Chapter 9
Discrete Mathematics

■ Section 9.1 Basic Combinatorics

Exploration 1

1. Six: ABC, ACB, BAC, BCA, CAB, CBA.

2. Approximately 1 person out of 6, which would mean 10 people out of 60.

3. No. If they all looked the same, we would expect approximately 10 people to get the order right simply by chance. The fact that this did not happen leads us to reject the "look-alike" conclusion.

4. It is likely that the salesman rigged the test to mislead the office workers. He might have put the copy from the more expensive machine on high-quality bond paper to make it look more like an original, or he might have put a tiny ink smudge on the original to make it look like a copy. You can offer your own alternate scenarios.

Quick Review 9.1

1. 52

2. 13

3. 6

4. 11

5. 10

6. 4

7. 11

8. 4

9. 64

10. 13

Section 9.1 Exercises

1. There are three possibilities for who stands on the left, and then two remaining possibilities for who stands in the middle, and then one remaining possibility for who stands on the right: $3 \cdot 2 \cdot 1 = 6$.

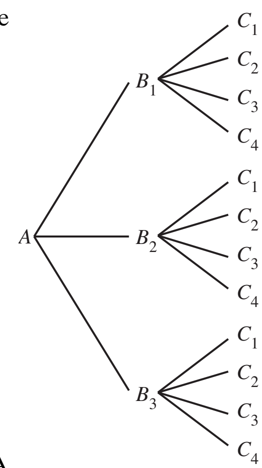
2. Any of the four jobs could be ranked most important, and then any of the remaining three jobs could be ranked second, and so on: $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

3. Any of the five books could be placed on the left, and then any of the four remaining books could be placed next to it, and so on: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

4. Any of the five dogs could be awarded 1st place, and then any of the remaining four dogs could be awarded 2nd place, and so on: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

5. There are $3 \cdot 4 = 12$ possible pairs: K_1Q_1, K_1Q_2, K_1Q_3, K_1Q_4, K_2Q_1, K_2Q_2, K_2Q_3, K_2Q_4, K_3Q_1, K_3Q_2, K_3Q_3, and K_3Q_4.

6. There are $3 \cdot 4 = 12$ possible routes. In the tree diagram, B_1 represents the first road from town A to town B, etc.



7. $9! = 362{,}880$ (ALGORITHM)

8. $22 \cdot 21 \cdot 20 = 9240$

9. There are 11 letters, where S and I each appear 4 times and P appears 2 times. The number of distinguishable permutations is $\dfrac{11!}{4!4!2!} = 34{,}650$.

10. There are 11 letters, where A appears 3 times and O and T each appear 2 times. The number of distinguishable permutations is $\dfrac{11!}{3!2!2!} = 1{,}663{,}200$.

11. The number of ways to fill 3 distinguishable offices from a pool of 13 candidates is ${}_{13}P_3 = \dfrac{13!}{10!} = 1716$.

12. The number of ways to select and prioritize 6 out of 12 projects is ${}_{12}P_6 = \dfrac{12!}{6!} = 665{,}280$.

13. $4 \cdot 3 \cdot 2 \cdot 1 = 24$

14. $(3 \cdot 2 \cdot 1)(1) = 6$

15. $\dfrac{6!}{(6-2)!} = \dfrac{6 \cdot 5 \cdot 4!}{4!} = 30$

16. $\dfrac{9!}{(9-2)!} = \dfrac{9 \cdot 8 \cdot 7!}{7!} = 72$

17. $\dfrac{10!}{7!(10-7)!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7!}{7! \cdot 3 \cdot 2 \cdot 1} = 120$

18. $\dfrac{10!}{3!(10-3)!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7!}{3! \cdot 7!} = \dfrac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$

19. combinations

20. permutations

21. combinations

22. permutations (different roles)

23. There are 10 choices for the first character, 9 for the second, 26 for the third, then 25, then 8, then 7, then 6: $10 \cdot 9 \cdot 26 \cdot 25 \cdot 8 \cdot 7 \cdot 6 = 19{,}656{,}000$.

24. There are 36 choices for each character: $36^5 = 60{,}466{,}176$.

25. There are 6 possibilities for the red die, and 6 for the green die: $6 \cdot 6 = 36$.

26. There are 2 possibilities for each flip: $2^{10} = 1024$.

27. ${}_{25}C_3 = \dfrac{25!}{3!(25-3)!} = \dfrac{25!}{3!22!} = 2300$

28. ${}_{52}C_5 = \dfrac{52!}{5!(52-5)!} = \dfrac{52!}{5!47!} = 2{,}598{,}960$

29. ${}_{48}C_3 = \dfrac{48!}{3!(48-3)!} = \dfrac{48!}{3!45!} = 17{,}296$

30. Choose 7 positions from the 20:
${}_{20}C_7 = \dfrac{20!}{7!(20-7)!} = \dfrac{20!}{7!13!} = 77{,}520$

31. Choose A♠ and K♠, and 11 cards from the other 50:
${}_2C_2 \cdot {}_{50}C_{11} = 1 \cdot {}_{50}C_{11} = \dfrac{50!}{11!(50-11)!} = \dfrac{50!}{11!39!}$
$= 37{,}353{,}738{,}800$

32. ${}_8C_3 = \dfrac{8!}{3!(8-3)!} = \dfrac{8!}{3!5!} = 56$

33. We either have 3, 2, or 1 student(s) nominated:
${}_6C_3 + {}_6C_2 + {}_6C_1 = 20 + 15 + 6 = 41$

34. We either have 3, 2, or 1 appetizer(s) represented:
${}_5C_3 + {}_5C_2 + {}_5C_1 = 10 + 10 + 5 = 25$

35. Each of the 5 dice have 6 possible outcomes: $6^5 = 7776$

36. ${}_{20}C_8 = \dfrac{20!}{8!(20-8)!} = \dfrac{20!}{8!12!} = 125,970$

37. $2^9 - 1 = 511$

38. $3 \times 4 \times 3 \times 2^6 = 2304$

39. Since each topping can be included or left off, the total number of possibilities with n toppings is 2^n. Since $2^{11} = 2048$ is less than 4000 but $2^{12} = 4096$ is greater than 4000, Luigi offers at least 12 toppings.

40. There are 2^n subsets, of which $2^n - 2$ are proper subsets.

41. $2^{10} = 1024$

42. $5^{10} = 9{,}765{,}625$

43. True. $\dbinom{n}{a} = \dfrac{n!}{a!(n-a)!} = \dfrac{n!}{a!b!} = \dfrac{n!}{(n-b)!b!} = \dbinom{n}{b}$.

44. False. For example, $\dbinom{5}{2} = 10$ is greater than $\dbinom{5}{4} = 5$.

45. There are $\dbinom{6}{2} = 15$ different combinations of vegetables.
The total number of entrée-vegetable-dessert variations is $4 \cdot 15 \cdot 6 = 360$. The answer is D.

46. ${}_{10}P_5 = 30{,}240$. The answer is D.

47. ${}_nP_n = \dfrac{n!}{(n-n)!} = n!$ The answer is B.

48. There are as many ways to vote as there are subsets of a set with 5 members. That is, there are 2^5 ways to fill out the ballot. The answer is C.

49. Answers will vary. Here are some possible answers:

(a) Number of 3-card hands that can be dealt from a deck of 52 cards

(b) Number of ways to choose 3 chocolates from a box of 12 chocolates

(c) Number of ways to choose a starting soccer team from a roster of 25 players (where position matters)

(d) Number of 5-digit numbers that can be formed using only the digits 1 and 2.

(e) Number of possible pizzas that can be ordered at a place that offers 3 different sizes and up to 10 different toppings.

50. Counting the number of ways to choose the two eggs you are going to have for breakfast is equivalent to counting the number of ways to choose the ten eggs you are *not* going to have for breakfast.

51. **(a)** Twelve

(b) Every 0 represents a factor of 10, or a factor of 5 multiplied by a factor of 2. In the product $50 \cdot 49 \cdot 48 \cdot \ldots \cdot 2 \cdot 1$, the factors 5, 10, 15, 20, 30, 35, 40, and 45 each contain 5 as a factor once, and 25 and 50 each contain 5 twice, for a total of twelve occurrences. Since there are 47 factors of 2 to pair up with the twelve factors of 5, 10 is a factor of 50! twelve times.

52. **(a)** Each combination of the n vertices taken two at a time determines a segment that is either an edge or a diagonal. There are ${}_nC_2$ such combinations.

(b) Subtracting the n edges from the answers in (a), we find that ${}_nC_2 - n = \dfrac{n!}{2!(n-2)!} - n$
$= \dfrac{n(n-1)}{2} - \dfrac{2n}{2} = \dfrac{n^2 - 3n}{2}$.

53. In the nth week, 5^n copies of the letter are sent. In the last week of the year, that's $5^{52} \approx 2.22 \times 10^{36}$ copies of the letter. This exceeds the population of the world, which is about 6×10^9, so someone (several people, actually) has had to receive a second copy of the letter.

54. Six. No matter where the first person sits, there are the $3! = 6$ ways to sit the others in different positions relative to the first person.

55. Three. This is equivalent to the round table problem (Exercise 42), except that the necklace can be *turned upside-down*. Thus, each different necklace accounts for two of the six different orderings.

56. The chart on the left is more reasonable. Each pair of actresses will require about the same amount of time to interview. If we make a chart showing n (the number of actresses) and ${}_nC_2$ (the number of pairings), we can see that chart 1 allows approximately 3 minutes per pair throughout, while chart 2 allows less and less time per pair as n gets larger.

Number n	Number of Pairs ${}_nC_2$	Time per Pair Chart 1	Time per Pair Chart 2
3	3	3.33	3.33
6	15	3	2
9	36	3.06	1.67
12	66	3.03	1.52
15	105	3.05	1.43

57. There are ${}_{52}C_{13} = 635{,}013{,}559{,}600$ distinct bridge hands. Every day has $60 \cdot 60 \cdot 24 = 86{,}400$ seconds; a year has 365.24 days, which is 31,556,736 seconds. Therefore it will take about $\dfrac{635{,}013{,}559{,}600}{31{,}556{,}736} \approx 20{,}123$ years. (Using 365 days per year, the computation gives about 20,136 years.)

58. Each team can choose 5 players in ${}_{12}C_5 = 792$ ways, so there are $792^2 = 627{,}264$ ways total.

■ Section 9.2 The Binomial Theorem

Exploration 1

1. ${}_3C_0 = \dfrac{3!}{0!3!} = 1$, ${}_3C_1 = \dfrac{3!}{1!2!} = 3$, ${}_3C_2 = \dfrac{3!}{2!1!} = 3$, ${}_3C_3 = \dfrac{3!}{3!0!} = 1$. These are (in order) the coefficients in the expansion of $(a + b)^3$.

2. $\{1\ 4\ 6\ 4\ 1\}$. These are (in order) the coefficients in the expansion of $(a + b)^4$.

3. $\{1\ 5\ 10\ 10\ 5\ 1\}$. These are (in order) the coefficients in the expansion of $(a + b)^5$.

