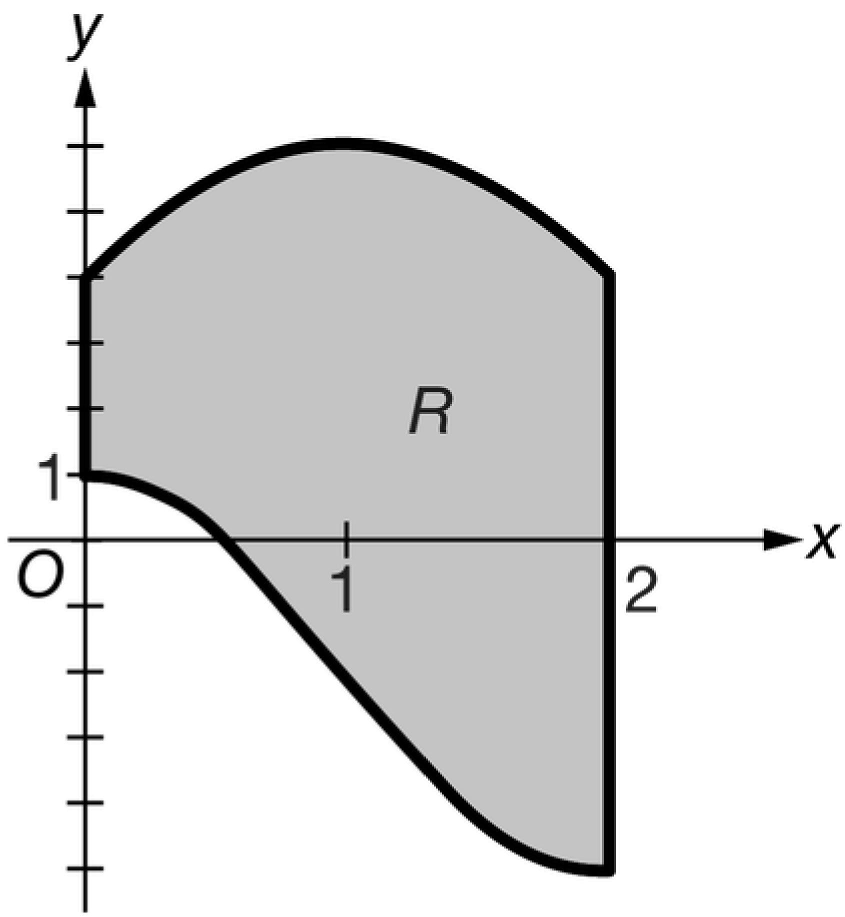
8.1-8.3

1. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.



Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and

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$h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

a) Find the area of R.

Please respond on separate paper, following directions from your teacher.

b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

Please respond on separate paper, following directions from your teacher.

c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

Please respond on separate paper, following directions from your teacher.

Part A

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

The first point is earned for an integrand that contains a form of $h(x) - g(x)$ or $g(x) - h(x)$.

The second point is earned for the antiderivative $\frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)$.

A response is eligible for the fourth point if (1) the first point is earned, (2) at least one of the second or third points is earned, (3) the integrand contains no copy errors or arithmetic errors. Simplification is not required to earn the fourth point. The limits of integration are part of the fourth point.

If the limits of integration are $x = 1$ to $x = 2$ in all parts of the question, the response is not eligible for the fourth point in Part A but eligible for points in Part B and Part C.

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0	1	2	3	4 ✔

The student response accurately includes **all four** of the criteria below.

