20. $f(x) = (3x + 4)^5 = (3x)^5 + 5(3x)^4 \cdot 4 + 10(3x)^3 \cdot 4^2 + 10(3x)^2 \cdot 4^3 + 5(3x) \cdot 4^4 + 4^5$
$= 243x^5 + 1620x^4 + 4320x^3 + 5760x^2 + 3840x + 1024$

21. $(2x + y)^4 = (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4 = 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

22. $(2y - 3x)^5 = (2y)^5 + 5(2y)^4(-3x) + 10(2y)^3(-3x)^2 + 10(2y)^2(-3x)^3 + 5(2y)(-3x)^4 + (-3x)^5$
$= 32y^5 - 240y^4x + 720y^3x^2 - 1080y^2x^3 + 810yx^4 - 243x^5$

23. $(\sqrt{x} - \sqrt{y})^6 = (\sqrt{x})^6 + 6(\sqrt{x})^5(-\sqrt{y}) + 15(\sqrt{x})^4 \cdot (-\sqrt{y})^2 + 20(\sqrt{x})^3(-\sqrt{y})^3 + 15(\sqrt{x})^2(-\sqrt{y})^4$
$+ 6(\sqrt{x})(-\sqrt{y})^5 + (-\sqrt{y})^6 = x^3 - 6x^{5/2}y^{1/2} + 15x^2y - 20x^{3/2}y^{3/2} + 15xy^2 - 6x^{1/2}y^{5/2} + y^3$

24. $(\sqrt{x} + \sqrt{3})^4 = (\sqrt{x})^4 + 4(\sqrt{x})^3(\sqrt{3}) + 6(\sqrt{x})^2 \cdot (\sqrt{3})^2 + 4(\sqrt{x})(\sqrt{3})^3 + (\sqrt{3})^4 = x^2 + 4x\sqrt{3x} + 18x + 12\sqrt{3x} + 9$

25. $(x^{-2} + 3)^5 = (x^{-2})^5 + 5(x^{-2})^4 \cdot 3 + 10(x^{-2})^3 \cdot 3^2 + 10(x^{-2})^2 \cdot 3^3 + 5(x^{-2}) \cdot 3^4 + 3^5$
$= x^{-10} + 15x^{-8} + 90x^{-6} + 270x^{-4} + 405x^{-2} + 243$

26. $(a - b^{-3})^7 = a^7 + 7a^6(-b^{-3}) + 21a^5(-b^{-3})^2 + 35a^4(-b^{-3})^3 + 35a^3(-b^{-3})^4 + 21a^2(-b^{-3})^5 + 7a(-b^{-3})^6 + (-b^{-3})^7$
$= a^7 - 7a^6b^{-3} + 21a^5b^{-6} - 35a^4b^{-9} + 35a^3b^{-12} - 21a^2b^{-15} + 7ab^{-18} - b^{-21}$

27. Answers will vary.

28. Answers will vary.

29. If $n >= 1$, $\binom{n}{1} = \dfrac{n!}{1!(n-1)!} = n = \dfrac{n!}{(n-1)!1!}$
$= \dfrac{n!}{(n-1)![n-(n-1)]!} = \binom{n}{n-1}$

30. If $n >= r >= 0$, $\binom{n}{r} = \dfrac{n!}{r!(n-r)!} = \dfrac{n!}{(n-r)!r!}$
$= \dfrac{n!}{(n-r)![n-(n-r)]!} = \binom{n}{n-r}$

31. $\binom{n-1}{r-1} + \binom{n-1}{r}$
$= \dfrac{(n-1)!}{(r-1)![(n-1)-(r-1)]!} + \dfrac{(n-1)!}{r!(n-1-r)!}$
$= \dfrac{r(n-1)!}{r(r-1)!(n-r)!} + \dfrac{(n-1)!(n-r)}{r!(n-r)(n-r-1)!}$
$= \dfrac{r(n-1)!}{r!(n-r)!} + \dfrac{(n-r)(n-1)!}{r!(n-r)!}$
$= \dfrac{(r+n-r)(n-1)!}{r!(n-r)!}$
$= \dfrac{n!}{r!(n-r)!}$
$= \binom{n}{r}$

32. (a) Any pair (n, m) of nonnegative integers — except for $(1, 1)$ — provides a counterexample. For example, $n = 2$ and $m = 3$: $(2 + 3)! = 5! = 120$, but $2! + 3! = 2 + 6 = 8$.

(b) Any pair (n, m) of nonnegative integers — except for $(0, 0)$ or any pair $(1, m)$ or $(n, 1)$ — provides a counterexample. For example, $n = 2$ and $m = 3$: $(2 \cdot 3)! = 6! = 720$, but, $2! \cdot 3! = 2 \cdot 6 = 12$.

33. Let $n >= 2$. $\binom{n}{2} + \binom{n+1}{2} = \dfrac{n!}{2!(n-2)!} + \dfrac{(n+1)!}{2!(n-1)!}$
$= \dfrac{n(n-1)}{2} + \dfrac{(n+1)(n)}{2}$
$= \dfrac{n^2 - n + n^2 + n}{2} = n^2$

34. Let $n >= 2$. $\binom{n}{n-2} + \binom{n+1}{n-1} = \dfrac{n!}{(n-2)![n-(n-2)]!}$
$+ \dfrac{(n+1)!}{(n-1)![(n+1)-(n-1)]!}$
$= \dfrac{n!}{(n-2)!2!} + \dfrac{(n+1)!}{(n-1)!2!}$
$= \dfrac{n(n-1)}{2} + \dfrac{(n+1)n}{2}$
$= \dfrac{n^2 - n + n^2 + n}{2} = n^2$

35. True. The signs of the coefficients are determined by the powers of the $(-y)$ terms, which alternate between odd and even.