Quick Review 9.2

1. $x^2 + 2xy + y^2$
2. $a^2 + 2ab + b^2$
3. $25x^2 - 10xy + y^2$
4. $a^2 - 6ab + 9b^2$
5. $9s^2 + 12st + 4t^2$
6. $9p^2 - 24pq + 16q^2$
7. $u^3 + 3u^2v + 3uv^2 + v^3$
8. $b^3 - 3b^2c + 3bc^2 - c^3$
9. $8x^3 - 36x^2y + 54xy^2 - 27y^3$
10. $64m^3 + 144m^2n + 108mn^2 + 27n^3$

Section 9.2 Exercises

1. $(a + b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

2. $(a + b)^6 = \binom{6}{0}a^6b^0 + \binom{6}{1}a^5b^1 + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 + \binom{6}{6}a^0b^6$
$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

3. $(x + y)^7 = \binom{7}{0}x^7y^0 + \binom{7}{1}x^6y^1 + \binom{7}{2}x^5y^2 + \binom{7}{3}x^4y^3 + \binom{7}{4}x^3y^4 + \binom{7}{5}x^2y^5 + \binom{7}{6}x^1y^6 + \binom{7}{7}x^0y^7$
$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

4. $(x + y)^{10} = \binom{10}{0}x^{10}y^0 + \binom{10}{1}x^9y^1 + \binom{10}{2}x^8y^2 + \binom{10}{3}x^7y^3 + \binom{10}{4}x^6y^4 + \binom{10}{5}x^5y^5 + \binom{10}{6}x^4y^6$
$+ \binom{10}{7}x^3y^7 + \binom{10}{8}x^2y^8 + \binom{10}{9}x^1y^9 + \binom{10}{10}x^0y^{10}$
$= x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6 + 120x^3y^7 + 45x^2y^8 + 10xy^9 + y^{10}$

5. Use the entries in row 3 as coefficients:
$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

6. Use the entries in row 5 as coefficients:
$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

7. Use the entries in row 8 as coefficients:
$(p + q)^8 = p^8 + 8p^7q + 28p^6q^2 + 56p^5q^3 + 70p^4q^4 + 56p^3q^5 + 28p^2q^6 + 8pq^7 + q^8$

8. Use the entries in row 9 as coefficients:
$(p + q)^9 = p^9 + 9p^8q + 36p^7q^2 + 84p^6q^3 + 126p^5q^4 + 126p^4q^5 + 84p^3q^6 + 36p^2q^7 + 9pq^8 + q^9$

9. $\binom{9}{2} = \dfrac{9!}{2!7!} = \dfrac{9 \cdot 8}{2 \cdot 1} = 36$

10. $\binom{15}{11} = \dfrac{15!}{11!4!} = \dfrac{15 \cdot 14 \cdot 13 \cdot 12}{4 \cdot 3 \cdot 2 \cdot 1} = 1365$

11. $\binom{166}{166} = \dfrac{166!}{166!0!} = 1$

12. $\binom{166}{0} = \dfrac{166!}{0!166!} = 1$

13. $\binom{14}{3} = \binom{14}{11} = 364$

14. $\binom{13}{8} = \binom{13}{5} = 1287$

15. $(-2)^8\binom{12}{8} = (-2)^8\binom{12}{4} = 126{,}720$

16. $(-3)^4\binom{11}{4} = (-3)^4\binom{11}{7} = 26{,}730$

17. $f(x) = (x - 2)^5 = x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + (-2)^5 = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

18. $g(x) = (x + 3)^6 = x^6 + 6x^5 \cdot 3 + 15x^4 \cdot 3^2 + 20x^3 \cdot 3^3 + 15x^2 \cdot 3^4 + 6x \cdot 3^5 + 3^6$
$= x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x + 729$

19. $h(x) = (2x - 1)^7$
$= (2x)^7 + 7(2x)^6(-1) + 21(2x)^5(-1)^2 + 35(2x)^4(-1)^3 + 35(2x)^3(-1)^4 + 21(2x)^2(-1)^5 + 7(2x)(-1)^6 + (-1)^7$
$= 128x^7 - 448x^6 + 672x^5 - 560x^4 + 280x^3 - 84x^2 + 14x - 1$

20. $f(x) = (3x + 4)^5 = (3x)^5 + 5(3x)^4 \cdot 4 + 10(3x)^3 \cdot 4^2 + 10(3x)^2 \cdot 4^3 + 5(3x) \cdot 4^4 + 4^5$
$= 243x^5 + 1620x^4 + 4320x^3 + 5760x^2 + 3840x + 1024$

21. $(2x + y)^4 = (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4 = 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

22. $(2y - 3x)^5 = (2y)^5 + 5(2y)^4(-3x) + 10(2y)^3(-3x)^2 + 10(2y)^2(-3x)^3 + 5(2y)(-3x)^4 + (-3x)^5$
$= 32y^5 - 240y^4x + 720y^3x^2 - 1080y^2x^3 + 810yx^4 - 243x^5$

23. $(\sqrt{x} - \sqrt{y})^6 = (\sqrt{x})^6 + 6(\sqrt{x})^5(-\sqrt{y}) + 15(\sqrt{x})^4 \cdot (-\sqrt{y})^2 + 20(\sqrt{x})^3(-\sqrt{y})^3 + 15(\sqrt{x})^2(-\sqrt{y})^4$
$+ 6(\sqrt{x})(-\sqrt{y})^5 + (-\sqrt{y})^6 = x^3 - 6x^{5/2}y^{1/2} + 15x^2y - 20x^{3/2}y^{3/2} + 15xy^2 - 6x^{1/2}y^{5/2} + y^3$

24. $(\sqrt{x} + \sqrt{3})^4 = (\sqrt{x})^4 + 4(\sqrt{x})^3(\sqrt{3}) + 6(\sqrt{x})^2 \cdot (\sqrt{3})^2 + 4(\sqrt{x})(\sqrt{3})^3 + (\sqrt{3})^4 = x^2 + 4x\sqrt{3x} + 18x + 12\sqrt{3x} + 9$

25. $(x^{-2} + 3)^5 = (x^{-2})^5 + 5(x^{-2})^4 \cdot 3 + 10(x^{-2})^3 \cdot 3^2 + 10(x^{-2})^2 \cdot 3^3 + 5(x^{-2}) \cdot 3^4 + 3^5$
$= x^{-10} + 15x^{-8} + 90x^{-6} + 270x^{-4} + 405x^{-2} + 243$

26. $(a - b^{-3})^7 = a^7 + 7a^6(-b^{-3}) + 21a^5(-b^{-3})^2 + 35a^4(-b^{-3})^3 + 35a^3(-b^{-3})^4 + 21a^2(-b^{-3})^5 + 7a(-b^{-3})^6 + (-b^{-3})^7$
$= a^7 - 7a^6b^{-3} + 21a^5b^{-6} - 35a^4b^{-9} + 35a^3b^{-12} - 21a^2b^{-15} + 7ab^{-18} - b^{-21}$

27. Answers will vary.

28. Answers will vary.

29. If $n >= 1$, $\binom{n}{1} = \dfrac{n!}{1!(n-1)!} = n = \dfrac{n!}{(n-1)!1!}$
$= \dfrac{n!}{(n-1)![n-(n-1)]!} = \binom{n}{n-1}$

30. If $n >= r >= 0$, $\binom{n}{r} = \dfrac{n!}{r!(n-r)!} = \dfrac{n!}{(n-r)!r!}$
$= \dfrac{n!}{(n-r)![n-(n-r)]!} = \binom{n}{n-r}$

31. $\binom{n-1}{r-1} + \binom{n-1}{r}$
$= \dfrac{(n-1)!}{(r-1)![(n-1)-(r-1)]!} + \dfrac{(n-1)!}{r!(n-1-r)!}$
$= \dfrac{r(n-1)!}{r(r-1)!(n-r)!} + \dfrac{(n-1)!(n-r)}{r!(n-r)(n-r-1)!}$
$= \dfrac{r(n-1)!}{r!(n-r)!} + \dfrac{(n-r)(n-1)!}{r!(n-r)!}$
$= \dfrac{(r+n-r)(n-1)!}{r!(n-r)!}$
$= \dfrac{n!}{r!(n-r)!}$
$= \binom{n}{r}$

32. **(a)** Any pair (n, m) of nonnegative integers — except for $(1, 1)$ — provides a counterexample. For example, $n = 2$ and $m = 3$: $(2 + 3)! = 5! = 120$, but $2! + 3! = 2 + 6 = 8$.

(b) Any pair (n, m) of nonnegative integers — except for $(0, 0)$ or any pair $(1, m)$ or $(n, 1)$ — provides a counterexample. For example, $n = 2$ and $m = 3$: $(2 \cdot 3)! = 6! = 720$, but, $2! \cdot 3! = 2 \cdot 6 = 12$.

33. Let $n >= 2$. $\binom{n}{2} + \binom{n+1}{2} = \dfrac{n!}{2!(n-2)!} + \dfrac{(n+1)!}{2!(n-1)!}$
$= \dfrac{n(n-1)}{2} + \dfrac{(n+1)(n)}{2}$
$= \dfrac{n^2 - n + n^2 + n}{2} = n^2$

34. Let $n >= 2$. $\binom{n}{n-2} + \binom{n+1}{n-1} = \dfrac{n!}{(n-2)![n-(n-2)]!}$
$+ \dfrac{(n+1)!}{(n-1)![(n+1)-(n-1)]!}$
$= \dfrac{n!}{(n-2)!2!} + \dfrac{(n+1)!}{(n-1)!2!}$
$= \dfrac{n(n-1)}{2} + \dfrac{(n+1)n}{2}$
$= \dfrac{n^2 - n + n^2 + n}{2} = n^2$

35. True. The signs of the coefficients are determined by the powers of the $(-y)$ terms, which alternate between odd and even.

36. True. In fact, the sum of every row is a power of 2.

37. The fifth term of the expansion is $\binom{8}{4}(2x)^4(1)^4 = 1120x^4$. The answer is C.

38. The two smallest numbers in row 10 are 1 and 10. The answer is B.

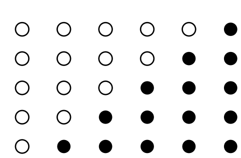
39. The sum of the coefficients of $(3x - 2y)^{10}$ is the same as the value of $(3x - 2y)^{10}$ when $x = 1$ and $y = 1$.
The answer is A.

40. The even-numbered terms in the two expressions are opposite-signed and cancel out, while the odd-numbered terms are identical and add together. The answer is D.

41. **(a)** 1, 3, 6, 10, 15, 21, 28, 36, 45, 55

(b) They appear diagonally down the triangle, starting with either of the 1's in row 2.

(c) Since n and $n + 1$ represent the sides of the given rectangle, then $n(n + 1)$ represents its area. The triangular number is $1/2$ of the given area. Therefore, the triangular number is $\dfrac{n(n+1)}{2}$.



(d) From **(c)**, we observe that the nth triangular number can be written as $\frac{n(n+1)}{2}$. We know that binomial coefficients are the values of $\binom{n}{r}$ for $r = 0, 1, 2, 3, \ldots, n$. We can show that $\frac{n(n+1)}{2} = \binom{n+1}{2}$ as follows:

$$\frac{n(n+1)}{2} = \frac{(n+1)n(n-1)!}{2(n-1)!} = \frac{(n+1)!}{2!(n-1)!} = \frac{(n+1)!}{2!((n+1)-2)!} = \binom{n+1}{2}.$$

So, to find the fourth triangular number, for example, compute $\binom{4+1}{2} = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot 3!}{2!3!} = \frac{5 \cdot 4}{2} = 10.$

42. **(a)** 2 (Every other number appears at least twice.)

(b) 1

(c) No (They all appear in order down the second diagonal.)

(d) 0 (See Exercise 44 for a proof.)

(e) All are divisible by p.

(f) Rows that are positive-integer powers of 2: 2, 4, 8, 16, etc.

(g) Rows that are 1 less than a power of 2: 0, 1, 3, 7, 15, etc.

(h) Answers will vary. One possible answer: For any prime numbered row, or row where the first element is a prime number, all the numbers in that row (excluding the 1's) are divisible by the prime. For example, in the seventh row (1 7 21 35 35 21 7 1) 7, 21, and 35 are all divisible by 7.

43. The sum of the entries in the nth row equals the sum of the coefficients in the expansion of $(x + y)^n$. But this sum, in turn, is equal to the value of $(x + y)^n$ when $x = 1$ and $y = 1$:

$$2^n = (1+1)^n = \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1^1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^01^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

44. $0 = (1-1)^n$

$$= \binom{n}{0}1^n + \binom{n}{1}1^{n-1}(-1) + \binom{n}{2}1^{n-2}(-1)^2 + \cdots + \binom{n}{n}(-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n\binom{n}{n}$$

45. $3^n = (1+2)^n$

$$= \binom{n}{0}1^n + \binom{n}{1}1^{n-1}2 + \binom{n}{2}1^{n-2}2^2 + \cdots + \binom{n}{n}2^n = \binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \cdots + 2^n\binom{n}{n}$$

■ Section 9.3 Sequences

Quick Review 9.3

1. $3 + (5-1)4 = 3 + 16 = 19$
2. $\frac{5}{2}[(6 + (5-1)4] = \frac{5}{2}(22) = 55$
3. $5 \cdot 4^2 = 80$
4. $\frac{5(1-4^3)}{(1-4)} = \frac{-315}{-3} = 105$
5. $a_{10} = \frac{10}{11}$
6. $a_{10} = 5 + (10-1)3 = 32$
7. $a_{10} = 5 \cdot 2^9 = 2560$
8. $a_{10} = \left(\frac{4}{3}\right)\left(\frac{1}{2}\right)^9 = \left(\frac{4}{3}\right)\left(\frac{1}{512}\right) = \frac{1}{384}$
9. $a_{10} = 32 - 17 = 15$
10. $a_{10} = \frac{10^2}{2^{10}} = \frac{100}{1024} = \frac{25}{256}$

Section 9.3 Exercises

For #1–4, substitute $n = 1, n = 2, \ldots, n = 6$, and $n = 100$.