- ☐ integrand
- ☐ antiderivative of $3\cos\left(\frac{\pi}{2}x\right)$
- ☐ antiderivative of remaining terms
- ☐ answer

Solution:

$$\int_0^2 (h(x) - g(x))\ dx = \int_0^2 \left(\left(6 - 2(x-1)^2\right) - \left(-2 + 3\cos\left(\frac{\pi}{2}x\right)\right)\right)\ dx$$

$$= \left[\left(6x - \tfrac{2}{3}(x-1)^3\right) - \left(-2x + \tfrac{6}{\pi}\sin\left(\tfrac{\pi}{2}x\right)\right)\right]_{x=0}^{x=2}$$

$$= \left(\left(12 - \tfrac{2}{3}\right) - (-4+0)\right) - \left(\left(0 + \tfrac{2}{3}\right) - (0+0)\right)$$

$$= 12 - \tfrac{2}{3} + 4 - \tfrac{2}{3} = \tfrac{44}{3}$$

The area of R is $\dfrac{44}{3}$.

Part B

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

The first point is earned for the definite integral $\int_0^2 A(x)\ dx$ or $\int_0^2 \frac{1}{x+3}\ dx$.

0	1	2 ✔

The student response accurately includes **both** of the criteria below.

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- ☐ integral
- ☐ answer

Solution:

$$\int_0^2 A(x)\ dx = \int_0^2 \frac{1}{x+3}\ dx$$
$$= [\ln(x+3)]_{x=0}^{x=2} = \ln 5 - \ln 3$$

The volume of the solid is $\ln 5 - \ln 3$.

Part C

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

The first point is earned for a definite integral, multiplied by π, with 0 and 2 as the lower and upper limits of integration, respectively. The π may appear in the integrand, but it must be correct to earn the first point.

The second point is earned with an integrand that is a difference of squares with a correct or consistent axis of rotation.

The third point is earned for the correct integrand. Note that a reversal of order within the squared terms, e.g. $\pi\int_0^2 \left((g(x)-6)^2-(h(x)-6)^2\right)\ dx$, is equivalent to the given solution and thus earns the third point.

0	1	2	3 ✔

The student response accurately includes **all three** of the criteria below.

- ☐ limits and constant
- ☐ form of integrand
- ☐ integrand

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Solution:

$$\pi \int_0^2 \left((6 - g(x))^2 - (6 - h(x))^2 \right) \, dx$$

2. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Let R be the region in the first quadrant bounded by the graph of $y = \frac{1}{x}$, the horizontal line $y = 1$, and the vertical line $x = e$.

(a) Find the area of region R.

Please respond on separate paper, following directions from your teacher.

(b) Find the volume of the solid generated when region R is revolved about the line $y = 1$.

Please respond on separate paper, following directions from your teacher.

(c) Region R is the base of a solid. For this solid, each cross section perpendicular to the x-

axis is a semicircle. Write, but do not evaluate, an expression involving an integral that gives the volume of the solid.

Please respond on separate paper, following directions from your teacher.

Part A

At most 2 out of 3 points earned if there is one error in the integrand OR 1 error with limits of integration. Numerical answers do not need to be simplified (e.g., $\ln\, e$)

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3 ✓

The student response accurately includes all three of the criteria below.

- ☐ definite integral
- ☐ antiderivative
- ☐ answer

Solution:

$$\text{Area} = \int_1^e \left(1 - \frac{1}{x}\right)\, dx = (x - \ln\, x)\,|_1^e = (e - \ln\, e) - (1 - \ln\, 1) = e - 2$$

Part B

At most 2 out of 3 points earned if there is one error in the integrand OR 1 error with limits of integration. Numerical answers do not need to be simplified (e.g., $\ln\, e$)

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3	4 ✓

The student response accurately includes all four of the criteria below.

- ☐ integrand
- ☐ limits and π
- ☐ antiderivative
- ☐ answer

Solution:

$$\text{Volume} = \pi \int_1^e \left(1 - \frac{1}{x}\right)^2 dx = \pi \int_1^e \left(1 - \frac{2}{x} + \frac{1}{x^2}\right) dx$$
$$= \pi\left(x - 2\ln x - \tfrac{1}{x}\right)\Big|_1^e = \pi\left(e - 2\ln e - \tfrac{1}{e}\right) = \pi\left(e - 2 - \tfrac{1}{e}\right)$$

Part C

Halving the diameter of the semicircle to find an expression for its radius may appear in the integrand, as presented. It may also appear as part of the constant multiple.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2 ✓

The student response accurately includes both of the criteria below.

