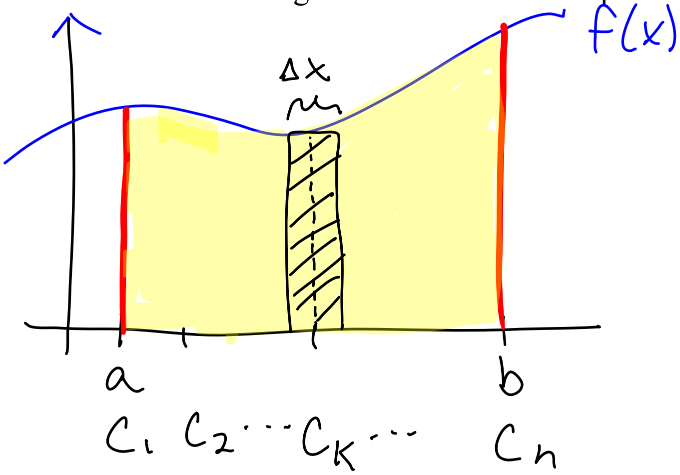
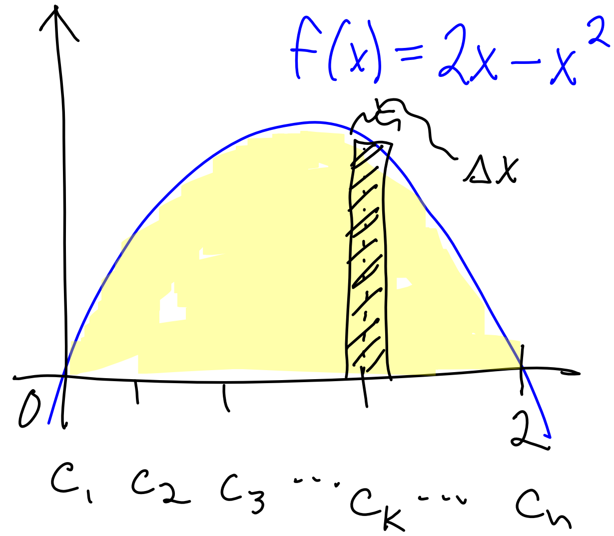
AP Calculus AB
6.2 Day 1

Name____________________

Let's sketch 1 generic function and 1 specific function:





What does each of these represent?

1. $f(c_k) \bullet \Delta x$

= height · base

= area of 1 rect.

4. $\underbrace{(2c_k - (c_k)^2)}\bullet \Delta x$

= height · base

= area of 1 rect.

2. $\underbrace{\sum_{k=1}^{n}}_{\text{sum}} \underbrace{f(c_k) \bullet \Delta x}_{\text{area of rect.}}$

= total area of all "n" rectangles

5. $\underbrace{\sum_{k=1}^{n}}_{\text{sum}} \underbrace{(2c_k - (c_k)^2) \bullet \Delta x}_{\text{area of 1 rect.}}$

= total area of "n" rectangles

3. $\underbrace{\lim_{\Delta x \to 0}} \sum_{k=1}^{n} f(c_k) \bullet \Delta x$

- base gets infinitesimally thin
- $n \to \infty$ # of rects.
- gives <u>exact</u> area

$$= \underbrace{\int_a^b}_{\text{sum}} \underbrace{f(x)}_{\text{height}} \cdot \underbrace{dx}_{\text{base}}$$

6. $\lim_{n \to \infty} \sum_{k=1}^{n} (2c_k - (c_k)^2) \bullet \Delta x$

(see # 5)

$$= \int_0^2 (2x - x^2)\,dx$$

The Definite Integral of a Continuous Function on $[a, b]$

Let f be continuous on $[a, b]$, and let $[a, b]$ be partitioned into n subintervals of equal length $\Delta x = (b - a)/n$. Then the definite integral of f over $[a, b]$ is given by

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x,$$

where each c_k is chosen arbitrarily in the k^{th} subinterval.

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x = \int_a^b f(x)\, dx.$$

Upper limit of integration

The function is the **integrand.**

Integral sign

$$\underbrace{\int_a^b f(x)\, dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

x is the **variable of integration.**

Lower limit of integration

Integral of f from a to b

When you find the value of the integral, you have **evaluated the integral.**

The value of the definite integral of a function over any particular interval depends on the function and not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)\, dt \quad \text{or} \quad \int_a^b f(u)\, du \quad \text{instead of} \quad \int_a^b f(x)\, dx.$$

No matter how we represent the integral, it is the same *number,* defined as a limit of Riemann sums. Since it does not matter what letter we use to run from a to b, the variable of integration is called a **dummy variable.**

A Definition of the Integral Based on RRAM   Name_______________

To find $\int_a^b f(x)dx$, the "area" between the x-axis and a curve $f(x)$ on an interval $[a,b]$ we can first divide the area between the x-axis and the curve $f(x)$ into "rectangles", each of width $\Delta x = \frac{b-a}{n}$ and "height" $f(c_k)$, where $c_k = a + k\Delta x$. We can then find the area under the curve by multiplying the width $\Delta x = \frac{b-a}{n}$ of each rectangle by its height $f(c_k)$ and adding/summing the areas of all the rectangles as we let the number of rectangles approach ∞.