1. $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}; \frac{101}{100}$
2. $\frac{4}{3}, 1, \frac{4}{5}, \frac{2}{3}, \frac{4}{7}, \frac{1}{2}; \frac{2}{51}$
3. 0, 6, 24, 60, 120, 210; 999,900
4. −4, −6, −6, −4, 0, 6; 9500

For #5–10, use previously computed values of the sequence to find the next term in the sequence.

5. 8, 4, 0, −4; −20
6. −3, 7, 17, 27; 67
7. 2, 6, 18, 54; 4374
8. 0.75, −1.5, 3, −6; −96
9. 2, −1, 1, 0; 3
10. −2, 3, 1, 4; 23
11. $\lim_{n\to\infty} n^2 = \infty$, so the sequence diverges.
12. $\lim_{n\to\infty} \frac{1}{2^n} = 0$, so the sequence converges to 0.
13. $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots, \frac{1}{n^2}, \ldots$

    $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so the sequence converges to 0.
14. $\lim_{n\to\infty} (3n - 1) = \infty$, so the sequence diverges.
15. Since the degree of the numerator is the same as the degree of the denominator, the limit is the ratio of the leading coefficients. Thus $\lim_{n\to\infty} \frac{3n-1}{2-3n} = -1$. The sequence converges to −1.

16. Since the degree of the numerator is the same as the degree of the denominator, the limit is the ratio of the leading coefficients. Thus $\lim_{n\to\infty} \frac{2n-1}{n+1} = 2$. The sequence converges to 2.

17. $\lim_{n\to\infty} (0.5)^n = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$, so the sequence converges to 0.

18. $\lim_{n\to\infty} (1.5)^n = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty$, so the sequence diverges.

19. $a_1 = 1$ and $a_{n+1} = a_n + 3$ for $n \geq 1$ yields $1, 4, 7, \ldots, (3n-2), \ldots$
$\lim_{n\to\infty} (3n-2) = \infty$, so the sequence diverges.

20. $u_1 = 1$ and $u_{n+1} = \frac{u_n}{3}$ for $n \geq 1$ yields 1, $\frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^{n-1}}, \ldots$
$\lim_{n\to\infty} \frac{1}{3^{n-1}} = 0$, so the sequence converges to 0.

For #21–24, subtract the first term from the second to find the common difference d. Use the formula $a_n = a_1 + (n-1)d$ with $n = 10$ to find the tenth term. The recursive rule for the nth term is $a_n = a_{n-1} + d$, and the explicit rule is the one given above.

21. (a) $d = 4$
(b) $a_{10} = 6 + 9(4) = 42$
(c) Recursive rule: $a_1 = 6$; $a_n = a_{n-1} + 4$ for $n \geq 2$
(d) Explicit rule: $a_n = 6 + 4(n-1)$

22. (a) $d = 5$
(b) $a_{10} = -4 + 9(5) = 41$
(c) Recursive rule: $a_1 = -4$; $a_n = a_{n-1} + 5$ for $n \geq 2$
(d) Explicit rule: $a_n = -4 + 5(n-1)$

23. (a) $d = 3$
(b) $a_{10} = -5 + 9(3) = 22$
(c) Recursive rule: $a_1 = -5$; $a_n = a_{n-1} + 3$ for $n \geq 2$
(d) Explicit rule: $a_n = -5 + 3(n-1)$

24. (a) $d = 11$
(b) $a_{10} = -7 + 9(11) = 92$
(c) Recursive rule: $a_1 = -7$; $a_n = a_{n-1} + 11$ for $n \geq 2$
(d) Explicit rule: $a_n = -7 + 11(n-1)$

For #25–28, divide the second term by the first to find the common ratio r. Use the formula $a_n = a_1 \cdot r^{n-1}$ with $n = 8$ to find the eighth term. The recursive rule for the nth term is $a_n = a_{n-1} \cdot r$, and the explicit rule is the one given above.

25. (a) $r = 3$
(b) $a_8 = 2 \cdot 3^7 = 4374$
(c) Recursive rule: $a_1 = 2$; $a_n = 3a_{n-1}$ for $n \geq 2$
(d) Explicit rule: $a_n = 2 \cdot 3^{n-1}$

26. (a) $r = 2$
(b) $a_8 = 3 \cdot 2^7 = 384$
(c) Recursive rule: $a_1 = 3$; $a_n = 2a_{n-1}$ for $n \geq 2$
(d) Explicit rule: $a_n = 3 \cdot 2^{n-1}$

27. (a) $r = -2$
(b) $a_8 = (-2)^7 = -128$
(c) Recursive rule: $a_1 = 1$; $a_n = -2a_{n-1}$ for $n \geq 2$
(d) Explicit rule: $a_n = (-2)^{n-1}$

28. (a) $r = -1$
(b) $a_8 = -2 \cdot (-1)^7 = 2$
(c) Recursive rule: $a_1 = -2$; $a_n = -1a_{n-1} = -a_{n-1}$ for $n \geq 2$
(d) Explicit rule: $a_n = -2 \cdot (-1)^{n-1} = 2 \cdot (-1)^n$

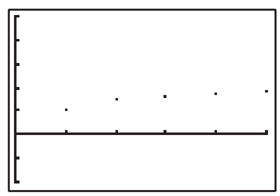
29. $a_4 = -8 = a_1 + 3d$ and $a_7 = 4 = a_1 + 6d$, so $a_7 - a_4 = 12 = 3d$. Therefore $d = 4$, so $a_1 = -8 - 3d = -20$ and $a_n = a_{n-1} + 4$ for $n \geq 2$.

30. $a_5 = -5 = a_1 + 4d$ and $a_9 = -17 = a_1 + 8d$, so $a_9 - a_5 = -12 = 4d$. Therefore $d = -3$, so $a_1 = -5 - 4d = 7$ and $a_n = a_{n-1} - 3$ for $n \geq 2$.

31. $a_2 = 3 = a_1 \cdot r^1$ and $a_8 = 192 = a_1 \cdot r^7$, so $a_8/a_2 = 64 = r^6$. Therefore $r = \pm 2$, so $a_1 = 3/(\pm 2) = \pm\frac{3}{2}$ and $a_n = -\frac{3}{2} \cdot (-2)^{n-1} = 3 \cdot (-2)^{n-2}$ or $a_n = \frac{3}{2} \cdot 2^{n-1} = 3 \cdot 2^{n-2}$.

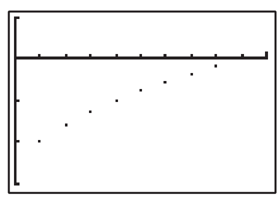
32. $a_3 = -75 = a_1 \cdot r^2$ and $a_6 = -9375 = a_1 \cdot r^5$, so $a_6/a_3 = 125 = r^3$. Therefore $r = 5$, so $a_1 = -75/5^2 = -3$ and $a_n = -3 \cdot 5^{n-1}$.

33.



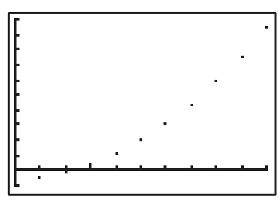
$[0, 5]$ by $[-2, 5]$

34.



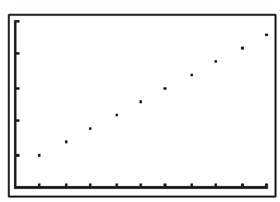
$[0, 10]$ by $[-3, 1]$

35.



$[0, 10]$ by $[-10, 100]$

36.



$[0, 10]$ by $[0, 25]$

37. The height (in cm) will be an arithmetic sequence with common difference $d = 2.3$ cm, so the height in week n is $700 + 2.3(n-1)$: 700, 702.3, 704.6, 706.9, …, 815, 817.3.

38. The first column is an arithmetic sequence with common difference $d = 14$. The second column is a geometric sequence with common ratio $r = \frac{1}{2}$.

Time (billions of years)	Mass (g)
0	16
14	8
28	4
42	2
56	1

39. The numbers of seats in each row form a finite arithmetic sequence with $a_1 = 7$, $d = 2$, and $n = 25$. The total number of seats is

$$\frac{25}{2}[2(7) + (25 - 1)(2)] = 775.$$

40. The numbers of tiles in each row form a finite arithmetic sequence with $a_1 = 15$, $a_n = 30$, and $n = 16$. The total number of tiles is

$$16\left(\frac{15 + 30}{2}\right) = 360.$$

41. The ten-digit numbers will vary; thus the sequences will vary. The end result will, however, be the same. Each limit will be 9. One example is:

Five random digits: 1, 4, 6, 8, 9
Five random digits: 2, 3, 4, 5, 6
List: 1, 2, 3, 4, 4, 5, 6, 6, 8, 9
Ten-digit number: 2, 416, 345, 689
Ten-digit number: 9, 643, 128, 564
a_1 = positive difference of the ten-digit numbers $= 7,226,782,875$
a_{n+1} = sum of the digits of a_n, so
a_2 = sum of the digits of $a_1 = 54$
a_3 = sum of the digits of $a_2 = 9$.
All successive sums of digits will be 9, so the sequence converges and the limit is 9.

42. Everyone should end up at the word "all."

43. True. Since two successive terms are negative, the common ratio r must be positive, and so the sign of the first term determines the sign of every number in the sequence.

44. False. For example, consider the sequence 5, 1, −3, −7, …

45. $a_1 = 2$ and $a_2 = 8$ implies $d = 8 - 2 = 6$
$c = a_1 - d$ so $c = 2 - 6 = -4$
$a_4 = 6 \cdot 4 + (-4) = 20$.
The answer is A.

46. $\lim_{n\to\infty} \sqrt{n} = \lim_{n\to\infty} n^{1/2} = \infty$, so the sequence diverges.
The answer is B.

47. $r = \frac{a_2}{a_1} = \frac{6}{2} = 3$
$a_6 = a_1 r^5 = 2 \cdot 3^5 = 486$ and $a_2 = 6$, so
$\frac{a_6}{a_2} = \frac{486}{6} = 81$.
The answer is E.

48. The geometric sequence will be defined by $a_{n+1} = a_n \div 3$ for $n \geq 1$ and $a_1 \neq 0$.

$a_2 = \frac{a_1}{3}$

$a_3 = \frac{a_2}{3} = \frac{a_1/3}{3} = \frac{a_1}{9}$

$a_4 = \frac{a_3}{3} = \frac{a_1/9}{3} = \frac{a_1}{27}$

$a_n = \frac{a_1}{3^{n-1}}$, which represents a geometric sequence.

The answer is C.

49. **(a)** $a_1 = 1$ because there is initially one male-female pair (this is the number of pairs after 0 months). $a_2 = 1$ because after one month, the original pair has only just become fertile. $a_3 = 2$ because after two months, the original pair produces a new male-female pair.

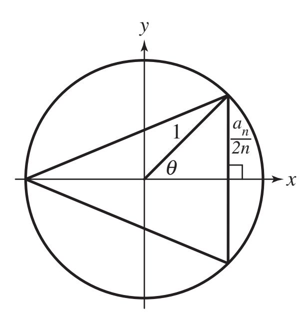
(b) Notice that after $n - 2$ months, there are a_{n-1} pairs, of which a_{n-2} (the number of pairs present one month earlier) are fertile. Therefore, after $n - 1$ months, the number of pairs will be $a_n = a_{n-1} + a_{n-2}$: to last month's total, we add the number of new pairs born. Thus $a_4 = 3$, $a_5 = 5$, $a_6 = 8$, $a_7 = 13$, $a_8 = 21$, $a_9 = 34$, $a_{10} = 55$, $a_{11} = 89$, $a_{12} = 144$, $a_{13} = 233$.

(c) Since a_1 is the initial number of pairs, and a_2 is the number of pairs after one month, we see that a_{13} is the number of pairs after 12 months.

50. Use a calculator: $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $a_5 = 5$, $a_6 = 8$, $a_7 = 13$. These are the first seven terms of the Fibonacci sequence.

51. **(a)** For a polygon with n sides, let A be the vertex in quadrant I at the top of the vertical segment, and let B be the point on the x-axis directly below A. Together with $(0, 0)$, these two points form a right triangle; the acute angle at the origin has measure $\theta = \frac{2\pi}{2n} = \frac{\pi}{n}$, since there are $2n$ such triangles making up the polygon. The length of the side opposite this angle is $\sin\theta = \sin\frac{\pi}{n}$, and there are $2n$ such sides making up the perimeter of the polygon, so $\sin\frac{\pi}{n} = \frac{a_n}{2n}$, or $a_n = 2n\sin(\pi/n)$.