36. True. In fact, the sum of every row is a power of 2.

37. The fifth term of the expansion is $\binom{8}{4}(2x)^4(1)^4 = 1120x^4$. The answer is C.

38. The two smallest numbers in row 10 are 1 and 10. The answer is B.

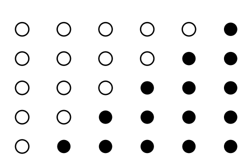
39. The sum of the coefficients of $(3x - 2y)^{10}$ is the same as the value of $(3x - 2y)^{10}$ when $x = 1$ and $y = 1$.
The answer is A.

40. The even-numbered terms in the two expressions are opposite-signed and cancel out, while the odd-numbered terms are identical and add together. The answer is D.

41. (a) 1, 3, 6, 10, 15, 21, 28, 36, 45, 55

(b) They appear diagonally down the triangle, starting with either of the 1's in row 2.

(c) Since n and $n + 1$ represent the sides of the given rectangle, then $n(n + 1)$ represents its area. The triangular number is $1/2$ of the given area. Therefore, the triangular number is $\dfrac{n(n+1)}{2}$.



(d) From **(c)**, we observe that the nth triangular number can be written as $\frac{n(n+1)}{2}$. We know that binomial coefficients are the values of $\binom{n}{r}$ for $r = 0, 1, 2, 3, \ldots, n$. We can show that $\frac{n(n+1)}{2} = \binom{n+1}{2}$ as follows:

$$\frac{n(n+1)}{2} = \frac{(n+1)n(n-1)!}{2(n-1)!} = \frac{(n+1)!}{2!(n-1)!} = \frac{(n+1)!}{2!((n+1)-2)!} = \binom{n+1}{2}.$$

So, to find the fourth triangular number, for example, compute $\binom{4+1}{2} = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot 3!}{2!3!} = \frac{5 \cdot 4}{2} = 10.$

42. **(a)** 2 (Every other number appears at least twice.)

(b) 1

(c) No (They all appear in order down the second diagonal.)

(d) 0 (See Exercise 44 for a proof.)

(e) All are divisible by p.

(f) Rows that are positive-integer powers of 2: 2, 4, 8, 16, etc.

(g) Rows that are 1 less than a power of 2: 0, 1, 3, 7, 15, etc.

(h) Answers will vary. One possible answer: For any prime numbered row, or row where the first element is a prime number, all the numbers in that row (excluding the 1's) are divisible by the prime. For example, in the seventh row (1 7 21 35 35 21 7 1) 7, 21, and 35 are all divisible by 7.

43. The sum of the entries in the nth row equals the sum of the coefficients in the expansion of $(x+y)^n$. But this sum, in turn, is equal to the value of $(x+y)^n$ when $x = 1$ and $y = 1$:

$$2^n = (1+1)^n = \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1^1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^01^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

44. $$0 = (1-1)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}(-1) + \binom{n}{2}1^{n-2}(-1)^2 + \cdots + \binom{n}{n}(-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n\binom{n}{n}$$

45. $$3^n = (1+2)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}2 + \binom{n}{2}1^{n-2}2^2 + \cdots + \binom{n}{n}2^n = \binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \cdots + 2^n\binom{n}{n}$$

■ Section 9.3 Sequences

Quick Review 9.3

1. $3 + (5-1)4 = 3 + 16 = 19$
2. $\frac{5}{2}[(6 + (5-1)4] = \frac{5}{2}(22) = 55$
3. $5 \cdot 4^2 = 80$
4. $\frac{5(1-4^3)}{(1-4)} = \frac{-315}{-3} = 105$
5. $a_{10} = \frac{10}{11}$
6. $a_{10} = 5 + (10-1)3 = 32$
7. $a_{10} = 5 \cdot 2^9 = 2560$
8. $a_{10} = \left(\frac{4}{3}\right)\left(\frac{1}{2}\right)^9 = \left(\frac{4}{3}\right)\left(\frac{1}{512}\right) = \frac{1}{384}$
9. $a_{10} = 32 - 17 = 15$
10. $a_{10} = \frac{10^2}{2^{10}} = \frac{100}{1024} = \frac{25}{256}$

Section 9.3 Exercises

For #1–4, substitute $n = 1, n = 2, \ldots, n = 6$, and $n = 100$.

1. $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}; \frac{101}{100}$
2. $\frac{4}{3}, 1, \frac{4}{5}, \frac{2}{3}, \frac{4}{7}, \frac{1}{2}; \frac{2}{51}$
3. 0, 6, 24, 60, 120, 210; 999,900
4. −4, −6, −6, −4, 0, 6; 9500

For #5–10, use previously computed values of the sequence to find the next term in the sequence.

5. 8, 4, 0, −4; −20
6. −3, 7, 17, 27; 67
7. 2, 6, 18, 54; 4374
8. 0.75, −1.5, 3, −6; −96
9. 2, −1, 1, 0; 3
10. −2, 3, 1, 4; 23
11. $\lim_{n\to\infty} n^2 = \infty$, so the sequence diverges.
12. $\lim_{n\to\infty} \frac{1}{2^n} = 0$, so the sequence converges to 0.
13. $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots, \frac{1}{n^2}, \ldots$
    $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so the sequence converges to 0.
14. $\lim_{n\to\infty} (3n-1) = \infty$, so the sequence diverges.
15. Since the degree of the numerator is the same as the degree of the denominator, the limit is the ratio of the leading coefficients. Thus $\lim_{n\to\infty} \frac{3n-1}{2-3n} = -1$. The sequence converges to −1.

16. Since the degree of the numerator is the same as the degree of the denominator, the limit is the ratio of the leading coefficients. Thus $\lim_{n\to\infty} \frac{2n-1}{n+1} = 2$. The sequence converges to 2.