In other words, $$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\bullet \Delta x \quad where \quad \Delta x = \frac{b-a}{n} \quad and \quad c_k = a + k\Delta x$$

Examples: Write the following integrals using the limit definition above.   RRAM n = number of intervals

1. $\int_0^5 \cos x\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \cos\left(\frac{5k}{n}\right)\cdot \frac{5}{n}$

$b-a = 5-0 = 5$

$\Delta x = \frac{5}{n}$

$c_k = 0 + k\left(\frac{5}{n}\right) = \frac{5k}{n}$

2. $\int_1^5 \cos x\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \cos\left(1 + \frac{4k}{n}\right)\cdot \frac{4}{n}$

$b-a = 4$

$\Delta x = \frac{4}{n}$

$c_k = 1 + k\left(\frac{4}{n}\right) = 1 + \frac{4k}{n}$

3. $\int_0^7 x^3\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{7k}{n}\right)^3 \frac{7}{n}$

$b-a = 7$

$\Delta x = 7/n$

$c_k = 0 + \frac{7k}{n} = \frac{7k}{n}$

4. $\int_1^6 x^3\,dx = \lim_{n\to\infty} \left(\left(1 + \frac{5k}{n}\right)\right)^3 \cdot \frac{5}{n}$

$b-a = 5$

$\Delta x = \frac{5}{n}$

$c_k = 1 + \frac{5k}{n}$

5. $\int_{-4}^4 x^3\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(-4 + \frac{8k}{n}\right)^3 \cdot \frac{8}{n}$

$b-a = 4-(-4) = 8$

$\Delta x = 8/n$

$c_k = -4 + \frac{8k}{n}$

Examples: Write the limit as a definite integral.

1) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{5}{n}\ln\left(2+\frac{5k}{n}\right)$ $= \int_2^7 \ln x\,dx$

$\Delta x = \frac{b-a}{n} = \frac{5}{n}$ $b-a=5$ $a=2$ $b=7$ $f(x)=\ln x$

$b=5+a$

2) $\lim_{n\to\infty}\frac{\pi}{3n}\sum_{k=1}^{n}\tan\left(\frac{k\pi}{3n}\right)$ $\int_0^{\frac{\pi}{3}} \tan x\,dx$

$\frac{\pi}{3n} = \frac{\pi/3}{n}$ $a+\frac{b-a}{n} = \frac{\frac{\pi}{3}}{n}$ $f(x)=\tan x$

$\frac{b-a}{n} = \frac{\pi/3}{n}$ $a=0$ $b=\pi/3$

Homework

(1-2) Multiple Choice

1. $\lim_{n\to\infty}\frac{1}{n}\left[\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\ldots+\sqrt{\frac{n}{n}}\right]=$

$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a+k\cdot\frac{b-a}{n}\right)\cdot\frac{b-a}{n}$

(A) $\frac{1}{2}\int_0^1 \frac{1}{\sqrt{x}}dx$ (B) $\int_0^1 \sqrt{x}\,dx$ (C) $\int_0^1 x\,dx$

(D) $\int_1^2 x\,dx$ (E) $2\int_1^2 x\sqrt{x}\,dx$

1. $\frac{b-a}{n} = \frac{1}{n}$

2. $a+\frac{b-a}{n} = \frac{1}{n}$

$a=0$ $b=1$

$f(x)=\sqrt{x}$

2. The expression $\frac{1}{50}\left(\sqrt{\frac{1}{50}}+\sqrt{\frac{2}{50}}+\sqrt{\frac{3}{50}}+\ldots+\sqrt{\frac{50}{50}}\right)$ is a Riemann sum Approximation for

(A) $\int_0^1 \sqrt{\frac{x}{50}}dx$ (B) $\int_0^1 \sqrt{x}\,dx$ (C) $\frac{1}{50}\int_0^1 \sqrt{\frac{x}{50}}dx$

(D) $\frac{1}{50}\int_0^1 \sqrt{x}\,dx$ (E) $\frac{1}{50}\int_0^{50} \sqrt{x}\,dx$

$\frac{b-a}{n} = \frac{1}{50}$

$a+\frac{b-a}{n} = \frac{1}{50}$ $a=0$ $b=1$

c. $f(x)=\sqrt{x}$

(3-6) Rewrite the given limit as a definite integral.

3. $\lim_{n\to\infty}\left(\frac{1}{n}\left(\left(9+\frac{1}{n}\right)^2+\left(9+\frac{2}{n}\right)^2+\left(9+\frac{3}{n}\right)^2+\left(9+\frac{4}{n}\right)^2+\ldots\ldots\ldots\ldots+\left(9+\frac{n}{n}\right)^2\right)\right)$

a) $\frac{b-a}{n}=\frac{1}{n}$ b. $\frac{b-a}{n}=9+\frac{1}{n}$

$a=9$

$b=10$

$f(x)=x^2$

$$\boxed{=\int_9^{10} x^2\,dx}$$

4. $\lim_{n\to\infty}\left(\frac{7}{n}\left(\sqrt[3]{-2+\frac{7}{n}}+\sqrt[3]{-2+\frac{14}{n}}+\sqrt[3]{-2+\frac{21}{n}}+\sqrt[3]{-2+\frac{28}{n}}+\ldots\ldots\ldots\ldots+\sqrt[3]{-2+\frac{7n}{n}}\right)\right)$

$\frac{b-a}{n}=\frac{7}{n}$ $f(x)=\sqrt[3]{x}$

$$\boxed{=\int_{-2}^{5}\sqrt[3]{x}\,dx}$$

$b-a=7$

$a=-2$ $b=5$

5. $\lim_{n\to\infty}\left(\frac{8}{n}\sum_{k=1}^{n}\left(4+\frac{8k}{n}\right)^3\right)$

$$\boxed{=\int_4^{12} x^3\,dx}$$

$\frac{b-a}{n}=\frac{8}{n}$ $a=4$ $f(x)=x^3$

$b-a=8$ $b=12$

6. $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{9}{n}\left(\frac{9k}{n}\right)^2\right)$ $=\int_0^9 x^2\,dx$

$\Delta x=\frac{b-a}{n}=\frac{9}{n}$

$b-a=9$ $f(x)=x^2$

$a=0$ $b=9$