(b) $a_{10} \approx 6.1803$, $a_{100} \approx 6.2822$, $a_{1000} \approx 6.2832$, $a_{10,000} \approx 6.2832 \approx 2\pi$. It appears that $a_n \to 2\pi$ as $n \to \infty$, which makes sense since the perimeter of the polygon should approach the circumference of the circle.



52. $P_1 = 525{,}000$; $P_n = 1.0175P_{n-1}$, $n \geq 2$

53. The difference of successive terms in $\{\log(a_n)\}$ will be of the form $\log(a_{n+1}) - \log(a_n) = \log\left(\frac{a_{n+1}}{a_n}\right)$. Since $\{a_n\}$ is geometric, $\frac{a_{n+1}}{a_n}$ is constant. This makes $\log\left(\frac{a_{n+1}}{a_n}\right)$ constant, so $\{\log(a_n)\}$ is a sequence with a constant difference (arithmetic).

54. The ratios of successive terms in $\{10^{b_n}\}$ will be of the form $10^{b_{n+1}} \div 10^{b_n} = 10^{b_{n+1}-b_n}$. Since $\{b_n\}$ is arithmetic, $b_{n+1} - b_n$ is constant. This makes $10^{b_{n+1}-b_n}$ constant, so $\{10^{b_n}\}$ is a sequence with a common ratio (geometric).

55. $a_1 = [1\ \ 1]$, $a_2 = [1\ \ 2]$, $a_3 = [2\ \ 3]$, $a_4 = [3\ \ 5]$, $a_5 = [5\ \ 8]$, $a_6 = [8\ \ 13]$, $a_7 = [13\ \ 21]$. The entries in the terms of this sequence are successive pairs of terms from the Fibonacci sequence.

56. $a_1 = [1\quad a]\quad r = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$
$a_2 = a_1 \cdot r = [1\quad a + d]$
$a_3 = a_2 \cdot r = [1\quad d + a + d] = [1\quad a + 2d]$
$a_4 = a_3 \cdot r = [1\quad d + a + 2d] = [1\quad a + 3d]$
$a_n = [1\quad a + (n-1)d]$.
So, the second entries of this geometric sequence of matrices form an arithmetic sequence with the first term a and common difference d.

■ Section 9.4 Series

Exploration 1

1. $3 + 6 + 9 + 12 + 15 = 45$

2. $5^2 + 6^2 + 7^2 + 8^2 = 25 + 36 + 49 + 64 = 174$

3. $\cos(0) + \cos(\pi) + \cdots + \cos(11\pi) + \cos(12\pi)$
$= 1 - 1 + 1 + \cdots - 1 + 1 = 1$

4. $\sin(0) + \sin(\pi) + \cdots + \sin(k\pi)$
$+ \cdots = 0 + 0 + \cdots + 0 + \cdots = 0$

5. $\frac{3}{10} + \frac{3}{100} + \frac{3}{1{,}000} + \cdots + \frac{3}{1{,}000{,}000} + \cdots = \frac{1}{3}$

Exploration 2

1. $1 + \ \ 2 + \ \ 3 + \cdots + 99 + 100$

2. $100 + \ 99 + \ 98 + \cdots + \ \ 2 + \ \ 1$

3. $101 + 101 + 101 + \cdots + 101 + 101$

4. $100(101) = 10{,}100$

5. The sum in Exercise 4 involves two copies of the same progression, so it doubles the sum of the progression. The answer that Gauss gave was 5050.

Quick Review 9.4

1. $a_1 = 4;\ d = 2$ so $a_{10} = a_1 + (n-1)d$
$a_{10} = 4 + (10-1)2 = 4 + 18 = 22$
$a_{10} = 22$

2. $a_1 = 3;\ a_2 = 1$ so $d = 1 - 3 = -2$
$a_{10} = a_1 + (n-1)d$
$a_{10} = 3 + (10-1)(-2) = 3 - 18 = -15$
$a_{10} = -15$

3. $a_3 = 6$ and $a_8 = 21$
$a_3 = a_1 + 2d$ and $a_8 = a_1 + 7d$
$(a_1 + 7d) - (a_1 + 2d) = 21 - 6$ so $5d = 15 \Rightarrow d = 3.$
$6 = a_1 + 2(3)$ so $a_1 = 0$
$a_{10} = 0 + 9(3) = 27$
$a_{10} = 27$

4. $a_5 = 3$, and $a_{n+1} = a_n + 5$ for $n \ge 1 \Rightarrow a_6 = 3 + 5 = 8$
$a_5 = 3$ and $a_6 = 8 \Rightarrow d = 5$
$a_5 = a_1 + 4d$ so $3 = a_1 + 4(5) \Rightarrow a_1 = -17$
$a_{10} = -17 + 9(5) = 28$
$a_{10} = 28$

5. $a_1 = 1$ and $a_2 = 2$ yields $r = \frac{2}{1} = 2$
$a_{10} = 1 \cdot 2^9 = 512$
$a_{10} = 512$

6. $a_4 = 1$ and $a_4 = a_1 \cdot r^3$; $a_6 = 2$ and $a_6 = a_1 \cdot r^5$
$\frac{a_1 \cdot r^5}{a_1 \cdot r^3} = \frac{2}{1}$
$r^2 = 2 \Rightarrow r = \sqrt{2}$
$1 = a_1(\sqrt{2})^3;\quad a_1 = \frac{1}{(\sqrt{2})^3} = \frac{1}{2\sqrt{2}}$
$a_{10} = \frac{1}{2\sqrt{2}}(\sqrt{2})^9 = \frac{16\sqrt{2}}{2\sqrt{2}} = 8$
$a_{10} = 8$

7. $a_7 = 5$ and $r = -2 \Rightarrow 5 = a_1(-2)^6$
$a_1 = \frac{5}{64};\quad a_{10} = \frac{5}{24}(-2)^9 = \frac{-2560}{24} = -40$
$a_{10} = -40$

8. $a_8 = 10$ and $a_8 = a_1 \cdot r^7$; $a_{12} = 40 \Rightarrow a_{12} = a_1 \cdot r^{11}$
$\frac{a_1 \cdot r^{11}}{a_1 \cdot r^7} = \frac{40}{10}$
$r^4 = 4$; so $r = (4)^{1/4}$
$10 = a_1((4)^{1/4})^7;\quad a_1 = \frac{10}{4^{7/4}}$
$a_{10} = \frac{10}{4^{7/4}}(4^{1/4})^9 = \frac{10(4^{9/4})}{4^{7/4}} = 10(4^{2/4}) = 10 \cdot 2 = 20$
$a_{10} = 20$

9. $\sum_{n=1}^{5} n^2 = 1 + 4 + 9 + 16 + 25 = 55$

10. $\sum_{n=1}^{5} (2n - 1) = 1 + 3 + 5 + 7 + 9 = 25$

Section 9.4 Exercises

1. $\sum_{k=1}^{11} (6k - 13)$

2. $\sum_{k=1}^{10} (3k - 1)$

3. $\sum_{k=1}^{n+1} k^2$

4. $\sum_{k=1}^{n+1} k^3$

5. $\sum_{k=0}^{\infty} 6(-2)^k$

6. $\sum_{k=0}^{\infty} 5(-3)^k$

For #7–12, use one of the formulas $S_n = n\left(\frac{a_1 + a_n}{2}\right)$ or $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$. In most cases, the first of these is easier (since the last term a_n is given); note that $n = \frac{a_n - a_1}{d} + 1$.

7. $6 \cdot \left(\frac{-7 + 13}{2}\right) = 6 \cdot 3 = 18$

8. $6 \cdot \left(\frac{-8 + 27}{2}\right) = 3 \cdot 19 = 57$

9. $80 \cdot \left(\frac{1 + 80}{2}\right) = 40 \cdot 81 = 3240$

10. $35 \cdot \left(\frac{2 + 70}{2}\right) = 35 \cdot 36 = 1260$

11. $13 \cdot \left(\frac{117 + 33}{2}\right) = 13 \cdot 75 = 975$

12. $29\left(\frac{111 + 27}{2}\right) = 29 \cdot 69 = 2001$

For #13–16, use the formula $S_n = \frac{a_1(1 - r^n)}{1 - r}$. Note that $n = 1 + \log_{|r|}\left|\frac{a_n}{a_1}\right| = 1 + \frac{\ln |a_n/a_1|}{\ln |r|}$.

13. $\frac{3(1 - 2^{13})}{1 - 2} = 24{,}573$

14. $\frac{5(1 - 3^{10})}{1 - 3} = 147{,}620$

15. $\frac{42[1 - (1/6)^9]}{1 - (1/6)} = 50.4(1 - 6^{-9}) \approx 50.4$

16. $\frac{42[1 - (-1/6)^{10}]}{1 - (-1/6)} = 36(1 - 6^{-10}) = 36 - 6^{-8} \approx 36$

For #17–22, use one of the formulas $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ or $S_n = \frac{a_1(1 - r^n)}{1 - r}$.

17. Arithmetic with $d = 3$: $\frac{10}{2}[2 \cdot 2 + (10 - 1)(3)]$
$= 5 \cdot 31 = 155$

18. Arithmetic with $d = -6$: $\frac{9}{2}[2 \cdot 14 + (9 - 1)(-6)]$
$= 9 \cdot (-10) = -90$

19. Geometric with $r = -\frac{1}{2}$: $\frac{4[1 - (-1/2)^{12}]}{1 - (-1/2)}$
$= \frac{8}{3} \cdot (1 - 2^{-12}) \approx 2.666$

20. Geometric with $r = -\frac{1}{2}$: $\frac{6[1 - (-1/2)^{11}]}{1 - (-1/2)}$
$= 4 \cdot (1 + 2^{-11}) \approx 4.002$

21. Geometric with $r = -11$: $\frac{-1[1 - (-11)^9]}{1 - (-11)}$
$= -\frac{1}{12} \cdot (1 + 11^9) = -196{,}495{,}641$

22. Geometric with $r = -12$: $\frac{-2[1 - (-12)^8]}{1 - (-12)}$
$= -\frac{2}{13} \cdot (1 - 12^8) = 66{,}151{,}030$

23. **(a)** The first six partial sums are {0.3, 0.33, 0.333, 0.3333, 0.33333, 0.333333}. The numbers appear to be approaching a limit of $0.\overline{3} = 1/3$. The series is convergent.

(b) The first six partial sums are {1, −1, 2, −2, 3, −3}. The numbers approach no limit. The series is divergent.

24. **(a)** The first six partial sums are {−2, 0, −2, 0, −2, 0}. The numbers approach no limit. The series is divergent.

(b) The first six partial sums are {1, 0.3, 0.23, 0.223, 0.2223, 0.22223}. The numbers appear to be approaching a limit of $0.\overline{2} = 2/9$. The series is convergent.

25. $r = \frac{1}{2}$, so it converges to $S = \frac{6}{1 - (1/2)} = 12$.

26. $r = \frac{1}{3}$, so it converges to $S = \frac{4}{1 - (1/3)} = 6$.

27. $r = 2$, so it diverges.

28. $r = 3$, so it diverges.

29. $r = \frac{1}{4}$, so it converges to $S = \frac{3/4}{1 - (1/4)} = 1$.

30. $r = \frac{2}{3}$, so it converges to $S = \frac{10/3}{1 - (2/3)} = 10$.

31. $7 + \frac{14}{99} = \frac{693}{99} + \frac{14}{99} = \frac{707}{99}$

32. $5 + \frac{93}{99} = 5 + \frac{31}{33} = \frac{196}{33}$

33. $-17 - \frac{268}{999} = -\frac{17{,}251}{999}$

34. $-12 - \frac{876}{999} = -12 - \frac{292}{333} = -\frac{4288}{333}$

35. **(a)** The ratio of any two successive account balances is $r = 1.1$. That is,
$\frac{\$22{,}000}{\$20{,}000} = \frac{\$24{,}200}{\$22{,}000} = \frac{\$26{,}620}{\$24{,}200} = \frac{\$29{,}282}{\$26{,}620} = 1.1$.

(b) Each year, the balance is 1.1 times as large as the year before. So, n years after the balance is \$20,000, it will be \$20,000 $(1.1)^n$.

(c) The sum of the eleven terms of the geometric sequence is $\frac{\$20{,}000(1 - 1.1^{11})}{1 - 1.1} = \$370{,}623.34$.

36. **(a)** The difference of any two successive account balances is $d = \$2016$. That is \$20,016 − \$18,000
= \$22,032 − \$20,016 = \$24,048 − \$22,032
= \$26,064 − \$24,048 = \$2016.