- ☐ integrand
- ☐ limits and constant

Solution:

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Volume $= \frac{\pi}{2}\int_1^e \left(\frac{1}{2}\left(1-\frac{1}{x}\right)\right)^2 dx$

3. A GRAPHING CALCULATOR IS REQUIRED FOR THIS QUESTION.

You are permitted to use your calculator to solve an equation, find the derivative of a function at a point, or calculate the value of a definite integral. However, you must clearly indicate the setup of your question, namely the equation, function, or integral you are using. If you use other built-in features or programs, you must show the mathematical steps necessary to produce your results. Your work must be expressed in standard mathematical notation rather than calculator syntax.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Let R be the region in the first quadrant enclosed by the graphs of $y = e^{(x^2)} - 1$ and $y = 4x$.

(a) Find the area of R.

Please respond on separate paper, following directions from your teacher.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

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Please respond on separate paper, following directions from your teacher.

(c) Region R forms the base of a solid whose cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

Please respond on separate paper, following directions from your teacher.

Part A

The first point requires a definite integral with a correct integrand. At most 2 out of 3 points earned for a copy error in the integrand OR a reversal of terms in the difference in the integrand. Subsequent points may be earned.

The second point requires correct antidifferentiation.

The third point is for limits and evaluation.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3 ✔

The student response accurately includes all three of the criteria below.

- ☐ integral
- ☐ antiderivative
- ☐ answer

Solution:

$$\text{Area} = \int_0^4 \left(4x - x^2\right)\ dx = \left(2x^2 - \frac{x^3}{3}\right)\Big|_{x=0}^{x=4} = 32 - \frac{64}{3} = \frac{32}{3}$$

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Part B

The first point requires a difference of squares for the integrand.

The second point requires both terms of the integrand.

At most 1 out of 2 integrand points earned if there is one error in the integrand OR a reversal of terms.

The third point requires the limits and π.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3 ✓

The student response accurately includes all three of the criteria below.

- ☐ integrand of difference of squares
- ☐ integrand terms
- ☐ limits and π

Solution:

$$\text{Volume} = \pi \int_0^4 \left(\left(16 - x^2\right)^2 - \left(16 - 4x\right)^2 \right) \, dx$$

Part C

The first point requires an integrand in terms of y that addresses the length of a side of the square cross section.

The second point requires the integrand to represent the cross sectional area.

At most 1 out of 2 integrand points earned if there is one error in the integrand.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

8.1-8.3

0	1	2	3 ✔

The student response accurately includes all three of the criteria below.

- ☐ integrand in terms of y for square cross section
- ☐ integrand terms
- ☐ limits

Solution:

$$\text{Volume} = \int_0^{16} \left(\sqrt{y} - \frac{y}{4}\right)^2 dy$$

The rate at which people enter an auditorium for a rock concert is modeled by the function R given by R(t) = $1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; R(t) is measured in people per hour. No one is in the auditorium at time t = 0, when the doors open. The doors close and the concert begins at time t = 2.

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4. The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time *t*. The derivative of w is given by $w'(t) = (2-t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditorium after time $t = 1$.

Please respond on separate paper, following directions from your teacher.

Part C

1 point is earned for integral

1 point is earned for answer

8.1-8.3

$$w(2) - w(1) = \int_1^2 w'(t)dt = \int_1^2 (2-t)R(t)d(t) = 387.5$$

The total wait time for those who enter the auditorium after time t = 1 is 387.5 hours.

0	1	2

The student response earns two of the following points:

1 point is earned for integral

1 point is earned for answer

$$w(2) - w(1) = \int_1^2 w'(t)dt = \int_1^2 (2-t)R(t)d(t) = 387.5$$

The total wait time for those who enter the auditorium after time t = 1 is 387.5 hours.