17. $\lim_{n\to\infty}(0.5)^n = \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$, so the sequence converges to 0.

18. $\lim_{n\to\infty}(1.5)^n = \lim_{n\to\infty}\left(\frac{3}{2}\right)^n = \infty$, so the sequence diverges.

19. $a_1 = 1$ and $a_{n+1} = a_n + 3$ for $n \ge 1$ yields $1, 4, 7, \ldots, (3n-2), \ldots$
$\lim_{n\to\infty}(3n-2) = \infty$, so the sequence diverges.

20. $u_1 = 1$ and $u_{n+1} = \frac{u_n}{3}$ for $n \ge 1$ yields 1, $\frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^{n-1}}, \ldots$
$\lim_{n\to\infty}\frac{1}{3^{n-1}} = 0$, so the sequence converges to 0.

For #21–24, subtract the first term from the second to find the common difference d. Use the formula $a_n = a_1 + (n-1)d$ with $n = 10$ to find the tenth term. The recursive rule for the nth term is $a_n = a_{n-1} + d$, and the explicit rule is the one given above.

21. **(a)** $d = 4$
(b) $a_{10} = 6 + 9(4) = 42$
(c) Recursive rule: $a_1 = 6$; $a_n = a_{n-1} + 4$ for $n \ge 2$
(d) Explicit rule: $a_n = 6 + 4(n-1)$

22. **(a)** $d = 5$
(b) $a_{10} = -4 + 9(5) = 41$
(c) Recursive rule: $a_1 = -4$; $a_n = a_{n-1} + 5$ for $n \ge 2$
(d) Explicit rule: $a_n = -4 + 5(n-1)$

23. **(a)** $d = 3$
(b) $a_{10} = -5 + 9(3) = 22$
(c) Recursive rule: $a_1 = -5$; $a_n = a_{n-1} + 3$ for $n \ge 2$
(d) Explicit rule: $a_n = -5 + 3(n-1)$

24. **(a)** $d = 11$
(b) $a_{10} = -7 + 9(11) = 92$
(c) Recursive rule: $a_1 = -7$; $a_n = a_{n-1} + 11$ for $n \ge 2$
(d) Explicit rule: $a_n = -7 + 11(n-1)$

For #25–28, divide the second term by the first to find the common ratio r. Use the formula $a_n = a_1 \cdot r^{n-1}$ with $n = 8$ to find the eighth term. The recursive rule for the nth term is $a_n = a_{n-1} \cdot r$, and the explicit rule is the one given above.

25. **(a)** $r = 3$
(b) $a_8 = 2 \cdot 3^7 = 4374$
(c) Recursive rule: $a_1 = 2$; $a_n = 3a_{n-1}$ for $n \ge 2$
(d) Explicit rule: $a_n = 2 \cdot 3^{n-1}$

26. **(a)** $r = 2$
(b) $a_8 = 3 \cdot 2^7 = 384$
(c) Recursive rule: $a_1 = 3$; $a_n = 2a_{n-1}$ for $n \ge 2$
(d) Explicit rule: $a_n = 3 \cdot 2^{n-1}$

27. **(a)** $r = -2$
(b) $a_8 = (-2)^7 = -128$
(c) Recursive rule: $a_1 = 1$; $a_n = -2a_{n-1}$ for $n \ge 2$
(d) Explicit rule: $a_n = (-2)^{n-1}$

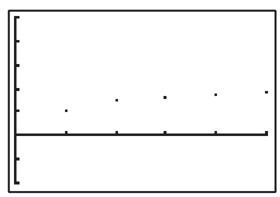
28. **(a)** $r = -1$
(b) $a_8 = -2 \cdot (-1)^7 = 2$
(c) Recursive rule: $a_1 = -2$; $a_n = -1a_{n-1} = -a_{n-1}$ for $n \ge 2$
(d) Explicit rule: $a_n = -2 \cdot (-1)^{n-1} = 2 \cdot (-1)^n$

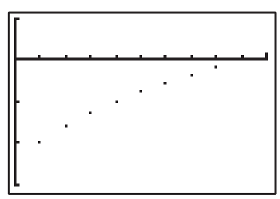
29. $a_4 = -8 = a_1 + 3d$ and $a_7 = 4 = a_1 + 6d$, so $a_7 - a_4 = 12 = 3d$. Therefore $d = 4$, so $a_1 = -8 - 3d = -20$ and $a_n = a_{n-1} + 4$ for $n \ge 2$.

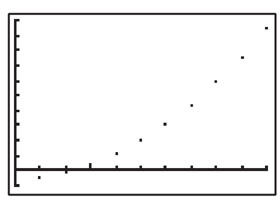
30. $a_5 = -5 = a_1 + 4d$ and $a_9 = -17 = a_1 + 8d$, so $a_9 - a_5 = -12 = 4d$. Therefore $d = -3$, so $a_1 = -5 - 4d = 7$ and $a_n = a_{n-1} - 3$ for $n \ge 2$.

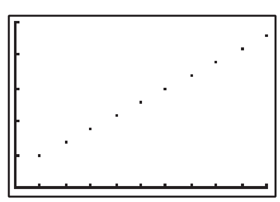
31. $a_2 = 3 = a_1 \cdot r^1$ and $a_8 = 192 = a_1 \cdot r^7$, so $a_8/a_2 = 64 = r^6$. Therefore $r = \pm 2$, so $a_1 = 3/(\pm 2)$ $= \pm\frac{3}{2}$ and $a_n = -\frac{3}{2} \cdot (-2)^{n-1} = 3 \cdot (-2)^{n-2}$ or $a_n = \frac{3}{2} \cdot 2^{n-1} = 3 \cdot 2^{n-2}$.