(b) Each year, the balance is \$2016 more than the year before. So, n years after the balance is \$18,000, it will be \$18,000 + \$2016n.

(c) The sum of the eleven terms of the arithmetic sequence is
$\frac{11}{2}[2(\$18{,}000) + (10)(\$2016)] = \$308{,}880$.

37. **(a)** The first term, $120(1 + 0.07/12)^0$, simplifies to 120. The common ratio of terms, r, equals $1 + 0.07/12$.

(b) The sum of the 120 terms is

$$\frac{120\,[1 - (1 + 0.07/12)^{120}]}{1 - (1 + 0.07/12)} = \$20{,}770.18.$$

38. **(a)** The first term, $100(1 + 0.08/12)^0$, simplifies to 100. The common ratio of terms, r, equals $1 + 0.08/12$.

(b) The sum of the 120 terms is

$$\frac{100\,[1 - (1 + 0.08/12)^{120}]}{1 - (1 + 0.08/12)} = \$18{,}294.60.$$

39. The heights of the ball on the bounces after the first bounce can be modeled by an infinite geometric series. The total height traveled by the ball on the subsequent bounces is:

$$\begin{aligned} &2 \cdot [2(0.9) + 2(0.9)^2 + 2(0.9)^3 + 2(0.9)^4 + \cdots] \\ &= 4 \cdot [(0.9) + (0.9)^2 + (0.9)^3 + (0.9)^4 + \cdots] \\ &= 4 \cdot \left[\frac{0.9}{1 - 0.9}\right] = 36 \text{ m.} \end{aligned}$$

Since the ball was dropped from 2 m, the total distance traveled by the ball is 36 m + 2 m = 38 m.

40. This is an example of a divergent infinite series; the ball would travel forever and traverse an infinite distance.

41. False. The series might diverge. For example, examine the series $1 + 2 + 3 + 4 + 5$ where all of the terms are positive. Consider the limit of the sequence of partial sums. The first five partial sums are {1, 3, 6, 10, 15}. These numbers increase without bound and do not approach a limit. Therefore, the series diverges and has no sum.

42. False. Justifications will vary. One example is to examine

$$\sum_{n=1}^{\infty} n \quad \text{and} \quad \sum_{n=1}^{\infty}(-n).$$

Both of these diverge, but $\sum_{n=1}^{\infty}(n + (-n)) = \sum_{n=1}^{\infty} 0 = 0$.

So the sum of the two divergent series converges.

43. $3^{-1} + 3^{-2} + 3^{-3} + \cdots + 3^{-n} + \cdots =$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots + \frac{1}{3^n} + \cdots$$

The first five partial sums are $\left\{\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}, \frac{121}{243}\right\}$. These appear to be approaching a limit of 1/2, which would suggest that the series converges to 1/2. The answer is A.

44. If $\sum_{n=1}^{\infty} x^n = 4$, then $x = 0.8$.

$$\sum_{n=1}^{\infty} 0.8^n = 0.8 + 0.64 + 0.512 + 0.4096 + 0.32768 + 0.262144 + \cdots$$

The first six partial sums are {0.8, 1.44, 1.952, 2.3616, 2.68928, 2.951424}. It appears from this sequence of partial sums that the series is converging. If the sequence of partial sums were extended to the 40th partial sum, you would see that the series converges to 4. The answer is D.

45. The common ratio is $0.75/3 = 0.25$, so the sum of the infinite series is $3/(1 - 0.25) = 4$. The answer is C.

46. The sum is an infinite geometric series with $|r| = 5/3 > 1$. The answer is E.

47. **(a)** Heartland: 20,505,437 people.

Southeast: 48,310,650 people.

(b) Heartland: 517,825 mi^2.

Southeast: 348,999 mi^2.

(c) Heartland: $\frac{20{,}505{,}437}{517{,}825} \approx 39.60$ people/mi^2.

Southeast: $\frac{48{,}310{,}650}{348{,}999} \approx 138.43$ people/mi^2.

(d) The table is shown below; the answer differs because the overall population density $\frac{\Sigma \text{ population}}{\Sigma \text{ area}}$ is generally not the same as the average of the population densities, $\frac{1}{n}\Sigma\left(\frac{\text{population}}{\text{area}}\right)$. The larger states within each group have a greater effect on the overall mean density. In a similar way, if a student's grades are based on a 100-point test and four 10-point quizzes, her overall average grade depends more on the test grade than on the four quiz grades.

Heartland:		Southeast:	
Iowa	≈54.13	Alabama	≈92.44
Kansas	≈34.68	Arkansas	≈54.82
Minnesota	≈62.84	Florida	≈320.60
Missouri	≈85.93	Georgia	≈164.45
Nebraska	≈23.61	Louisiana	≈94.94
N. Dakota	≈9.51	Mississippi	≈62.22
S. Dakota	≈10.56	S. Carolina	≈148.66
Total	≈281.26	Total	≈938.14
Average	≈40.18	Average	≈134.02

48. $\sum_{k=1}^{8}(k^2 - 2)$

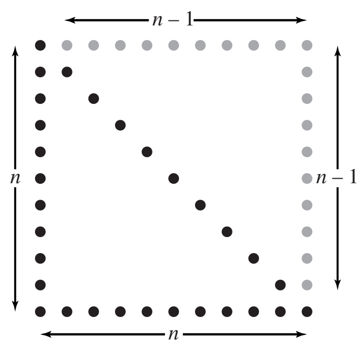
49. The table suggests that $S_n = \sum_{k=1}^{n} F_k = F_{n+2} - 1$.

n	F_n	S_n	$F_{n+2} - 1$
1	1	1	1
2	1	2	2
3	2	4	4
4	3	7	7
5	5	12	12
6	8	20	20
7	13	33	33
8	21	54	54
9	34	88	88

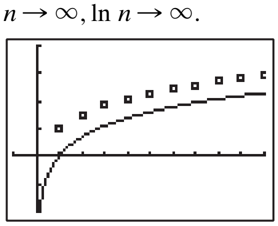
50. The nth triangular number is simply the sum of the first n consecutive positive integers:
$1 + 2 + 3 + \cdots + n = n\left(\frac{1+n}{2}\right) = \frac{n(n+1)}{2}$

51. Algebraically: $T_{n-1} + T_n = \frac{(n-1)n}{2} + \frac{n(n+1)}{2}$
$= \frac{n^2 - n + n^2 + n}{2} = n^2$

Geometrically: The array of black dots in the figure represents $T_n = 1 + 2 + 3 + \cdots + n$ (that is, there are T_n dots in the array). The array of gray dots represents $T_{n-1} = 1 + 2 + 3 + \cdots + (n - 1)$. The two triangular arrays fit together to form an $n \times n$ square array, which has n^2 dots.



52. If $\sum_{k=1}^{n} \frac{1}{k} \geq \ln n$ for all n, then the sum diverges since as $n \to \infty$, $\ln n \to \infty$.



$[-1, 10]$ by $[-2, 4]$

■ Section 9.5 Mathematical Induction

Exploration 1

1. Start with the rightmost peg if n is odd and the middle peg if n is even. From that point on, the first move for moving any smaller stack to a destination peg should be directly to the destination peg if the smaller stack's size n is odd and to the other available peg if n is even. The fact that the winning strategy follows such predictable rules is what makes it so interesting to students of computer programming.

Exploration 2

1. 43, 47, 53, 61, 71, 83, 97, 113, 131, 151. Yes.

2. 173, 197, 223, 251, 281, 313, 347, 383, 421, 461. Yes.

3. 503, 547, 593, 641, 691, 743, 797, 853, 911, 971. Yes. Inductive thinking might lead to the conjecture that $n^2 + n + 41$ is prime for all n, but we have no proof as yet!

4. The next 9 numbers are all prime, but $40^2 + 40 + 41$ is not. Quite obviously, neither is the number $41^2 + 41 + 41$.

Quick Review 9.5

1. $n^2 + 5n$

2. $n^2 - n - 6$

3. $k^3 + 3k^2 + 2k$

4. $(n + 3)(n - 1)$

5. $(k + 1)^3$

6. $(n - 1)^3$

7. $f(1) = 1 + 4 = 5, f(t) = t + 4,$
$f(t + 1) = t + 1 + 4 = t + 5$

8. $f(1) = \frac{1}{1+1} = \frac{1}{2}, f(k) = \frac{k}{k+1},$
$f(k+1) = \frac{k+1}{k+1+1} = \frac{k+1}{k+2}$

9. $P(1) = \frac{2 \cdot 1}{3 \cdot 1 + 1} = \frac{1}{2},$
$P(k) = \frac{2k}{3k+1}; P(k+1) = \frac{2(k+1)}{3(k+1)+1} = \frac{2k+2}{3k+4}$

10. $P(1) = 2(1)^2 - 1 - 3 = -2, P(k) = 2k^2 - k - 3,$
$P(k + 1) = 2(k + 1)^2 - (k + 1) - 3 = 2k^2 + 3k - 2$

Section 9.5 Exercises

1. P_n: $2 + 4 + 6 + \cdots + 2n = n^2 + n$.
P_1 is true: $2(1) = 1^2 + 1$.
Now assume P_k is true: $2 + 4 + 6 + \ldots + 2k$
$= k^2 + k$. Add $2(k + 1)$ to both sides:
$2 + 4 + 6 + \cdots + 2k + 2(k + 1)$
$= k^2 + k + 2(k + 1) = k^2 + 3k + 2$
$= k^2 + 2k + 1 + k + 1 = (k + 1)^2 + (k + 1)$, so
P_{k+1} is true. Therefore, P_n is true for all $n \geq 1$.

2. P_n: $8 + 10 + 12 + \cdots + (2n + 6) = n^2 + 7n$.
P_1 is true: $2(1) + 6 = 1^2 + 7 \cdot 1$.
Now assume P_k is true:
$8 + 10 + 12 + \cdots + (2k + 6) = k^2 + 7k$.
Add $2(k + 1) + 6 = 2k + 8$ to both sides:
$8 + 10 + 12 + \cdots + (2k + 6) + [2(k + 1) + 6]$
$= k^2 + 7k + 2k + 8 = (k^2 + 2k + 1) + 7k + 7$
$= (k + 1)^2 + 7(k + 1)$, so P_{k+1} is true.
Therefore, P_n is true for all $n \geq 1$.

3. P_n: $6 + 10 + 14 + \cdots + (4n + 2) = n(2n + 4)$.
P_1 is true: $4(1) + 2 = 1(2 + 4)$.

Now assume P_k is true:
$6 + 10 + 14 + \cdots + (4k + 2) = k(2k + 4)$.
Add $4(k + 1) + 2 = 4k + 6$ to both sides:
$6 + 10 + 14 + \cdots + (4k + 2) + [4(k + 1) + 2]$
$= k(2k + 4) + 4k + 6 = 2k^2 + 8k + 6$
$= (k + 1)(2k + 6) = (k + 1)[2(k + 1) + 4]$, so P_{k+1} is true. Therefore, P_n is true for all $n \geq 1$.

4. P_n: $14 + 18 + 22 + \cdots + (4n + 10) = 2n(n + 6)$.
P_1 is true: $4(1) + 10 = 2 \cdot 1(1 + 6)$. Now assume P_k is true:
$14 + 18 + 22 + \cdots + (4k + 10) = 2k(k + 6)$. Add $4(k + 1) + 10 = 4k + 14$ to both sides:
$14 + 18 + 22 + \cdots + (4k + 10) + [4(k + 1) + 10]$
$= 2k(k + 6) + (4k + 14)$
$= 2(k^2 + 8k + 7) = 2(k + 1)(k + 7)$
$= 2(k + 1)(k + 1 + 6)$, so P_{k+1} is true. Therefore, P_n is true for all $n \geq 1$.

5. P_n: $5n - 2$. P_1 is true: $a_1 = 5 \cdot 1 - 2 = 3$.
Now assume P_k is true: $a_k = 5k - 2$.
To get a_{k+1}, add 5 to a_k; that is,
$a_{k+1} = (5k - 2) + 5 = 5(k + 1) - 2$ This shows that P_{k+1} is true. Therefore, P_n is true for all $n \geq 1$.

6. P_n: $a_n = 2n + 5$. P_1 is true: $a_1 = 2 \cdot 1 + 5 = 7$.
Now assume P_k is true: $a_k = 2k + 5$.
To get a_{k+1}, add 2 to a_k; that is,
$a_{k+1} = (2k + 5) + 2 = 2(k + 1) + 5$. This shows that P_{k+1} is true. Therefore, P_n is true for all $n \geq 1$.

