Chapter 7

Differential Equations and Mathematical Modeling

Section 7.1 Slope Fields and Euler's Method (pp. 329–339)

Exploration 1 Seeing the Slopes

1. Since $\frac{dy}{dx}=0$ represents a line with a slope of 0, we should expect to see horizontal slope lines. We see this at odd multiples of $\frac{\pi}{2}$.

2. The formula $\frac{dy}{dx}=\cos x$ depends only on *x*, not on *y*.

3. Yes; the curves are vertical translations of each other, so they all have the same slope at any given value of *x*.

4. At $x=0$, $\frac{dy}{dx}=\cos 0=1$, so the slope at 0 should be 1. That appears to be the slope of each curve as it crosses the *y*-axis.

5. At $x=\pi$, $\frac{dy}{dx}=\cos\pi=-1$, so the slope should be –1. That appears to be the slope of each curve at $x=\pi$.

6. Yes; the curves themselves are graphs of odd functions, but we see that the *slopes* at the points (x, y) and $(-x, -y)$ are the same.

Quick Review 7.1

1. Yes; $\frac{d}{dx}e^x=e^x$

2. Yes; $\frac{d}{dx}e^{4x}=4e^{4x}$

3. No; $\frac{d}{dx}(x^2e^x)=2xe^x+x^2e^x$

4. Yes; $\frac{d}{dx}e^{x^2}=2xe^{x^2}$

5. No; $\frac{d}{dx}(e^{x^2}+5)=2xe^{x^2}$

6. Yes; $\frac{d}{dx}\sqrt{2x}=\frac{1}{2\sqrt{2x}}(2)=\frac{1}{\sqrt{2x}}$

7. Yes; $\frac{d}{dx}\sec x=\sec x\tan x$

8. No; $\frac{d}{dx}x^{-1}=-x^{-2}$

9. $y=3x^2+4x+C$
   $2=3(1)^2+4(1)+C$
   $C=-5$

10. $y=2\ \sin x-3\ \cos x+C$
    $4=2\ \sin(0)-3\ \cos(0)+C$
    $C=7$

11. $y=e^{2x}+\sec x+C$
    $5=e^{2(0)}+\sec(0)+C$
    $C=3$

12. $y=\tan^{-1}x+\ln(2x-1)+C$
    $\pi=\tan^{-1}(1)+\ln(2(1)-1)+C$
    $C=\frac{3\pi}{4}$

Section 7.1 Exercises

1. $\int dy=\int(5x^4-\sec^2 x)\,dx$
   $y=x^5-\tan x+C$

2. $\int dy=\int(\sec x\tan x-e^x)\,dx$
   $y=\sec x-e^x+C$

3. $\int dy=\int(\sin x-e^{-x}+8x^3)\,dx$
   $y=-\cos x+e^{-x}+2x^4+C$

4. $\int dy=\int\left(\frac{1}{x}-\frac{1}{x^2}\right)dx=\ln x+\frac{1}{x}+C$

5. $\int dy=\int\left(5^x\ln 5+\frac{1}{x^2+1}\right)dx=5^x+\tan^{-1}x+C$

6. $\int dy = \int\left(\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{x}}\right)dx$

$= \sin^{-1} x - 2\sqrt{x} + C$

7. $\int dy = \int (3t\cos(t^3))\,dt = \sin(t^3) + C$

8. $\int dy = \int \cos t\, e^{\sin t}\,dt = e^{\sin t} + C$

9. $\int du = \int (\sec^2(x^5)(5x^4))\,dx = \tan(x^5) + C$

10. $\int dy = \int 4(\sin u)^3 \cos u\, du$

$= (\sin u)^4 + C$

$= \sin^4 u + C$

11. $\int dy = \int 3\sin x\, dx = -3\cos x + C$

$2 = -3\ \cos(0) + C,\ C = 5$

$y = -3\ \cos x + 5$

12. $\int dy = \int 2e^x - \cos x\, dx = 2e^x - \sin x + C$

$3 = 2e^0 - \sin(0) + C,\ C = 1$

$y = 2e^x - \sin x + 1$

13. $\int du = \int (7x^6 - 3x^2 + 5)\,dx = x^7 - x^3 + 5x + C$

$1 = 1^7 - 1^3 + 5 + C,\ C = -4$

$u = x^7 - x^3 + 5x - 4$

14. $\int dA = \int (10x^9 + 5x^4 - 2x + 4)\ dx$

$= x^{10} + x^5 - x^2 + 4x + C$

$6 = 1^{10} + 1^5 - 1^2 + 4(1) + C,\ C = 1$

$A = x^{10} + x^5 - x^2 + 4x + 1$

15. $\int dy = \int\left(-\frac{1}{x^2} - \frac{3}{x^4} + 12\right)\ dx$

$= x^{-1} + x^{-3} + 12x + C$

$3 = 1^{-1} + 1^{-3} + 12(1) + C,\ C = -11$

$y = x^{-1} + x^{-3} + 12x - 11\ \ (x > 0)$

16. $\int dy = \int\left(5\sec^2 x - \frac{3}{2}\sqrt{x}\right)\ dx$

$= 5\tan x - x^{3/2} + C$

$7 = 5\tan(0) - (0)^{3/2} + C,\ C = 7$

$y = 5\tan x - x^{3/2} + 7$

17. $\int dy = \int\left(\frac{1}{1+t^2} + 2^t \ln 2\right)dt = \tan^{-1} t + 2^t + C$