32. $a_3 = -75 = a_1 \cdot r^2$ and $a_6 = -9375 = a_1 \cdot r^5$, so $a_6/a_3 = 125 = r^3$. Therefore $r = 5$, so $a_1 = -75/5^2 = -3$ and $a_n = -3 \cdot 5^{n-1}$.

33.

[0, 5] by [−2, 5]

34.

[0, 10] by [−3, 1]

35.

[0, 10] by [−10, 100]

36.

[0, 10] by [0, 25]

37. The height (in cm) will be an arithmetic sequence with common difference $d = 2.3$ cm, so the height in week n is $700 + 2.3(n-1)$: 700, 702.3, 704.6, 706.9, … , 815, 817.3.

38. The first column is an arithmetic sequence with common difference $d = 14$. The second column is a geometric sequence with common ratio $r = \frac{1}{2}$.

Time (billions of years)	Mass (g)
0	16
14	8
28	4
42	2
56	1

39. The numbers of seats in each row form a finite arithmetic sequence with $a_1 = 7$, $d = 2$, and $n = 25$. The total number of seats is

$$\frac{25}{2}[2(7) + (25 - 1)(2)] = 775.$$

40. The numbers of tiles in each row form a finite arithmetic sequence with $a_1 = 15$, $a_n = 30$, and $n = 16$. The total number of tiles is

$$16\left(\frac{15 + 30}{2}\right) = 360.$$

41. The ten-digit numbers will vary; thus the sequences will vary. The end result will, however, be the same. Each limit will be 9. One example is:

Five random digits: 1, 4, 6, 8, 9
Five random digits: 2, 3, 4, 5, 6
List: 1, 2, 3, 4, 4, 5, 6, 6, 8, 9
Ten-digit number: 2, 416, 345, 689
Ten-digit number: 9, 643, 128, 564
a_1 = positive difference of the ten-digit numbers
= 7, 226, 782, 875
a_{n+1} = sum of the digits of a_n, so
a_2 = sum of the digits of $a_1 = 54$
a_3 = sum of the digits of $a_2 = 9$.
All successive sums of digits will be 9, so the sequence converges and the limit is 9.

42. Everyone should end up at the word "all."

43. True. Since two successive terms are negative, the common ratio r must be positive, and so the sign of the first term determines the sign of every number in the sequence.

44. False. For example, consider the sequence 5, 1, −3, −7, …

45. $a_1 = 2$ and $a_2 = 8$ implies $d = 8 - 2 = 6$
$c = a_1 - d$ so $c = 2 - 6 = -4$
$a_4 = 6 \cdot 4 + (-4) = 20$.
The answer is A.

46. $\lim_{n\to\infty} \sqrt{n} = \lim_{n\to\infty} n^{1/2} = \infty$, so the sequence diverges.
The answer is B.

47. $r = \frac{a_2}{a_1} = \frac{6}{2} = 3$
$a_6 = a_1 r^5 = 2 \cdot 3^5 = 486$ and $a_2 = 6$, so
$\frac{a_6}{a_2} = \frac{486}{6} = 81$.
The answer is E.

48. The geometric sequence will be defined by $a_{n+1} = a_n \div 3$ for $n \geq 1$ and $a_1 \neq 0$.

$a_2 = \frac{a_1}{3}$

$a_3 = \frac{a_2}{3} = \frac{a_1/3}{3} = \frac{a_1}{9}$

$a_4 = \frac{a_3}{3} = \frac{a_1/9}{3} = \frac{a_1}{27}$

$a_n = \frac{a_1}{3^{n-1}}$, which represents a geometric sequence.

The answer is C.

49. **(a)** $a_1 = 1$ because there is initially one male-female pair (this is the number of pairs after 0 months). $a_2 = 1$ because after one month, the original pair has only just become fertile. $a_3 = 2$ because after two months, the original pair produces a new male-female pair.

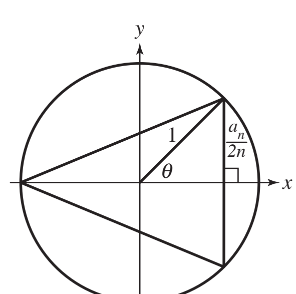
(b) Notice that after $n - 2$ months, there are a_{n-1} pairs, of which a_{n-2} (the number of pairs present one month earlier) are fertile. Therefore, after $n - 1$ months, the number of pairs will be $a_n = a_{n-1} + a_{n-2}$: to last month's total, we add the number of new pairs born. Thus $a_4 = 3$, $a_5 = 5$, $a_6 = 8$, $a_7 = 13$, $a_8 = 21$, $a_9 = 34$, $a_{10} = 55$, $a_{11} = 89$, $a_{12} = 144$, $a_{13} = 233$.

(c) Since a_1 is the initial number of pairs, and a_2 is the number of pairs after one month, we see that a_{13} is the number of pairs after 12 months.

50. Use a calculator: $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $a_5 = 5$, $a_6 = 8$, $a_7 = 13$. These are the first seven terms of the Fibonacci sequence.

51. **(a)** For a polygon with n sides, let A be the vertex in quadrant I at the top of the vertical segment, and let B be the point on the x-axis directly below A. Together with (0, 0), these two points form a right triangle; the acute angle at the origin has measure $\theta = \frac{2\pi}{2n} = \frac{\pi}{n}$, since there are $2n$ such triangles making up the polygon. The length of the side opposite this angle is $\sin\theta = \sin\frac{\pi}{n}$, and there are $2n$ such sides making up the perimeter of the polygon, so $\sin\frac{\pi}{n} = \frac{a_n}{2n}$, or $a_n = 2n\sin(\pi/n)$.

(b) $a_{10} \approx 6.1803$, $a_{100} \approx 6.2822$, $a_{1000} \approx 6.2832$, $a_{10,000} \approx 6.2832 \approx 2\pi$. It appears that $a_n \to 2\pi$ as $n \to \infty$, which makes sense since the perimeter of the polygon should approach the circumference of the circle.