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature *T(x)*, in degrees Celsius (˚ C), of the wire *x* cm from the heated end. The function *T* is decreasing and twice differentiable.

5. Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

Please respond on separate paper, following directions from your teacher.

Part C

1 point is earned for the value

1 point is earned for the meaning

$$\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45^\circ C$$

The temperature drops 45˚C from the heated end of the wire of the other end of the wire.

0	1	2 ✓

The student response earns all of the following points:

1 point is earned for the value

1 point is earned for the meaning

$$\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45^\circ C$$

The temperature drops 45˚C from the heated end of the wire of the other end of the wire.

Let *R* be the region in the first quadrant under the graph of $y = \frac{1}{\sqrt{x}}$ for 4 ≤ x ≤ 9 .

6. If the line *x*=*k* divides the region *R* into two regions of equal area, what is the value of *k*?

Please respond on separate paper, following directions from your teacher.

Part B

8.1-8.3

1 point is earned for the correct demonstration of variable limit using *k* as upper or lower limit with 4 or 9 (or imported limit)

$$\int_4^k \frac{dx}{\sqrt{x}} \text{ or } \int_k^9 \frac{dx}{\sqrt{x}}$$

1 point is earned for the correct equation involving the two halves of *R* that when solved will give answer with demonstrated algorithm that knows when to stop

1 point is earned for the correct answer

0/1 if answer from equation not involving relevant areas

1/3 bald answer

$$\int_4^k \frac{dx}{\sqrt{x}} = 1$$
$$2\sqrt{x}]_4^k = 1$$
$$2\sqrt{k} - 2\sqrt{4} = 1$$
$$k = \tfrac{25}{4}$$
$$\left(\text{or } \int_k^9 \frac{dx}{\sqrt{x}} = 1 \text{ or } \int_4^k \frac{dx}{\sqrt{x}} = \int_k^9 \frac{dx}{\sqrt{x}}\right)$$

0	1	2	3 ✓

The student response earns three of the following points:

1 point is earned for the correct demonstration of variable limit using *k* as upper or lower limit with 4 or 9 (or imported limit)

$$\int_4^k \frac{dx}{\sqrt{x}} \text{ or } \int_k^9 \frac{dx}{\sqrt{x}}$$

1 point is earned for the correct equation involving the two halves of *R* that when solved will give answer with demonstrated algorithm that knows when to stop

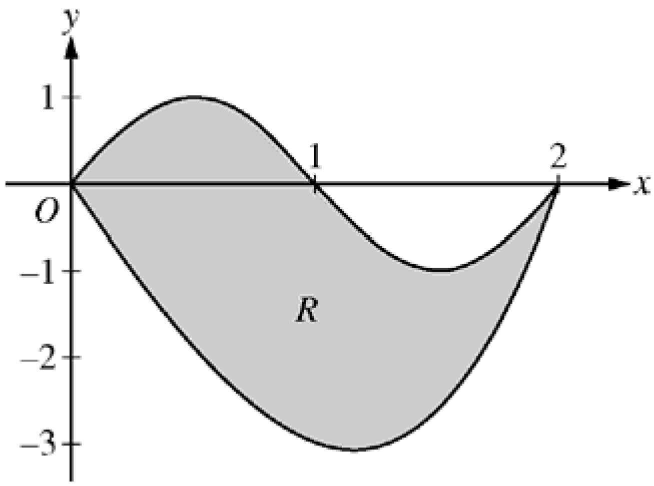
1 point is earned for the correct answer

0/1 if answer from equation not involving relevant areas

8.1-8.3

1/3 bald answer

$$\int_4^k \frac{dx}{\sqrt{x}} = 1$$
$$2\sqrt{x}]_4^k = 1$$
$$2\sqrt{k} - 2\sqrt{4} = 1$$
$$k = \frac{25}{4}$$
$$\left(\text{or } \int_k^9 \frac{dx}{\sqrt{x}} = 1 \text{ or } \int_4^k \frac{dx}{\sqrt{x}} = \int_k^9 \frac{dx}{\sqrt{x}}\right)$$



Let *R* be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

7. The horizontal line *y* = -2 splits the region *R* into two parts. Write, but do not evaluate, an integral expression for the area of the part of *R* that is below this horizontal line.

