9.6 Ratio Test

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Let $\sum a_n$ be an infinite series

and $\displaystyle \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$, then

a) series converges if $L < 1$

b) series diverges if $L > 1$

c) inconclusive if $L = 1$ (must try something else)

ex $\displaystyle \sum_{n=1}^{\infty} \frac{n^2}{n!} \longrightarrow \lim_{n\to\infty} \frac{\left(\frac{(n+1)^2}{(n+1)!}\right)}{\left(\frac{n^2}{n!}\right)}$

$$\lim_{n\to\infty} \frac{(n+1)^{\cancel{2}}}{\underbrace{(n+1)!}_{\cancel{(n+1)}\cdot \cancel{n!}}} \cdot \frac{\cancel{n!}}{n^2} = \lim_{n\to\infty} \frac{n+1}{n^2} = 0 < 1$$

$\displaystyle \sum_{n=1}^{\infty}$ converges by the ratio test.

Ex! $\displaystyle \sum_{n=1}^{\infty} \frac{(n+1)!}{n^3} \Rightarrow \lim_{n\to\infty} \left(\frac{\overset{(n+2)\cancel{(n+1)!}}{(n+2)!}}{(n+1)^3} \cdot \frac{n^3}{\cancel{(n+1)!}} \right) = \lim_{n\to\infty} \frac{\overset{n^4}{n^3(n+2)}}{(n+1)^3} = \infty \geq 1$

$$\lim_{n\to\infty} \frac{\left(\frac{((n+1)+1)!}{(n+1)^3}\right)}{\left(\frac{(n+1)!}{n^3}\right)}$$

$\therefore \displaystyle \sum_{n=1}^{\infty} \frac{(n+1)!}{n^3}$ diverges by the ratio test

Ex: $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 \cdot 2^n}{3^{n+1}}$

$a_{n+1} = \dfrac{(n+1)^2 \cdot 2^{n+1}}{3^{(n+1)+1}}$

$$\lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n^2 \cdot 2^n}$$

Fun Fact

$3^a \cdot 3^b = 3^{a+b}$

$3^{n+1} = 3^n \cdot 3$

$$\lim_{n\to\infty} \frac{(n+1)^2 \cdot \cancel{2^n} \cdot 2}{\cancel{3^n} \cdot 3^2} \cdot \frac{\cancel{3^n} \cdot 3}{n^2 \cancel{2^n}} = \frac{2}{3} < 1$$

$\therefore \displaystyle\sum_{n=0}^{\infty} \frac{n^2 \cdot 2^n}{3^{n+1}}$ converges by the ratio test

you try...

$\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(n+1)!}$

$$\lim_{n\to\infty} \frac{3^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{3^n}$$

$$\lim_{n\to\infty} \frac{3}{(n+2)} = 0 < 1$$

$\to$ converges by ratio test

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1} \Rightarrow \lim_{n\to\infty} \frac{\sqrt{n+1}}{n+2} \cdot \frac{n+1}{\sqrt{n}} = 1$!!

1. $\dfrac{\sqrt{n}}{n+1} \geq 0$ ✓

2. $\dfrac{\sqrt{n}}{n+1} \geq \dfrac{\sqrt{n+1}}{n+2}$ ✓

converges by

2. $\frac{\sqrt{n}}{n+1} \geq \frac{\sqrt{n+1}}{n+2}$ ✓

3. $\lim_{n \to \infty} \frac{\sqrt{n}}{n+1} = 0$ ✓

converges by alt series test