52. $P_1 = 525{,}000$; $P_n = 1.0175P_{n-1}$, $n \geq 2$

53. The difference of successive terms in $\{\log(a_n)\}$ will be of the form $\log(a_{n+1}) - \log(a_n) = \log\left(\frac{a_{n+1}}{a_n}\right)$. Since $\{a_n\}$ is geometric, $\frac{a_{n+1}}{a_n}$ is constant. This makes $\log\left(\frac{a_{n+1}}{a_n}\right)$ constant, so $\{\log(a_n)\}$ is a sequence with a constant difference (arithmetic).

54. The ratios of successive terms in $\{10^{b_n}\}$ will be of the form $10^{b_{n+1}} \div 10^{b_n} = 10^{b_{n+1}-b_n}$. Since $\{b_n\}$ is arithmetic, $b_{n+1} - b_n$ is constant. This makes $10^{b_{n+1}-b_n}$ constant, so $\{10^{b_n}\}$ is a sequence with a common ratio (geometric).

55. $a_1 = [1\ \ 1], a_2 = [1\ \ 2], a_3 = [2\ \ 3], a_4 = [3\ \ 5]$, $a_5 = [5\ \ 8], a_6 = [8\ \ 13], a_7 = [13\ \ 21]$. The entries in the terms of this sequence are successive pairs of terms from the Fibonacci sequence.

56. $a_1 = [1\quad a] \quad r = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$

$a_2 = a_1 \cdot r = [1\quad a + d]$
$a_3 = a_2 \cdot r = [1\quad d + a + d] = [1\quad a + 2d]$
$a_4 = a_3 \cdot r = [1\quad d + a + 2d] = [1\quad a + 3d]$
$a_n = [1\quad a + (n-1)d]$.
So, the second entries of this geometric sequence of matrices form an arithmetic sequence with the first term a and common difference d.

■ Section 9.4 Series

Exploration 1

1. $3 + 6 + 9 + 12 + 15 = 45$
2. $5^2 + 6^2 + 7^2 + 8^2 = 25 + 36 + 49 + 64 = 174$
3. $\cos(0) + \cos(\pi) + \cdots + \cos(11\pi) + \cos(12\pi)$ $= 1 - 1 + 1 + \cdots - 1 + 1 = 1$
4. $\sin(0) + \sin(\pi) + \cdots + \sin(k\pi)$ $+ \cdots = 0 + 0 + \cdots + 0 + \cdots = 0$
5. $\frac{3}{10} + \frac{3}{100} + \frac{3}{1{,}000} + \cdots + \frac{3}{1{,}000{,}000} + \cdots = \frac{1}{3}$

Exploration 2

1. $1 + \ \ 2 + \ \ 3 + \cdots + 99 + 100$
2. $100 + \ 99 + \ 98 + \cdots + \ 2 + \ \ 1$
3. $101 + 101 + 101 + \cdots + 101 + 101$
4. $100(101) = 10{,}100$
5. The sum in Exercise 4 involves two copies of the same progression, so it doubles the sum of the progression. The answer that Gauss gave was 5050.

Quick Review 9.4

1. $a_1 = 4;\ d = 2$ so $a_{10} = a_1 + (n-1)d$
$a_{10} = 4 + (10-1)2 = 4 + 18 = 22$
$a_{10} = 22$

2. $a_1 = 3;\ a_2 = 1$ so $d = 1 - 3 = -2$
$a_{10} = a_1 + (n-1)d$
$a_{10} = 3 + (10-1)(-2) = 3 - 18 = -15$
$a_{10} = -15$

3. $a_3 = 6$ and $a_8 = 21$
$a_3 = a_1 + 2d$ and $a_8 = a_1 + 7d$
$(a_1 + 7d) - (a_1 + 2d) = 21 - 6$ so $5d = 15 \Rightarrow d = 3.$
$6 = a_1 + 2(3)$ so $a_1 = 0$
$a_{10} = 0 + 9(3) = 27$
$a_{10} = 27$

4. $a_5 = 3$, and $a_{n+1} = a_n + 5$ for $n \geq 1 \Rightarrow a_6 = 3 + 5 = 8$
$a_5 = 3$ and $a_6 = 8 \Rightarrow d = 5$
$a_5 = a_1 + 4d$ so $3 = a_1 + 4(5) \Rightarrow a_1 = -17$
$a_{10} = -17 + 9(5) = 28$
$a_{10} = 28$

5. $a_1 = 1$ and $a_2 = 2$ yields $r = \frac{2}{1} = 2$
$a_{10} = 1 \cdot 2^9 = 512$
$a_{10} = 512$

6. $a_4 = 1$ and $a_4 = a_1 \cdot r^3$; $a_6 = 2$ and $a_6 = a_1 \cdot r^5$
$\frac{a_1 \cdot r^5}{a_1 \cdot r^3} = \frac{2}{1}$
$r^2 = 2 \Rightarrow r = \sqrt{2}$
$1 = a_1(\sqrt{2})^3;\quad a_1 = \frac{1}{(\sqrt{2})^3} = \frac{1}{2\sqrt{2}}$
$a_{10} = \frac{1}{2\sqrt{2}}(\sqrt{2})^9 = \frac{16\sqrt{2}}{2\sqrt{2}} = 8$
$a_{10} = 8$

7. $a_7 = 5$ and $r = -2 \Rightarrow 5 = a_1(-2)^6$
$a_1 = \frac{5}{64};\quad a_{10} = \frac{5}{24}(-2)^9 = \frac{-2560}{24} = -40$
$a_{10} = -40$

