L'Hop and more FRQs (Wednesday(12/11))

Wednesday, December 11, 2019 9:57 AM

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

L'Hospital's Rule Free Response Practice (NO CALCULATOR)

1. Functions *f*, *g*, and *h* are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x-2)$ is tangent to both the graph of *g* at $x = 2$ and the graph of *h* at $x = 2$.

   a) Find $h'(2)$.

   $h'(2) = 2/3$ 1pt

   b) Let *a* be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

   $a'(x) = 3x^3 \cdot h'(x) + h(x) \cdot 9x^2$ 2pts

   $a'(2) = 3(2)^3 h'(2) + h(2) \cdot 9(2)^2 = 160$ 1pt

   c) The function *h* satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

   $\lim_{x \to 2} h(2) = 4$ because h twice diff. and $h(2) = 4$

   1pt

   $\lim_{x \to 2} x^2 - 4 = 0$

   $\lim_{x \to 2} 1 - (f(x))^3 = 0$

   so $1 - (f(2))^3 = 0$

   $(f(2))^3 = 1$

   $\boxed{f(2) = 1}$ 1pt

   $\lim_{x \to 2} \dfrac{2x}{-3f'(x) \cdot (f(x))^2} = \dfrac{4}{1}$ 1pt

   $-3f'(2) \cdot (f(2))^2 = 1$

   $-3f'(2) = 1$

   $f'(2) = -1/3$

   1pt

   ☆ note -1 if a student states $\lim \dfrac{x^2-4}{1-(f(x))^3} = \dfrac{0}{0}$ only 3 out of 4 if!!

   d) It is known that $g(x) \le h(x)$ for $1 < x < 3$. Let *k* be a function satisfying $g(x) \le k(x) \le h(x)$ for $1 < x < 3$. Is *k* continuous at $x = 2$? Justify your answer.

   g & h are continuous b/c differentiable

   $\lim_{x \to 2} g(x) = 4$ and $\lim_{x \to 2} = 4$ $\therefore$ $\lim_{x \to 2} k(x) = 4$

   Squeeze theorem

2. The function g is continuous for all real numbers x and is defined by $g(x) = \frac{\cos(2x)-1}{x^2}$ for $x \neq 0$. Use L'Hospital's Rule to find the value of $g(0)$. Show the work that leads to your answer.

b/c continuous $\lim_{x\to 0} g(x) = g(0)$

$\lim_{x\to 0} \frac{\cos(2x)-1}{x^2}$ $\to 0$, $\to 0$

$\lim_{x\to 0} \frac{-2\sin(2x)}{2x}$ $\to 0$, $\to 0$ 1pt

$\lim_{x\to 0} \frac{-4\cos(2x)}{2} = \frac{-4}{2}$

$= -2$

$g(0) = -2$ 1pt

3. A particle moves along the y-axis so that its position at time t is given by $y(t) = t^2 \tan\left(\frac{1}{t}\right)$ for $t > 1$.

(t, y)

(a) Show that the velocity of the particle at time t is given by $v(t) = 2t\tan\left(\frac{1}{t}\right) - \sec^2\left(\frac{1}{t}\right)$ for $t > 1$.

$y'(t) = v(t) = 2t\left(\tan\left(\frac{1}{t}\right)\right) + t^2 \sec^2\left(\frac{1}{t}\right)\cdot\left(\frac{-1}{t^2}\right)$ 3pts

(b) At time $t = \frac{4}{\pi}$, is the particle moving toward the origin or away from the origin? Give a reason for your answer.

