9.4 Notes

Friday, February 7, 2020 8:21 AM

Comparison Tests

Direct Comparison Test

Let $\sum a_n$ be an infinite series w/ non-negative terms:

1. $\sum a_n$ converge if there is a convergent series $\sum c_n$ where $a_n \le c_n$ for all n

2. $\sum a_n$ diverge if there is a divergent series $\sum d_n$ where $a_n \ge d_n$ for all n.

Diverge or converge?

Ex: $\sum_{n=0}^{\infty} \frac{3^n+1}{2^n}$

$\star$ $\frac{3^n+1}{2^n} \ge \frac{3^n}{2^n}$ ✓

$\star$ $\sum_{n=0}^{\infty} \frac{3^n}{2^n} \Rightarrow \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$ Geometric $|r|>1$ ✓ Diverges

$\sum \frac{3^n+1}{2^n}$ diverges by Direct comparison Test ✓

Ex: $\sum_{n=1}^{\infty} \frac{1}{4+2^n}$

$\frac{1}{4+2^n} \le \frac{1}{2^n}$

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n}_{\text{converges since } r=\frac{1}{2}<1}$ $r = \frac{1}{2} < 1$

$\therefore$ $\sum \frac{1}{4+2^n}$ converges by the direct comparison test.

Ex. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+3}}$

$\frac{1}{\sqrt{n^3+3}} \le \frac{1}{\sqrt{n^3}} =$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \Rightarrow$ converges by the p-series test b/c $\frac{3}{2} > 1$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+3}}$ converges by the direct comparison test

Try... $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n-2}}$

$\frac{1}{\sqrt{n-2}} \ge \frac{1}{\sqrt{n}}$

Try... $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n-2}}$ $\qquad \frac{1}{\sqrt{n-2}} \geq \frac{1}{\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \Rightarrow$ diverges by the p-series test $\frac{1}{2} \leq 1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n-2}}$ diverges by the Direct comparison test.

Ex $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt[3]{n}}$ $\qquad \frac{1}{2+\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}}$ Diverges p-series !!

Limit Comparison Test → use when Direct Comparison fail

Suppose $\sum a_n$ an $\sum b_n$ have non-negative terms for all n.

then:

1. $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $0 < c < \infty$ then both $\sum a_n$ and $\sum b_n$ both converge or diverge.

2. $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum b_n$ converges then $\sum a_n$ converges also

3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then if $\sum b_n$ diverges, then $\sum a_n$ diverges

$$\lim_{n\to\infty} \frac{\frac{1}{2+\sqrt[3]{n}}}{\frac{1}{\sqrt[3]{n}}} = \lim_{n\to\infty} \frac{\sqrt[3]{n}}{2+\sqrt[3]{n}} = 1$$

$\Sigma Bm = 1$

$\sum \frac{1}{2+\sqrt[3]{n}}$ diverges by the limit comparison test

$\frac{n^{1/2}}{?} = n^{1/2-2} = n^{-3/2}$

(2). $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2-1}$ $\qquad \displaystyle\frac{\sqrt{n}}{n^2-1} \geq \underbrace{\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}}$ converges p-series

$$\frac{n^{1/2}}{n^2} = n^{1/2-2} = n^{-3/2}$$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2-1}}{\frac{\sqrt{n}}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1$$

$\therefore \displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2-1}$ converges by limit comparison test!