8. $a_8 = 10$ and $a_8 = a_1 \cdot r^7$; $a_{12} = 40 \Rightarrow a_{12} = a_1 \cdot r^{11}$
$\frac{a_1 \cdot r^{11}}{a_1 \cdot r^7} = \frac{40}{10}$
$r^4 = 4$; so $r = (4)^{1/4}$
$10 = a_1((4)^{1/4})^7;\quad a_1 = \frac{10}{4^{7/4}}$
$a_{10} = \frac{10}{4^{7/4}}(4^{1/4})^9 = \frac{10(4^{9/4})}{4^{7/4}} = 10(4^{2/4}) = 10 \cdot 2 = 20$
$a_{10} = 20$

9. $\sum_{n=1}^{5} n^2 = 1 + 4 + 9 + 16 + 25 = 55$

10. $\sum_{n=1}^{5} (2n-1) = 1 + 3 + 5 + 7 + 9 = 25$

Section 9.4 Exercises

1. $\sum_{k=1}^{11} (6k - 13)$
2. $\sum_{k=1}^{10} (3k - 1)$
3. $\sum_{k=1}^{n+1} k^2$
4. $\sum_{k=1}^{n+1} k^3$
5. $\sum_{k=0}^{\infty} 6(-2)^k$
6. $\sum_{k=0}^{\infty} 5(-3)^k$

For #7–12, use one of the formulas $S_n = n\left(\frac{a_1 + a_n}{2}\right)$ or $S_n = \frac{n}{2}[2a_1 + (n-1)d]$. In most cases, the first of these is easier (since the last term a_n is given); note that $n = \frac{a_n - a_1}{d} + 1$.

7. $6 \cdot \left(\frac{-7+13}{2}\right) = 6 \cdot 3 = 18$

8. $6 \cdot \left(\frac{-8+27}{2}\right) = 3 \cdot 19 = 57$

9. $80 \cdot \left(\frac{1+80}{2}\right) = 40 \cdot 81 = 3240$

10. $35 \cdot \left(\frac{2+70}{2}\right) = 35 \cdot 36 = 1260$

11. $13 \cdot \left(\frac{117+33}{2}\right) = 13 \cdot 75 = 975$

12. $29\left(\frac{111+27}{2}\right) = 29 \cdot 69 = 2001$

For #13–16, use the formula $S_n = \frac{a_1(1-r^n)}{1-r}$. Note that $n = 1 + \log_{|r|}\left|\frac{a_n}{a_1}\right| = 1 + \frac{\ln|a_n/a_1|}{\ln|r|}$.

13. $\frac{3(1-2^{13})}{1-2} = 24{,}573$

14. $\frac{5(1-3^{10})}{1-3} = 147{,}620$

15. $\frac{42[1-(1/6)^9]}{1-(1/6)} = 50.4(1-6^{-9}) \approx 50.4$

16. $\frac{42[1-(-1/6)^{10}]}{1-(-1/6)} = 36(1-6^{-10}) = 36 - 6^{-8} \approx 36$

For #17–22, use one of the formulas $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ or $S_n = \frac{a_1(1-r^n)}{1-r}$.

17. Arithmetic with $d = 3$: $\frac{10}{2}[2 \cdot 2 + (10-1)(3)]$
$= 5 \cdot 31 = 155$

18. Arithmetic with $d = -6$: $\frac{9}{2}[2 \cdot 14 + (9-1)(-6)]$
$= 9 \cdot (-10) = -90$

19. Geometric with $r = -\frac{1}{2}$: $\frac{4[1-(-1/2)^{12}]}{1-(-1/2)}$
$= \frac{8}{3} \cdot (1 - 2^{-12}) \approx 2.666$

20. Geometric with $r = -\frac{1}{2}$: $\frac{6[1-(-1/2)^{11}]}{1-(-1/2)}$
$= 4 \cdot (1 + 2^{-11}) \approx 4.002$

21. Geometric with $r = -11$: $\frac{-1[1-(-11)^9]}{1-(-11)}$
$= -\frac{1}{12} \cdot (1 + 11^9) = -196{,}495{,}641$

22. Geometric with $r = -12$: $\frac{-2[1-(-12)^8]}{1-(-12)}$
$= -\frac{2}{13} \cdot (1 - 12^8) = 66{,}151{,}030$

23. (a) The first six partial sums are {0.3, 0.33, 0.333, 0.3333, 0.33333, 0.333333}. The numbers appear to be approaching a limit of $0.\overline{3} = 1/3$. The series is convergent.

(b) The first six partial sums are {1, −1, 2, −2, 3, −3}. The numbers approach no limit. The series is divergent.

24. (a) The first six partial sums are {−2, 0, −2, 0, −2, 0}. The numbers approach no limit. The series is divergent.

(b) The first six partial sums are {1, 0.3, 0.23, 0.223, 0.2223, 0.22223}. The numbers appear to be approaching a limit of $0.\overline{2} = 2/9$. The series is convergent.

25. $r = \frac{1}{2}$, so it converges to $S = \frac{6}{1-(1/2)} = 12$.

26. $r = \frac{1}{3}$, so it converges to $S = \frac{4}{1-(1/3)} = 6$.

27. $r = 2$, so it diverges.

28. $r = 3$, so it diverges.

29. $r = \frac{1}{4}$, so it converges to $S = \frac{3/4}{1-(1/4)} = 1$.

30. $r = \frac{2}{3}$, so it converges to $S = \frac{10/3}{1-(2/3)} = 10$.