Please respond on separate paper, following directions from your teacher.

Part B

The response can earn up to 2 points:

8.1-8.3

1 point: For the correct limits

$x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$

1 point: For the correct integrand

The area of the shaded region is $\int_s^r (-2 - (x^3 - 4x))dx$

0	1	2

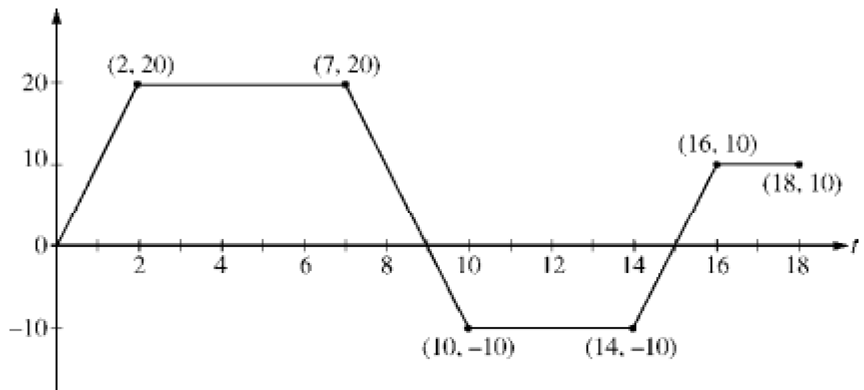
The response earns two of the following points:

1 point: For the correct limits

$x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$

1 point: For the correct integrand

The area of the shaded region is $\int_s^r (-2 - (x^3 - 4x))dx$



Graph of v

A squirrel starts at building *A* at time *t*=0 and travels along a straight,horizontal wire connected to building *B*. For 0≤*t*≤18 , the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.

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8. Write expressions for the squirrel's acceleration *a*(*t*), velocity *v*(*t*), and distance *x*(*t*) from building *A* that are valid for the time interval 7<*t*<10.

Please respond on separate paper, following directions from your teacher.

Part D

1 point is earned for *a*(*t*)

1 point is earned for *v*(*t*)

2 points are earned for *x*(*t*)

$$For 7 < t < 10, a(t) = \frac{20-(-10)}{7-10} = -10$$
$$v(t) = 20 - 10(t-7) = -10t + 90$$
$$x(7) = \frac{7+5}{2}.20 = 120$$
$$x(t) = x(7) + \int_7^t (-10u + 90)du$$
$$= 120 + (-5u^2 + 90u)\Big|_{u=7}^{u=t}$$
$$= -5t^2 + 90t - 265$$

0	1	2	3	4

The student response earns all of the following points:

1 point is earned for *a*(*t*)

1 point is earned for *v*(*t*)

2 points are earned for *x*(*t*)

8.1-8.3

$For 7 < t < 10, a(t) = \frac{20-(-10)}{7-10} = -10$

$v(t) = 20 - 10(t-7) = -10t + 90$

$x(7) = \frac{7+5}{2}.20 = 120$

$$x(t) = x(7) + \int_{7}^{t} (-10u + 90)du$$

$= 120 + (-5u^2 + 90u)|_{u=7}^{u=t}$

$= -5t^2 + 90t - 265$

The following are related to this scenario:

A store is having a 12-hour sale. The total number of shoppers who have entered the store t hours after the sale begins is modeled by the function S defined by $S(t) = 0.5t^4 - 16t^3 + 144t^2$ for $0 \le t \le 12$. At time $t=0$, when the sale begins, there are no shoppers in the store.