7. P_n: $a_n = 2 \cdot 3^{n-1}$.
P_1 is true: $a_1 = 2 \cdot 3^{1-1} = 2 \cdot 3^0 = 2$.
Now assume P_k is true: $a_k = 2 \cdot 3^{k-1}$.
To get a_{k+1}, multiply a_k by 3; that is,
$a_{k+1} = 3 \cdot 2 \cdot 3^{k-1} = 2 \cdot 3k = 2 \cdot 3^{(k+1)-1}$. This shows that P_{k+1} is true. Therefore, P_n is true for all $n \geq 1$.

8. P_n: $a_n = 3 \cdot 5^{n-1}$.
P_1 is true: $a_1 = 3 \cdot 5^{1-1} = 3 \cdot 5^0 = 3$.
Now assume P_k is true: $a_k = 3 \cdot 5^{k-1}$.
To get a_{k+1}, multiply a_k by 5; that is,
$a_{k+1} = 5 \cdot 3 \cdot 5^{k-1} = 3 \cdot 5^k = 3 \cdot 5^{(k+1)-1}$. This shows that P_{k+1} is true. Therefore, P_n is true for all $n \geq 1$.

9. P_1: $1 = \dfrac{1(1 + 1)}{2}$.

P_k: $1 + 2 + \cdots + k = \dfrac{k(k + 1)}{2}$

P_{k+1}: $1 + 2 + \cdots + k + (k + 1) = \dfrac{(k + 1)(k + 2)}{2}$.

10. P_1: $(2(1) - 1)^2 = \dfrac{1(2 - 1)(2 + 1)}{3}$.

P_k: $1^2 + 3^2 + \cdots + (2k - 1)^2 = \dfrac{k(2k - 1)(2k + 1)}{3}$.

P_{k+1}: $1^2 + 3^2 + \cdots + (2k - 1)^2 + (2k + 1)^2$
$= \dfrac{(k + 1)(2k + 1)(2k + 3)}{3}$.

11. P_1: $\dfrac{1}{1 \cdot 2} = \dfrac{1}{1 + 1}$.

P_k: $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{k(k + 1)} = \dfrac{k}{k + 1}$.

P_{k+1}: $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{k(k + 1)}$
$+ \dfrac{1}{(k + 1)(k + 2)} = \dfrac{k + 1}{k + 2}$.

12. P_1: $1^4 = \dfrac{1(1 + 1)(2 + 1)(3 + 3 - 1)}{30}$.

P_k: $1^4 + 2^4 + \cdots + k^4$
$= \dfrac{k(k + 1)(2k + 1)(3k^2 + 3k - 1)}{30}$.

P_{k+1}: $1^4 + 2^4 + \cdots + k^4 + (k + 1)^4$
$= \dfrac{(k + 1)(k + 2)(2k + 3)(3k^2 + 9k + 5)}{30}$.

13. P_n: $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$.
P_1 is true: $4(1) - 3 = 1 \cdot (2 \cdot 1 - 1)$.
Now assume P_k is true:
$1 + 5 + 9 + \cdots + (4k - 3) = k(2k - 1)$.
Add $4(k + 1) - 3 = 4k + 1$ to both sides:
$1 + 5 + 9 + \cdots + (4k - 3) + [4(k + 1) - 3]$
$= k(2k - 1) + 4k + 1 = 2k^2 + 3k + 1$
$= (k + 1)(2k + 1) = (k + 1)[2(k + 1) - 1]$,
so P_{k+1} is true.
Therefore, P_n is true for all $n \geq 1$.

14. P_n: $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$.
P_1 is true: $2^{1-1} = 2^1 - 1$.
Now assume P_k is true:
$1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1$.
Add 2^k to both sides,
$1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^k$
$= 2^k - 1 + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1$, so P_{k+1} is true.
Therefore, P_n is true for all $n \geq 1$.

15. P_n: $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$.

P_1 is true: $\dfrac{1}{1 \cdot 2} = \dfrac{1}{1 + 1}$.

Now assume P_k is true:
$\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{k(k + 1)}$.

Add $\dfrac{1}{(k + 1)(k + 2)}$ to both sides:

$\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{k(k + 1)} + \dfrac{1}{(k + 1)(k + 2)}$

$= \dfrac{k}{k + 1} + \dfrac{1}{(k + 1)(k + 2)} = \dfrac{k(k + 2) + 1}{(k + 1)(k + 2)}$

$= \dfrac{(k + 1)(k + 1)}{(k + 1)(k + 2)} = \dfrac{k + 1}{k + 2} = \dfrac{k + 1}{(k + 1) + 1}$,
so P_{k+1} is true.
Therefore, P_n is true for all $n \geq 1$.

16. P_n: $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$.

P_1 is true: it says that $\dfrac{1}{1 \cdot 3} = \dfrac{1}{2 \cdot 1 + 1}$.

Now assume P_k is true:

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}.$$

Add $\dfrac{1}{[2(k+1)-1][2(k+1)+1]}$

$=\dfrac{1}{(2k+1)(2k+3)}$ to both sides, and we have

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}=\frac{k(2k+3)+1}{(2k+1)(2k+3)}\cdots$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}=\frac{k+1}{2(k+1)+1},$$

so P_{k+1} is true.
Therefore, P_n is true for all $n \ge 1$.

17. P_n: $2^n \ge 2n$.
P_1 is true: $2^1 \ge 2\cdot 1$ (in fact, they are equal). Now assume P_k is true: $2^k \ge 2k$.
Then $2^{k+1} = 2\cdot 2^k \ge 2\cdot 2k = 2\cdot(k+k) \ge 2(k+1)$, so P_{k+1} is true. Therefore, P_n is true for all $n \ge 1$.

18. P_n: $3^n \ge 3n$.
P_1 is true: $3^1 \ge 3\cdot 1$ (in fact, they are equal). Now assume P_k is true: $3^k \ge 3k$.
Then $3^{k+1} = 3\cdot 3^k \ge 3\cdot 3k = 3\cdot(k+2k) \ge 3(k+1)$, so P_{k+1} is true. Therefore, P_n is true for all $n \ge 1$.

19. P_n: 3 is a factor of $n^3 + 2n$.
P_1 is true: 3 is a factor of $1^3 + 2\cdot 1 = 3$.
Now assume P_k is true: 3 is a factor of $k^3 + 2k$.
Then $(k+1)^3 + 2(k+1)$
$= (k^3 + 3k^2 + 3k + 1) + (2k + 2)$
$= (k^3 + 2k) + 3(k^2 + k + 1)$.
Since 3 is a factor of both terms, it is a factor of the sum, so P_{k+1} is true. Therefore, P_n is true for all $n \ge 1$.

20. P_n: 6 is a factor of $7^n - 1$.
P_1 is true: 6 is a factor of $7^1 - 1 = 6$.
Now assume P_k is true, so that 6 is a factor of $7^k - 1 = 6$.
Then $7^{k+1} - 1 = 7\cdot 7^k - 1 = 7(7^k - 1) + 6$. Since 6 is a factor of both terms of this sum, it is a factor of the sum, so P_{k+1} is true.
Therefore, P_n is true for all positive integers n.

21. P_n: The sum of the first n terms of a geometric sequence with first term a_1 and common ratio $r \ne 1$ is $\dfrac{a_1(1-r^n)}{1-r}$.
P_1 is true: $a_1 = \dfrac{a_1(1-r^1)}{1-r}$.
Now assume P_k is true so that
$a_1 + a_1r + \cdots = a_1r^{k-1} = \dfrac{a_1(1-r^k)}{(1-r)}$.
Add a_1r^k to both sides:
$a_1 + a_1r + \cdots = a_1r^{k-1}$
$+ a_1r^k = \dfrac{a_1(1-r^k)}{(1-r)} + a_1r^k$

$$=\frac{a_1(1-r^k)+a_1r^k(1-r)}{1-r}$$

$$=\frac{a_1-a_1r^k-a_1r^{k+1}}{1-r}=\frac{a_1-a_1r^{k+1}}{1-r},$$

so P_{k+1} is true. Therefore, P_n is true for all positive integers n.

22. P_n: $S_n = \dfrac{n}{2}[2a_1 + (n-1)d]$.
First note that $a_n = a_1 + (n-1)d$. P_1 is true:
$S_1 = \dfrac{1}{2}[2a_1 + (1-1)d] = \dfrac{1}{2}(2a_1) = a_1$.

Now assume P_k is true: $S_k = \dfrac{k}{2}[2a_1 + (k-1)d]$.

Add $a_{k+1} = a_1 + kd$ to both sides, and observe that $S_k + a_{k+1} = S_{k+1}$.
Then we have

$$S_{k+1}=\frac{k}{2}[2a_1+(k-1)d]+a_1+kd$$

$$=ka_1+\frac{1}{2}k(k-1)d+a_1+kd$$

$$=(k+1)a_1+\frac{1}{2}k(k+1)d$$

$$=\frac{k+1}{2}[2a_1+(k+1-1)d].$$

Therefore, P_{k+1} is true, so P_n is true for all $n \ge 1$.

23. P_n: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

P_1 is true: $\displaystyle\sum_{k=1}^{1} k = 1 = \frac{1\cdot 2}{2}$.

Now assume P_k is true: $\displaystyle\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Add $(k+1)$ to both sides, and we have

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$$

$$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$

$=\dfrac{(k+1)(k+1+1)}{2}$, so P_{k+1} is true.

Therefore, P_n is true for all $n \ge 1$.

24. P_n: $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$. P_1 is true: $1^3 = \dfrac{1^2 2^2}{4}$.

Now assume P_k is true so that
$1^3 + 2^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$. Add $(k+1)^3$ to both sides and we have
$1^3 + 2^3 + \cdots + k^3 + (k+1)^3$

$$=\frac{k^2(k+1)^2}{4}+(k+1)^3=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$=\frac{(k+1)^2(k^2+4k+4)}{4}=\frac{(k+1)^2((k+1)+1)^2}{4}$$

so P_{k+1} is true. Therefore, P_n is true for all positive integers.

25. Use the formula in 23: $\sum_{k=1}^{500} k = \frac{(500)(501)}{2} = 125{,}250$

26. Use the formula in Example 2: $\sum_{k=1}^{250} k^2 = \frac{(250)(251)(501)}{6}$ $= 5{,}239{,}625$

27. Use the formula in 23: $\sum_{k=4}^{n} k = \sum_{k=1}^{n} k - \sum_{k=1}^{3} k$
$= \frac{n(n+1)}{2} - \frac{3 \cdot 4}{2} = \frac{n^2+n-12}{2} = \frac{(n-3)(n+4)}{2}$

28. Use the formula in 24: $\sum_{k=1}^{75} k^3 = \frac{(75^2)(76^2)}{4} = 8{,}122{,}500$

29. Use the formula in 14: $\sum_{k=1}^{35} 2^{k-1} = 2^{35} - 1 \approx 3.44 \times 10^{10}$

30. Use the formula in 24: $\sum_{k=1}^{15} k^3 = \frac{(15^2)(16^2)}{4}$ $= 14{,}400$

31. $\sum_{k=1}^{n}(k^2 - 3k + 4) = \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 4$
$= \frac{n(n+1)(2n+1)}{6} - 3\left[\frac{n(n+1)}{2}\right] + 4n$
$= \frac{n(n^2 - 3n + 8)}{3}$

32. $\sum_{k=1}^{n}(2k^2 + 5k - 2) = \sum_{k=1}^{n} 2k^2 + \sum_{k=1}^{n} 5k - \sum_{k=1}^{n} 2 = 2$
$= 2\left[\frac{n(n+1)(2n+1)}{6}\right] + 5\left[\frac{n(n+1)}{2}\right] - 2n$
$= \frac{n(4n^2 + 21n + 5)}{6} = \frac{n(n+5)(4n+1)}{6}$

33. $\sum_{k=1}^{n}(k^3 - 1) = \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} 1 = \frac{n^2(n+1)^2}{4} - n$
$= \frac{n(n^3 + 2n^2 + n - 4)}{4} = \frac{n(n-1)(n^2 + 3n + 4)}{4}$

34. $\sum_{k=1}^{n}(k^3 + 4k - 5) = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 4k - \sum_{k=1}^{n} 5$
$= \frac{n^2(n+1)^2}{4} + 4\left[\frac{n(n+1)}{2}\right] - 5n$
$= \frac{n(n^3 + 2n^2 + 9n - 12)}{4} = \frac{n(n-1)(n^2 + 3n + 12)}{4}$

35. The inductive step does not work for two people. Sending them alternately out of the room leaves one person (and one blood type) each time, but we cannot conclude that their blood types will match *each other.*

36. The number k is a fixed number for which the statement P_k is known to be true. Once the anchor is established, we can assume that such a number k exists. We cannot assume that P_n is true, because n is not fixed.