$3 = \tan^{-1}(0) + 2^0 + C,\ C = 2$

$y = \tan^{-1} t + 2^t + 2$

18. $\int dx = \int\left(\frac{1}{t} - \frac{1}{t^2} + 6\right)\ dt = \ln t + t^{-1} + 6t + C$

$0 = \ln(1) + 1^{-1} + 6(1) + C,\ C = -7$

$x = \ln t + t^{-1} + 6t - 7 \qquad (t > 0)$

19. $\int dv = \int (4\sec t\ \tan t + e^t + 6t)\ dt$

$= 4\sec t + e^t + 3t^2 + C$

$5 = 4\ \sec(0) + e^0 + 3(0)^2 + C,\ C = 0$

$v = 4\ \sec t + e^t + 3t^2$

$\left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)$

20. $\int ds = \int t(3t-2)\ dt = t^3 - t^2 + C$

$0 = (1)^3 - (1)^2 + C,\ C = 0$

$s = t^3 - t^2$

21. $\frac{dy}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\int_1^x \sin(t^2)\,dt$

$y = \int_1^x \sin(t^2)\ dt + 5$

22. $\frac{du}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\int_0^x \sqrt{2+\cos t}\,dt$

$u = \int_0^x \sqrt{2+\cos t}\ dt - 3$

23. $F'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = \frac{d}{dx}\int_2^x e^{\cos t}\,dt$

$F(x) = \int_2^x e^{\cos t}\,dt + 9$

24. $G'(s) = \frac{d}{ds}\int_a^s f(t)\,dt = \frac{d}{ds}\int_0^s \sqrt[3]{\tan t}\ dt$

$G(s) = \int_0^s \sqrt[3]{\tan t}\ dt + 4$

25. Graph (b).

$(\sin 0)^2 = 0$

$(\sin 1)^2 > 0$

$(\sin(-1))^2 > 0$

26. Graph (c).

$(\sin 0)^3 = 0$

$(\sin 1)^3 > 0$

$(\sin(-1))^3 < 0$

27. Graph (a).

$(\cos 0)^2 > 0$

$(\cos 1)^2 > 0$

$(\cos(-1))^2 > 0$

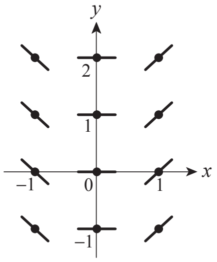
28. Graph (d).

$(\cos 0)^3 > 0$

$(\cos 1)^3 > 0$

$(\cos(-2))^3 < 0$

29.



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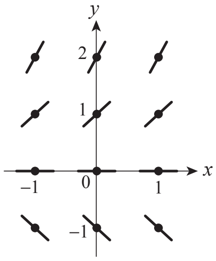

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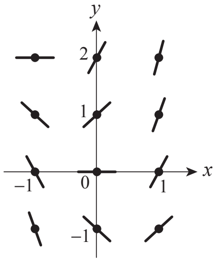

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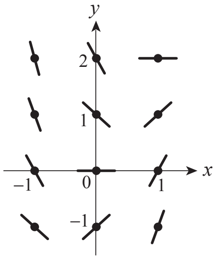

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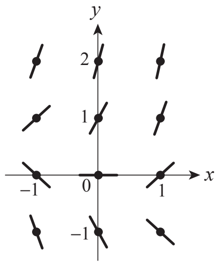

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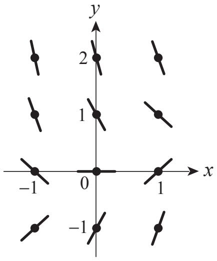
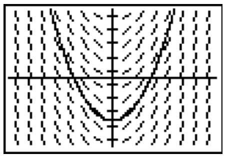
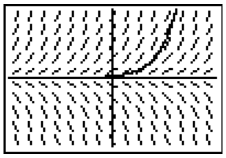


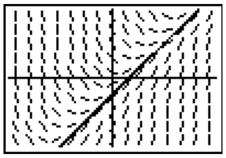
35. When $x = 0$, $\dfrac{dy}{dx} = 0$. The only graph satisfying this condition is graph (c).



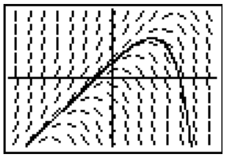
36. When $y > 0$, $\dfrac{dy}{dx} > 0$ and when $y < 0$, $\dfrac{dy}{dx} < 0$. The only graph satisfying these conditions is graph (e).