31. $7 + \frac{14}{99} = \frac{693}{99} + \frac{14}{99} = \frac{707}{99}$

32. $5 + \frac{93}{99} = 5 + \frac{31}{33} = \frac{196}{33}$

33. $-17 - \frac{268}{999} = -\frac{17{,}251}{999}$

34. $-12 - \frac{876}{999} = -12 - \frac{292}{333} = -\frac{4288}{333}$

35. (a) The ratio of any two successive account balances is $r = 1.1$. That is,
$$\frac{\$22{,}000}{\$20{,}000} = \frac{\$24{,}200}{\$22{,}000} = \frac{\$26{,}620}{\$24{,}200} = \frac{\$29{,}282}{\$26{,}620} = 1.1.$$

(b) Each year, the balance is 1.1 times as large as the year before. So, n years after the balance is \$20,000, it will be $\$20{,}000\,(1.1)^n$.

(c) The sum of the eleven terms of the geometric sequence is $\frac{\$20{,}000(1-1.1^{11})}{1-1.1} = \$370{,}623.34$.

36. (a) The difference of any two successive account balances is $d = \$2016$. That is \$20,016 − \$18,000 = \$22,032 − \$20,016 = \$24,048 − \$22,032 = \$26,064 − \$24,048 = \$2016.

(b) Each year, the balance is \$2016 more than the year before. So, n years after the balance is \$18,000, it will be $\$18{,}000 + \$2016n$.

(c) The sum of the eleven terms of the arithmetic sequence is
$$\frac{11}{2}[2(\$18{,}000) + (10)(\$2016)] = \$308{,}880.$$

37. (a) The first term, $120(1 + 0.07/12)^0$, simplifies to 120. The common ratio of terms, r, equals $1 + 0.07/12$.

(b) The sum of the 120 terms is

$$\frac{120\,[1 - (1 + 0.07/12)^{120}]}{1 - (1 + 0.07/12)} = \$20{,}770.18.$$

38. (a) The first term, $100(1 + 0.08/12)^0$, simplifies to 100. The common ratio of terms, r, equals $1 + 0.08/12$.

(b) The sum of the 120 terms is

$$\frac{100\,[1 - (1 + 0.08/12)^{120}]}{1 - (1 + 0.08/12)} = \$18{,}294.60.$$

39. The heights of the ball on the bounces after the first bounce can be modeled by an infinite geometric series. The total height traveled by the ball on the subsequent bounces is:

$$2 \cdot [2(0.9) + 2(0.9)^2 + 2(0.9)^3 + 2(0.9)^4 + \cdots]$$
$$= 4 \cdot [(0.9) + (0.9)^2 + (0.9)^3 + (0.9)^4 + \cdots]$$
$$= 4 \cdot \left[\frac{0.9}{1 - 0.9}\right] = 36 \text{ m}.$$

Since the ball was dropped from 2 m, the total distance traveled by the ball is 36 m + 2 m = 38 m.

40. This is an example of a divergent infinite series; the ball would travel forever and traverse an infinite distance.

41. False. The series might diverge. For example, examine the series $1 + 2 + 3 + 4 + 5$ where all of the terms are positive. Consider the limit of the sequence of partial sums. The first five partial sums are {1, 3, 6, 10, 15}. These numbers increase without bound and do not approach a limit. Therefore, the series diverges and has no sum.

42. False. Justifications will vary. One example is to examine

$$\sum_{n=1}^{\infty} n \quad \text{and} \quad \sum_{n=1}^{\infty} (-n).$$

Both of these diverge, but $\sum_{n=1}^{\infty}(n + (-n)) = \sum_{n=1}^{\infty} 0 = 0$.

So the sum of the two divergent series converges.

43. $3^{-1} + 3^{-2} + 3^{-3} + \cdots + 3^{-n} + \cdots =$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots + \frac{1}{3^n} + \cdots$$

The first five partial sums are $\left\{\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}, \frac{121}{243}\right\}$. These appear to be approaching a limit of 1/2, which would suggest that the series converges to 1/2. The answer is A.

44. If $\sum_{n=1}^{\infty} x^n = 4$, then $x = 0.8$.

$$\sum_{n=1}^{\infty} 0.8^n = 0.8 + 0.64 + 0.512 + 0.4096 + 0.32768 + 0.262144 + \cdots$$

The first six partial sums are {0.8, 1.44, 1.952, 2.3616, 2.68928, 2.951424}. It appears from this sequence of partial sums that the series is converging. If the sequence of partial sums were extended to the 40th partial sum, you would see that the series converges to 4. The answer is D.

45. The common ratio is $0.75/3 = 0.25$, so the sum of the infinite series is $3/(1 - 0.25) = 4$. The answer is C.

46. The sum is an infinite geometric series with $|r| = 5/3 > 1$. The answer is E.

47. (a) Heartland: 20,505,437 people.
Southeast: 48,310,650 people.

(b) Heartland: 517,825 mi^2.
Southeast: 348,999 mi^2.

(c) Heartland: $\frac{20{,}505{,}437}{517{,}825} \approx 39.60$ people/mi^2.

Southeast: $\frac{48{,}310{,}650}{348{,}999} \approx 138.43$ people/mi^2.

(d) The table is shown below; the answer differs because the overall population density $\frac{\Sigma \text{ population}}{\Sigma \text{ area}}$ is generally not the same as the average of the population densities, $\frac{1}{n}\Sigma\left(\frac{\text{population}}{\text{area}}\right)$. The larger states within each group have a greater effect on the overall mean density. In a similar way, if a student's grades are based on a 100-point test and four 10-point quizzes, her overall average grade depends more on the test grade than on the four quiz grades.