37. False. Mathematical induction is used to show that a statement P_n is true for all positive integers.

38. True. $(1 + 1)^2 = 4 = 4(1)$. P_n is false, however, for all other values of n.

39. The inductive step assumes that the statement is true for some positive integer k. The answer is E.

40. The anchor step, proving P_1, comes first. The answer is A.

41. Mathematical induction could be used, but the formula for a finite arithmetic sequence with $a_1 = 1, d = 2$ would also work. The answer is B.

42. The first two partial sums are 1 and 9. That eliminates all answers except C. Mathematical induction can be used to show directly that C is the correct answer.

43. P_n: 2 is a factor of $(n + 1)(n + 2)$. P_1 is true because 2 is a factor of $(2)(3)$. Now assume P_k is true so that 2 is a factor of $(k + 1)(k + 2)$. Then
$[(k + 1) + 1][(k + 1) + 2]$
$= (k + 2)(k + 3) = k^2 + 5k + 6$
$= k^2 + 3k + 2 + 2k + 4$
$= (k + 1)(k + 2) + 2(x + 2)$. Since 2 is a factor of both terms of this sum, it is a factor of the sum, and so P_{k+1} is true. Therefore, P_n is true for all positive integers n.

44. P_n: 6 is a factor of $n(n + 1)(n + 2)$. P_1 is true because 6 is a factor of $(1)(2)(3)$. Now assume P_k is true so that 6 is a factor of $k(k + 1)(k + 2)$. Then $(k + 1)[(k + 1) + 1]$ $[(k + 1) + 2] = k(k + 1)(k + 2) + 3(k + 1)(k + 2)$. Since 2 is a factor of $(k + 1)(k + 2)$, 6 is a factor of both terms of the sum and thus of the sum itself, and so P_{k+1} is true.

45. Given any two consecutive integers, one of them must be even. Therefore, their product is even. Since $n + 1$ and $n + 2$ are consecutive integers, their product is even. Therefore, 2 is a factor of $(n + 1)(n + 2)$.

46. Given any three consecutive integers, one of them must be a multiple of 3, and at least one of them must be even. Therefore, their product is a multiple of 6. Since n, $n + 1$ and $n + 2$ are three consecutive integers, 6 is a factor of $n(n + 1)(n + 2)$.

47. P_n: $F_{n+2} - 1 = \sum_{k=1}^{n} F_k$. P_1 is true since
$F_{1+2} = 1 = F_3 - 1 = 2 - 1 = 1$, which equals
$\sum_{k=1}^{1} F_k = 1$. Now assume that P_k is true:
$F_{k+2} - 1 = \sum_{i=1}^{k} F_i$. Then $F_{(k+1)+2} - 1$
$= F_{k+3} - 1 = F_{k+1} + F_{k+2} - 1$
$= (F_{k+2} - 1) + F_{k+1} = \left(\sum_{i=1}^{k} F_i\right) + F_{k+1}$
$= \sum_{i=1}^{k+1} F_i$, so P_{k+1} is true. Therefore, P_n is true for all $n \geq 1$.

48. P_n: $a_n < 2$. P_1 is easy: $a_1 = \sqrt{2} < 2$. Now assume that P_k is true: $a_k < 2$. Note that $a_{k+1} = \sqrt{2 + a_k}$, so that $a^2_{k+1} < 2 + a_k < 2 + 2 = 4$; therefore $a_{k+1} < 2$, so P_{k+1} is true. Therefore, P_n is true for all $n \geq 1$.

49. P_n: $a - 1$ is a factor of $a^n - 1$. P_1 is true because $a - 1$ is a factor of $a - 1$. Now assume P_k is true so that $a - 1$ is a factor of $a^k - 1$. Then $a^{k+1} - 1 = a \cdot ak - 1$ $= a(a^k - 1) + (a - 1)$. Since $a - 1$ is a factor of both terms in the sum, it is a factor of the sum, and so P_{k+1} is true. Therefore, P_n is true for all positive integers n.

50. Let $P(a) = a^n - 1$. Since $P(1) = 1^n - 1 = 0$, the Factor Theorem for polynomials allows us to conclude that $a - 1$ is a factor of P.

51. P_n: $3n - 4 \geq n$ for $n \geq 2$. P_2 is true since $3 \cdot 2 - 4 \leq 2$. Now assume that P_k is true: $3k - 4 \geq 2$. Then $3(k + 1) - 4 = 3k + 3 - 4$ $= (3k - 4) + 3k \geq k + 3 \geq k + 1$, so P_{k+1} is true. Therefore, P_n is true for all $n \geq 2$.

52. P_n: $2^n \geq n^2$ for $n \geq 4$. P_4 is true since $2^4 \geq 4^2$. Now assume that P_k is true: $2^k \geq k^2$. Then $2^{k+1} = 2 \cdot 2^k \geq 2 \cdot k^2 \geq 2 \cdot k^2 \geq k^2 + 2k + 1$ $= (k + 1)^2$. The inequality $2k^2 \geq k^2 + 2k + 1$, or equivalently, $k^2 \geq 2k + 1$, is true for all $k \geq 4$ because $k^2 = k \cdot k \geq 4k = 2k + 2k > 2k + 1$.) Thus P_{k+1} is true, so P_n is true for all $n \geq 4$.

53. Use P_3 as the anchor and obtain the inductive step by representing any n-gon as the union of a triangle and an $(n - 1)$-gon.

■ Chapter 9 Review

1. $\binom{12}{5} = \dfrac{12!}{5!(12 - 5)!} = \dfrac{12!}{5!7!} = 792$

2. $\binom{789}{787} = \dfrac{789!}{787!(789 - 787)!} = \dfrac{789!}{787!2!} = 310{,}866$

3. ${}_{18}C_{12} = \dfrac{18!}{12!(18 - 12)!} = \dfrac{18!}{12!6!} = 18{,}564$

4. ${}_{35}C_{28} = \dfrac{35!}{28!(35 - 28)!} = \dfrac{35!}{28!7!} = 6{,}724{,}520$

5. ${}_{12}P_7 = \dfrac{12!}{(12 - 7)!} = \dfrac{12!}{5!} = 3{,}991{,}680$

6. ${}_{15}P_8 = \dfrac{15!}{(15 - 8)!} = \dfrac{15!}{7!} = 259{,}459{,}200$

7. $26 \cdot 36^4 = 43{,}670{,}016$ code words

8. $3 + (3 \cdot 4) = 15$ trips

9. ${}_{26}P_2 \cdot {}_{10}P_4 + {}_{10}P_3 \cdot {}_{26}P_3 = 14{,}508{,}000$ license plates

10. ${}_{45}C_3 = 14{,}190$ committees

11. Choose 10 more cards from the other 49: ${}_3C_3 \cdot {}_{49}C_{10} = {}_{49}C_{10} = 8{,}217{,}822{,}536$ hands

12. Choose a king, then 8 more cards from the other 44: ${}_4C_4 \cdot {}_4C_1 \cdot {}_{44}C_8 = {}_4C_1 \cdot {}_{44}C_8 = 708{,}930{,}508$ hands

13. ${}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5 = 2^5 - {}_5C_0 - {}_5C_1 = 26$ outcomes

14. ${}_{21}C_2 \cdot {}_{14}C_2 = 19{,}110$ committees

15. ${}_5P_1 + {}_5P_2 + {}_5P_3 + {}_5P_4 + {}_5P_5 = 325$

16. $2^4 = 16$ (This includes the possibility that he has *no* coins in his pocket.)

17. **(a)** There are 7 letters, all different. The number of distinguishable permutations is $7! = 5040$. (GERMANY can be rearranged to spell MEG RYAN.)

(b) There are 13 letters, where E, R, and S each appear twice. The number of distinguishable permutations is $\dfrac{13!}{2!2!2!} = 778{,}377{,}600$.
(PRESBYTERIANS can be rearranged to spell BRITNEY SPEARS.)

18. **(a)** There are 7 letters, all different. The number of distinguishable permutations is $7! = 5040$.

(b) There are 11 letters, where A appears 3 times and L, S, and E each appear 2 times. The number of distinguishable permutations is $\dfrac{11!}{3!2!2!2!} = 831{,}600$.

19. $(2x + y)^5 = (2x)^5 + 5(2x)^4y + 10(2x)^3y^2 + 10(2x)^2y^3 + 5(2x)y^4 + y^5 = 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$

20. $(4a - 3b)^7 = (4a)^7 + 7(4a)^6(-3b) + 21(4a)^5(-3b)^2 + 35(4a)^4(-3b)^3 + 35(4a)^3(-3b)^4 + 21(4a)^2(-3b)^5 + 7(4a)(-3b)^6 + (-3b)^7$
$= 16{,}384a^7 - 86{,}016a^6b + 193{,}536a^5b^2 - 241{,}920a^4b^3 + 181{,}440a^3b^4 - 81{,}648a^2b^5 + 20{,}412ab^6 - 2187b^7$

21. $(3x^2 + y^3)^5 = (3x^2)^5 + 5(3x^2)^4(y^3) + 10(3x^2)^3(y^3)^2 + 10(3x^2)^2(y^3)^3 + 5(3x^2)(y^3)^4 + (y^3)^5$
$= 243x^{10} + 405x^8y^3 + 270x^6y^6 + 90x^4y^9 + 15x^2y^{12} + y^{15}$

22. $\left(1 + \dfrac{1}{x}\right)^6 = 1 + 6(x^{-1}) + 15(x^{-1})^2 + 20(x^{-1})^3 + 15(x^{-1})^4 + 6(x^{-1})^5 + (x^{-1})^6$
$= 1 + 6x^{-1} + 15x^{-2} + 20x^{-3} + 15x^{-4} + 6x^{-5} + x^{-6}$

23. $(2a^3 - b^2)^9 = (2a^3)^9 + 9(2a^3)^8(-b^2) + 36(2a^3)^7(-b^2)^2 + 84(2a^3)^6(-b^2)^3 + 126(2a^3)^5(-b^2)^4 + 126(2a^3)^4(-b^2)^5 + 84(2a^3)^3(-b^2)^6$
$+ 36(2a^3)^2(-b^2)^7 + 9(2a^3)(-b^2)^8 + (-b^2)^9 = 512a^{27} - 2304a^{24}b^2 + 4608a^{21}b^4 - 5376a^{18}b^6 + 4032a^{15}b^8$
$- 2016a^{12}b^{10} + 672a^9b^{12} - 144a^6b^{14} + 18a^3b^{16} - b^{18}$

24. $(x^{-2} + y^{-1})^4 = (x^{-2})^4 + 4(x^{-2})^3(y^{-1}) + 6(x^{-2})^2(y^{-1})^2 + 4(x^{-2})(y^{-1})^3 + (y^{-1})^4 = x^{-8} + 4x^{-6}y^{-1} + 6x^{-4}y^{-2} + 4x^{-2}y^{-3} + y^{-4}$

25. $\binom{11}{8}(1)^8(-2)^3 = -\dfrac{11!8}{8!3!} = -\dfrac{11 \cdot 10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = -1320$

26. $\binom{8}{2}(2)^2(1)^6 = \dfrac{8!4}{2!6!} = \dfrac{8 \cdot 7 \cdot 4}{2 \cdot 1} = 112$

27. $\{1, 2, 3, 4, 5, 6\}$

28. $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), \ldots, (6, 6)\}$

29. $\{13, 16, 31, 36, 61, 63\}$

30. {Defective, Nondefective}

31. {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

32. {HHT, HTH, THH, TTH, THT, HTT}

33. {HHH, TTT}

34. {HHH, HHT, HTH, THH}

For #35 and 36, substitute $n = 1, n = 2, \ldots, n = 6$, and $n = 40$.

35. 0, 1, 2, 3, 4, 5; 39

36. $-1, \frac{4}{3}, -2, \frac{16}{5}, -\frac{16}{3}, \frac{64}{7}$; $\approx 2.68 \times 10^{10}$

For #37–42, use previously computed values of the sequence to find the next term in the sequence.

37. $-1, 2, 5, 8, 11, 14$; 32

38. 5, 10, 20, 40, 80, 160; 10,240

39. $-5, -3.5, -2, -0.5, 1, 2.5$; 11.5

40. $3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$; $3^{-10} = \frac{1}{59{,}049}$

41. $-3, 1, -2, -1, -3, -4$; -76

42. $-3, 2, -1, 1, 0, 1$; 13

For #43–50, check for common difference or ratios between successive terms.