1pt $y\left(\frac{4}{\pi}\right) = \left(\frac{4}{\pi}\right)^2 \tan\left(\frac{\pi}{4}\right) > 0$ ⇐ starts out above origin

1pt $y'\left(\frac{4}{\pi}\right) = \frac{8}{\pi}\tan\left(\frac{\pi}{4}\right) + \frac{16}{\pi^2}\sec^2\left(\frac{\pi}{4}\right)\cdot\frac{-16}{\pi^2} > 0$ ⇒ moves up (away) from origin since started above 1pt

(c) The velocity of the particle at time t can be written as $v(t) = \frac{2\tan\left(\frac{1}{t}\right)}{\frac{1}{t}} - \sec^2\left(\frac{1}{t}\right)$ for $t > 1$.

Find $\lim_{t\to\infty} v(t)$. Show the work that leads to your answer.

1pt $\lim_{t\to\infty} 2\tan\left(\frac{1}{t}\right) = 2\tan(0) = 0$

$\lim_{t\to\infty} \frac{1}{t} = 0$

$\lim_{t\to\infty} \frac{2\sec^2\left(\frac{1}{t}\right)\cdot\cancel{(-t^{-2})}}{\cancel{(-t)^{-2}}}$

$= 2\sec^2(0) = 2(1) = 2$

☆ −1 if student writes $\lim_{t\to\infty} v(t) = \frac{0}{0}$ ☆ no!

1pt $\lim_{t\to\infty} \sec^2\left(\frac{1}{t}\right) = \sec^2(0) = 1$

$2 - 1 = 1$ 1pt c

4. Particle P moves along the y-axis so that its position at time t is given by $y(t) = 4t - \frac{2}{3}$ for all times t. A second particle, particle Q, moves along the x-axis so that its position at time t is given by $x(t) = \frac{\sin(\pi t)}{2-t}$ for all times $t \neq 2$.

(a) As time t approaches 2, what is the limit of the position of particle Q? Show the work that leads to your answer.

1pt $\lim_{x\to 2} \sin(\pi t) = 0$

$\lim_{x\to 2} (2-t) = 0$

$\lim_{t\to 2} \frac{\pi\cos(\pi t)}{(-1)} = \frac{\pi(1)}{-1} = -\pi$ 2pts

(b) Show that the velocity of particle Q is given by $v_Q(t) = \frac{2\pi\cos(\pi t) - \pi t\cos(\pi t) + \sin(\pi t)}{(2-t)^2}$ for all times $t \neq 2$.

(b) Show that the velocity of particle Q is given by $v_Q(t) = \dfrac{2\pi\cos(\pi t) - \pi t\cos(\pi t) + \sin(\pi t)}{(2-t)^2}$

for all times $t \neq 2$.

$$x'(t) = v(t) = \frac{(2-t)(\pi\cos(\pi t)) - \sin(\pi t)(-1)}{(2-t)^2} \quad \Big\} \text{ 3pts}$$

Quotient Rule!

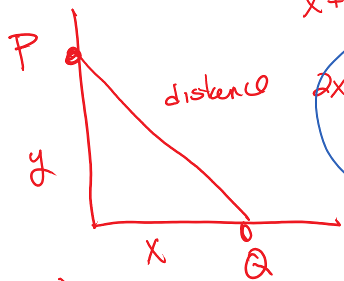
(c) Find the rate of change of the distance between particle P and particle Q at time $t = \frac{1}{2}$.

Show the work that leads to your answer.

time

Related Rate

$x^2 + y^2 = d^2$



$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2d\frac{dd}{dt}$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = d\frac{dd}{dt}$$

1pt

$y(t) = 4t - 2/3$

$\frac{dy}{dt} = 4\frac{dt}{dt} = 4$ 1pt

use $x'(t)$ from part b

$x'(1/2) = \frac{4}{9}$ 1pt

$y(1/2) = 4/3 \quad x(1/2) = 2/3 \quad d = \frac{\sqrt{20}}{3}$

$$\frac{2}{3}\left(\frac{4}{9}\right) + \frac{4}{3}(4) = \frac{\sqrt{20}}{3}\frac{dd}{dt}$$

$$\frac{dd}{dt} = \frac{152}{9\sqrt{20}} \approx 3.776 \text{ 1pt}$$