Heartland:		Southeast:	
Iowa	≈54.13	Alabama	≈92.44
Kansas	≈34.68	Arkansas	≈54.82
Minnesota	≈62.84	Florida	≈320.60
Missouri	≈85.93	Georgia	≈164.45
Nebraska	≈23.61	Louisiana	≈94.94
N. Dakota	≈9.51	Mississippi	≈62.22
S. Dakota	≈10.56	S. Carolina	≈148.66
Total	≈281.26	Total	≈938.14
Average	≈40.18	Average	≈134.02

48. $\sum_{k=1}^{8}(k^2 - 2)$

49. The table suggests that $S_n = \sum_{k=1}^{n} F_k = F_{n+2} - 1$.

n	F_n	S_n	$F_{n+2} - 1$
1	1	1	1
2	1	2	2
3	2	4	4
4	3	7	7
5	5	12	12
6	8	20	20
7	13	33	33
8	21	54	54
9	34	88	88

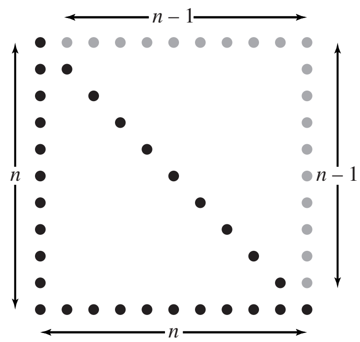
50. The nth triangular number is simply the sum of the first n consecutive positive integers:

$$1 + 2 + 3 + \cdots + n = n\left(\frac{1+n}{2}\right) = \frac{n(n+1)}{2}$$

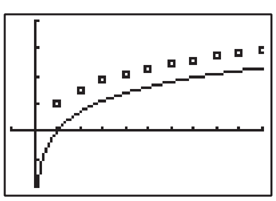
51. Algebraically: $T_{n-1} + T_n = \frac{(n-1)n}{2} + \frac{n(n+1)}{2}$

$$= \frac{n^2 - n + n^2 + n}{2} = n^2$$

Geometrically: The array of black dots in the figure represents $T_n = 1 + 2 + 3 + \cdots + n$ (that is, there are T_n dots in the array). The array of gray dots represents $T_{n-1} = 1 + 2 + 3 + \cdots + (n-1)$. The two triangular arrays fit together to form an $n \times n$ square array, which has n^2 dots.



52. If $\sum_{k=1}^{n} \frac{1}{k} \geq \ln n$ for all n, then the sum diverges since as $n \to \infty$, $\ln n \to \infty$.



[−1, 10] by [−2, 4]

■ Section 9.5 Mathematical Induction

Exploration 1

1. Start with the rightmost peg if n is odd and the middle peg if n is even. From that point on, the first move for moving any smaller stack to a destination peg should be directly to the destination peg if the smaller stack's size n is odd and to the other available peg if n is even. The fact that the winning strategy follows such predictable rules is what makes it so interesting to students of computer programming.

Exploration 2

1. 43, 47, 53, 61, 71, 83, 97, 113, 131, 151. Yes.

2. 173, 197, 223, 251, 281, 313, 347, 383, 421, 461. Yes.

3. 503, 547, 593, 641, 691, 743, 797, 853, 911, 971. Yes. Inductive thinking might lead to the conjecture that $n^2 + n + 41$ is prime for all n, but we have no proof as yet!

4. The next 9 numbers are all prime, but $40^2 + 40 + 41$ is not. Quite obviously, neither is the number $41^2 + 41 + 41$.

Quick Review 9.5

1. $n^2 + 5n$

2. $n^2 - n - 6$

3. $k^3 + 3k^2 + 2k$

4. $(n + 3)(n - 1)$

5. $(k + 1)^3$

6. $(n - 1)^3$

7. $f(1) = 1 + 4 = 5$, $f(t) = t + 4$,
$f(t + 1) = t + 1 + 4 = t + 5$

8. $f(1) = \frac{1}{1+1} = \frac{1}{2}$, $f(k) = \frac{k}{k+1}$,
$f(k+1) = \frac{k+1}{k+1+1} = \frac{k+1}{k+2}$

9. $P(1) = \frac{2 \cdot 1}{3 \cdot 1 + 1} = \frac{1}{2}$,
$P(k) = \frac{2k}{3k+1}$; $P(k+1) = \frac{2(k+1)}{3(k+1)+1} = \frac{2k+2}{3k+4}$

10. $P(1) = 2(1)^2 - 1 - 3 = -2$, $P(k) = 2k^2 - k - 3$,
$P(k + 1) = 2(k + 1)^2 - (k + 1) - 3 = 2k^2 + 3k - 2$

Section 9.5 Exercises

1. P_n: $2 + 4 + 6 + \cdots + 2n = n^2 + n$.
P_1 is true: $2(1) = 1^2 + 1$.
Now assume P_k is true: $2 + 4 + 6 + \ldots + 2k$
$= k^2 + k$. Add $2(k + 1)$ to both sides:
$2 + 4 + 6 + \cdots + 2k + 2(k + 1)$
$= k^2 + k + 2(k + 1) = k^2 + 3k + 2$
$= k^2 + 2k + 1 + k + 1 = (k + 1)^2 + (k + 1)$, so
P_{k+1} is true. Therefore, P_n is true for all $n \geq 1$.

2. P_n: $8 + 10 + 12 + \cdots + (2n + 6) = n^2 + 7n$.
P_1 is true: $2(1) + 6 = 1^2 + 7 \cdot 1$.
Now assume P_k is true:
$8 + 10 + 12 + \cdots + (2k + 6) = k^2 + 7k$.
Add $2(k + 1) + 6 = 2k + 8$ to both sides:
$8 + 10 + 12 + \cdots + (2k + 6) + [2(k + 1) + 6]$
$= k^2 + 7k + 2k + 8 = (k^2 + 2k + 1) + 7k + 7$
$= (k + 1)^2 + 7(k + 1)$, so P_{k+1} is true.
Therefore, P_n is true for all $n \geq 1$.

3. P_n: $6 + 10 + 14 + \cdots + (4n + 2) = n(2n + 4)$.
P_1 is true: $4(1) + 2 = 1(2 + 4)$.