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9. The rate at which shoppers leave the store, measured in shoppers per hour, is modeled by the function L defined by $L(t) = -80 + \dfrac{4400}{t^2 - 14t + 55}$ for $0 \le t \le 12$. According to the model, how many shoppers are in the store at the end of the sale (time $t=12$)? Give your answers to the nearest whole number.

Please respond on separate paper, following directions from your teacher.

Part C

The response can earn up to 3 points:

1 point: For correct integral

1 point: For use of S(12)

1 point: For the correct answer

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$$S(12) - \int_0^{12} L(t)dt = 195.701684$$

Therefore there are approximately 196 shoppers in the store at the end of the sale.

0	1	2	3 ✓

The response earns all three of the following points:

1 point: For correct integral

1 point: For use of S(12)

1 point: For the correct answer

$$S(12) - \int_0^{12} L(t)dt = 195.701684$$

Therefore there are approximately 196 shoppers in the store at the end of the sale.

For $0 \leq t \leq 6$, a particle is moving along the *x*-axis. The particle's position, *x(t)*, is not explicitly given. The velocity of the particle is given by $v(t) = 2\sin(e^{t/4}) + 1$. The acceleration of the particle is given by $a(t) = 1/2\, e^{t/4}\cos(e^{t/4})$ and $x(0) = 2$.

10. Find the average velocity of the particle for the time period $0 \leq t \leq 6$.

Please respond on separate paper, following directions from your teacher.

Part B

1 point is earned for integral

1 point is earned for the answer

Average velocity $= \frac{1}{6}\int_0^6 v(t)dt = 1.949$

8.1-8.3

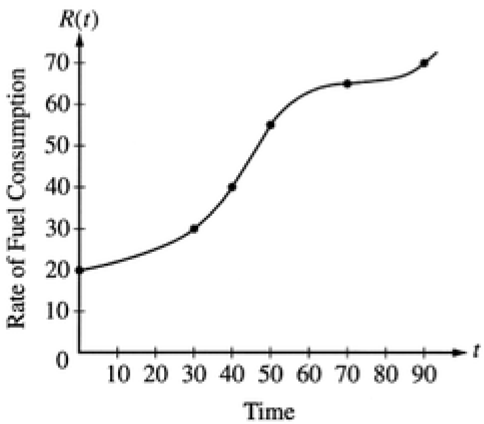
0	1	2

The student response earns all of the following points:

1 point is earned for integral

1 point is earned for the answer

Average velocity = $= \frac{1}{6}\int_0^6 v(t)dt = 1.949$



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.

11. For $0 < b \le 90$ minutes, explain the meaning of $\int_0^b R(t)\,\mathrm{dt}$ in terms of fuel consumption for the plane.

Explain the meaning of $\frac{1}{b}\int_0^b R(t)\,\mathrm{dt}$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

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Please respond on separate paper, following directions from your teacher.

Part D

1 point is earned for the meaning of $\int_0^b R(t)dt$

$\int_0^b R(t)dt$ is the total amount of fuel in gallons consumed for the first *b* minutes.

< -1 > if no reference to time *b*

1 point is earned for the meaning of $\frac{1}{b}\int_0^b R(t)dt$

$\frac{1}{b}\int_0^b R(t)dt$ is the average value of the rate of fuel consumption in gallons/min during the first *b* minutes.

< -1 > if no reference to time *b*

1 point is earned for the correct units in both answers

0	1	2	3

The student response earns three of the following points:

1 point is earned for the meaning of $\int_0^b R(t)dt$

$\int_0^b R(t)dt$ is the total amount of fuel in gallons consumed for the first *b* minutes.

< -1 > if no reference to time *b*

1 point is earned for the meaning of $\frac{1}{b}\int_0^b R(t)dt$

$\frac{1}{b}\int_0^b R(t)dt$ is the average value of the rate of fuel consumption in gallons/min during the first *b* minutes.

< -1 > if no reference to time *b*

1 point is earned for the correct units in both answers