43. Arithmetic with $d = -2.5$;
$a_n = 12 + (-2.5)(n - 1) = 14.5 - 2.5n$

44. Arithmetic with $d = 4$;
$a_n = -5 + 4(n - 1) = 4n - 9$

45. Geometric with $r = 1.2$;
$a_n = 10 \cdot (1.2)^{n-1}$

46. Geometric with $r = -2$;
$a_n = \frac{1}{8} \cdot (-2)^{n-1} = -\frac{1}{16}(-2)^n$

47. Arithmetic with $d = 4.5$;
$a_n = -11 + 4.5(n - 1) = 4.5n - 15.5$

48. Geometric with $r = \frac{1}{4}$;
$b_n = 7 \cdot \left(\frac{1}{4}\right)^n = 28 \cdot \left(\frac{1}{4}\right)^n$

49. $a_n = a_1 r^{n-1}$, so $-192 = a_1 r^3$ and $196{,}608 = a_1 r^8$. Then $r^5 = -1024$, so $r = -4$, and $a_1 = \frac{-192}{(-4)^3} = 3$;
$a_n = 3(-4)^{n-1}$.

50. $a_n = a_1 + (n - 1)d$, so $14 = a_1 + 2d$, and $-3.5 = a_1 + 7d$. Then $5d = -17.5$, so $d = -3.5$, and $a_1 = 14 - 2(-3.5) = 21$;
$a_n = 21 - 3.5(n - 1) = 24.5 - 3.5n$.

For #51–54, use one of the formulas $S_n = n\left(\frac{a_1 + a_n}{2}\right)$ or $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$. In most cases, the first of these is easier (since the last term a_n is given); note that $n = \frac{a_n - a_1}{d} + 1$.

51. $8 \cdot \left(\frac{-11 + 10}{2}\right) = 4 \cdot (-1) = -4$

52. $7 \cdot \left(\frac{13 - 11}{2}\right) = 7$

53. $27 \cdot \left(\frac{2.5 - 75.5}{2}\right) = \frac{1}{2} \cdot 27 \cdot (-73) = -985.5$

54. $31 \cdot \left(\frac{-5 + 55}{2}\right) = 31 \cdot 25 = 775$

For #55–58, use the formula $S_n = \frac{a_1(1 - r^n)}{1 - r}$. Note that $n = 1 + \log_{|r|}\left|\frac{a_n}{a_1}\right| = 1 + \frac{\ln|a_n/a_1|}{\ln|r|}$.

55. $\frac{4(1 - (-1/2)^6)}{1 - (-1/2)} = \frac{21}{8}$

56. $\frac{-3(1 - (1/3)^5)}{1 - (1/3)} = -\frac{121}{27}$

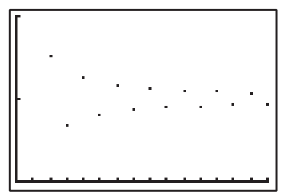
57. $\frac{2(1 - 3^{10})}{1 - 3} = 59{,}048$

58. $\frac{1(1 - (-2)^{14})}{1 - (-2)} = -5461$

59. Geometric with $r = \frac{1}{3}$:
$S_{10} = \frac{2187(1 - (1/3)^{10})}{1 - (1/3)} = \frac{29{,}524}{9} = 3280.\overline{4}$

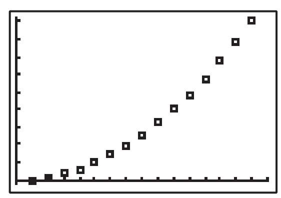
60. Arithmetic with $d = -3$:
$S_{10} = \frac{10}{2}[2(94) + 9(-3)] = 5 \cdot 161 = 805$

61.



[0, 15] by [0, 2]

62.



[0, 16] by [−10, 460]

63. With $a_1 = \$150$, $r = 1 + 0.08/12$, and $n = 120$, the sum becomes
$\frac{\$150\,[1 - (1 + 0.08/12)^{120}]}{1 - (1 + 0.08/12)} = \$27{,}441.91$.

64. The payment amount P must be such that
$$P\left(1 + \frac{0.08}{12}\right)^0 + P\left(1 + \frac{0.08}{12}\right)^1 + \ldots + P\left(1 + \frac{0.08}{12}\right)^{119} \geq \$30{,}000.$$
Using the formula for the sum of a finite geometric series,
$$\frac{P\,[1 - (1 + 0.08/12)^{120}]}{1 - (1 + 0.08/12)} \geq \$30{,}000$$
or $P \geq \$30{,}000 \frac{-0.08/12}{1 - (1 + 0.08/12)^{120}}$
$\approx \$163.983$
$\approx \$163.99$ rounded up.

65. Converges: geometric with $a_1 = \frac{3}{2}$ and $r = \frac{3}{4}$, so
$S = \frac{3/2}{1 - (3/4)} = \frac{3/2}{1/4} = 6.$

66. Converges: geometric with $a_1 = -\frac{2}{3}$ and $r = -\frac{1}{3}$, so $S = \frac{-2/3}{1-(-1/3)} = \frac{-2/3}{4/3} = -\frac{1}{2}$.

67. Diverges: geometric with $r = -\frac{4}{3}$

68. Diverges: geometric with $r = \frac{6}{5}$

69. Converges: geometric with $a_1 = 1.5$ and $r = 0.5$, so $S = \frac{1.5}{1-0.5} = \frac{1.5}{0.5} = 3$.

70. Diverges; geometric with $r = 1.2$

71. $\sum_{k=1}^{21}[-8+5(k-1)] = \sum_{k=1}^{21}(5k-13)$

72. $\sum_{k=1}^{10}4(-2)^{k-1} = \sum_{k=1}^{10}(-2)^{k+1}$

73. $\sum_{k=0}^{\infty}(2k+1)^2$ or $\sum_{k=1}^{\infty}(2k-1)^2$

74. $\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k$ or $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}$

75. $\sum_{k=1}^{n}(3k+1) = 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}1$
$= 3\cdot\frac{n(n+1)}{2} + n = \frac{3n^2+5n}{2} = \frac{n(3n+5)}{2}$

76. $\sum_{k=1}^{n}3k^2 = 3\sum_{k=1}^{n}k^2 = 3\cdot\frac{n(n+1)(2n+1)}{6}$
$= \frac{n(n+1)(2n+1)}{2}$

77. $\sum_{k=1}^{25}(k^2-3k+4) = \frac{25\cdot 26\cdot 51}{6} - 3\cdot\frac{25\cdot 26}{2}$
$+ 4\cdot 25 = 4650$

78. $\sum_{k=1}^{175}(3k^2-5k+1) = 3\cdot\frac{175\cdot 176\cdot 351}{6}$
$- 5\cdot\frac{175\cdot 176}{2} + 175 = 5{,}328{,}575$

79. P_n: $1+3+6+\cdots+\frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$.
P_1 is true: $\frac{1(1+1)}{2} = \frac{1(1+1)(1+2)}{6}$.
Now assume P_k is true: $1+3+6+\cdots+\frac{k(k+1)}{2}$
$= \frac{k(k+1)(k+2)}{6}$. Add $\frac{(k+1)(k+2)}{2}$ to both sides:
$1+3+6+\cdots+\frac{k(k+1)}{2}+\frac{(k+1)(k+2)}{2}$
$= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2}$
$= (k+1)(k+2)\left(\frac{k}{6}+\frac{1}{2}\right)$
$= (k+1)(k+2)\left(\frac{k+3}{6}\right)$
$= \frac{(k+1)((k+1)+1)((k+1)+2)}{6}$,
so P_{k+1} is true. Therefore, P_n is true for all $n \ge 1$.

80. P_n: $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1)$
$= \frac{n(n+1)(n+2)}{3}$. P_1 is true:
$1(1+1) = \frac{1(1+1)(1+2)}{3}$.
Now assume P_k is true: $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots$
$+ k(k+1)$
$= \frac{k(k+1)(k+2)}{3}$.
Add $(k+1)(k+2)$ to both sides:
$1\cdot 2 + 2\cdot 3 + \cdots + k(k+1) + (k+1)(k+2)$
$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$
$= (k+1)(k+2)\left(\frac{k}{3}+1\right)$
$= (k+1)(k+2)\left(\frac{k+3}{3}\right)$
$= \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$;
so P_k is true. Therefore, P_n is true for all $n \ge 1$.

81. P_n: $2^{n-1} \le n!$. P_1 is true: it says that $2^{1-1} \le 1!$ (they are equal). Now assume P_k is true: $2^{k-1} \le k!$. Then $2^{k+1-1} = 2\cdot 2^{k-1} \le 2\cdot k! \le$ $(k+1)k! = (k+1)!$, so P_{k+1} is true. Therefore, P_n is true for all $n \ge 1$.

82. P_n: n^3+2n is divisible by 3. P_1 is true because $1^3 + 2\cdot 1 = 3$ is divisible by 3. Now assume P_k is true: k^3+2k is divisible by 3. Then note that
$(k+1)^3 + 2(k+1) = (k^3+3k^2+3k+1)$
$+ (2k+2) = (k^3+2k) + 3(k^2+k+1)$.
Since both terms are divisible by 3, so is the sum, so P_{k+1} is true. Therefore, P_n is true for all $n \ge 1$.

83. 1 9 36 84 126 126 84 36 9 1

84. ${}_nP_k \times {}_{n-k}P_j = \frac{n!}{(n-k)!}\,\frac{(n-k)!}{[(n-k)-j]!}$
$= \frac{n!}{(n-k-j)!}$
$= \frac{n!}{[n-(k+j)]!} = {}_nP_{k+j}$

Chapter 9 Project

Answers are based on the sample data shown in the table.

1. (a) $\frac{308.7-248.7}{2010-1990} = 3.0$ million persons/year

(b) Since the year 2000 was 0 years after 2000, $p_0 = 281.4$.

(c) $p_n = p_{n-1} + 3$

(d) $p_n = 3n + 281.4$

(e) 2010: $p_n = 3(10) + 281.4 = 311.4$ million
2015: $p_n = 3(15) + 281.4 = 326.4$ million
2020: $p_n = 3(20) + 281.4 = 341.4$ million

2. (a) $w_n = w_{n-1}\cdot 1.0162$

(b) $w_n = 3.0\cdot 1.0162^n$

(c) 2010: $w_n = 3.0\cdot 1.0162^{51} \approx 6.81$ billion
2015: $w_n = 3.0\cdot 1.0162^{56} \approx 7.38$ billion
2020: $w_n = 3.0\cdot 1.0162^{61} \approx 8.00$ billion

(d) $w_n = \dfrac{11.511}{1 + 2.849e^{0.0281n}}$

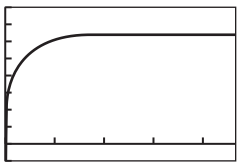
2010: $w_n = \dfrac{11.511}{1 + 2.849e^{0.0281(51)}} \approx 6.85$ billion

2015: $w_n = \dfrac{11.511}{1 + 2.849e^{0.0281(56)}} \approx 7.24$ billion

2020: $w_n = \dfrac{11.511}{1 + 2.849e^{0.0281(61)}} \approx 7.61$ billion

3. (a) $b_n = 0.75 \cdot b_{n-1} + 800$

(b)



[0, 47] by [–500, 4000]

(c) P_n: $b_n = 3200 - 2100(0.75)^n$

P_1 is true: $b_0 = 3200 - 2100(0.75)^0 = 3200 - 2100 = 1100$

P_k: $b_k = 3200 - 2100(0.75)^k$

P_{k+1}: Since $b_n = 0.75 \cdot b_{n-1} + 800$,

$b_{k+1} = 0.75\left(3200 - 2100(0.75)^k\right) + 800$

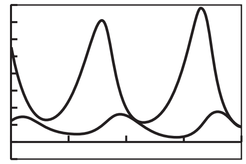
$b_{k+1} = 0.75(3200) - 2100(0.75)(0.75)^k + 800$

$b_{k+1} = 3200 - 2100(0.75)(0.75)^{k+1}$

So, P_{n+1} is true. Thus, P_n is true for all $n \geq 0$.

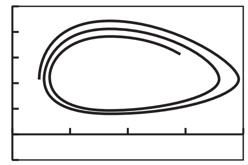
(d) Since $(0.75)^n$ approaches zero as n increases, the long-run population of blue gill is 3200.

4. (a)



[0, 400] by [–50, 400]

(b)



[0, 400] by [–20, 100]

(c) The populations move in cycles, with each rising and falling in turn. If the current trend continues, neither population would die out. If foxes die out, the rabbit population would begin to grow, since there would no longer be foxes to eat the rabbits. The population would grow exponentially until lack of resources moved the growth to a logistic model.