd} order Taylor Polynomial centered @ $x = 1$. $f(x) = \frac{1}{1+x}$

$f(x) = (1+x)^{-1}$

$$P_3(x) = \frac{1}{2} - \frac{\frac{1}{4}(x-1)}{1!} + \frac{\frac{2}{8}(x-1)^2}{2!} - \frac{\frac{6}{16}(x-1)^3}{3!}$$

$$\left\{\begin{array}{l} f(1) = \frac{1}{2} \\ f'(x) = -(1+x)^{-2} \Rightarrow f'(1) = -\frac{1}{4} \\ f''(x) = 2(1+x)^{-3} \Rightarrow f''(1) = \frac{2}{8} \\ f'''(x) = -6(1+x)^{-4} = f'''(1) = -6/16 \end{array}\right.$$