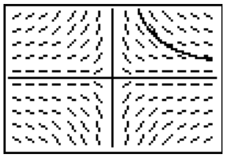
37. When $x > y$, $\dfrac{dy}{dx} > 0$ and when $x < y$, $\dfrac{dy}{dx} < 0$. The only graph satisfying these conditions is graph (a).



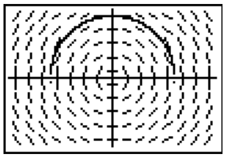
38. When $y > x$, $\frac{dy}{dx} > 0$ and when $y < x$, $\frac{dy}{dx} < 0$. The only graph satisfying these conditions is graph (d).



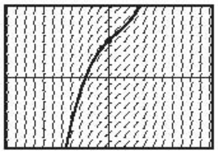
39. In quadrants I and III, $\frac{dy}{dx} < 0$ while in quadrants II and IV, $\frac{dy}{dx} > 0$. When $x = 0$, $\frac{dy}{dx}$ is undefined. The only graph satisfying these conditions is graph (b).



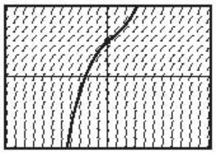
40. In quadrants I and III, $\frac{dy}{dx} < 0$ while in quadrants II and IV, $\frac{dy}{dx} > 0$. When $y = 0$, $\frac{dy}{dx}$ is undefined. The only graph satisfying these conditions is graph (f).



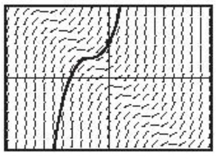
41. When $x = 0$ or $x = 1$, $\frac{dy}{dx} = 1$. The only graph satisfying this condition is graph (d).



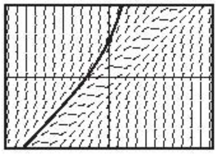
42. When $y = 0$, $\frac{dy}{dx} = \sqrt{5}$. The only graph satisfying this condition is graph (f).



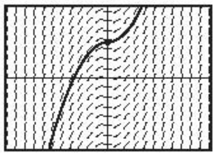
43. When $y = -x$, $\frac{dy}{dx} = 0$. The only graph satisfying this condition is graph (c).



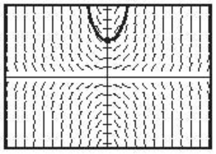
44. When $y = x$, $\frac{dy}{dx} = 0$. The only graph satisfying this condition is graph (e).



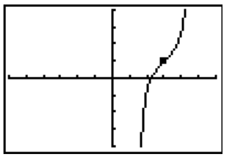
45. When $x = 0$, $\frac{dy}{dx} = 0$. When $x = 1$, $\frac{dy}{dx} = 1$. The only graph satisfying these conditions is graph (b).



46. When $x = 0$ or $y = 0$, $\frac{dy}{dx} = 0$. The only graph satisfying this condition is graph (a).

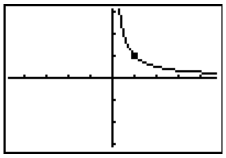


47. **(a)**



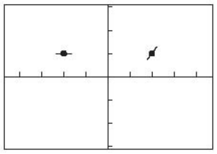
(b) The solution to the initial value problem includes only the continuous portion of the function $y = \tan x + 1$ that passes through the point $(\pi, 1)$.

48. **(a)**

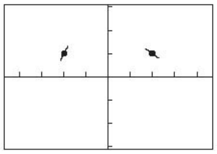


(b) The solution to the initial value problem includes only the continuous portion of the function $y = \frac{1}{x}$ that passes through the point (1, 1).

49. At (–2, 1), $\frac{dy}{dx} = 2(1) + (-2) = 0$ so the correct graph is (c). At (2, 1), $\frac{dy}{dx} = 2(1) + 2 = 4.$



50. At (2, 1), $\frac{dy}{dx} = (1)^2 - 2 = -1$ so the correct graph is (b). At (–2, 1), $\frac{dy}{dx} = (1)^2 - (-2) = 3.$



51.

(x, y)	$\frac{dy}{dx} = x - 1$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 2)	0.0	0.1	0	(1.1, 2)
(1.1, 2)	0.1	0.1	0.01	(1.2, 2.01)
(1.2, 2.01)	0.2	0.1	0.02	(1.3, 2.03)

$y = 2.03$

52.

(x, y)	$\frac{dy}{dx} = y - 1$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 3)	2.0	0.1	0.2	(1.1, 3.2)
(1.1, 3.2)	2.2	0.1	0.22	(1.2, 3.42)
(1.2, 3.42)	2.42	0.1	0.242	(1.3, 3.662)

$y = 3.662$

53.

(x, y)	$\frac{dy}{dx} = y - x$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 2)	1.0	0.1	0.1	(1.1, 2.1)
(1.1, 2.1)	1.0	0.1	0.1	(1.2, 2.2)
(1.2, 2.2)	1.0	0.1	0.1	(1.3, 2.3)

$y = 2.3$

54.

(x, y)	$\frac{dy}{dx} = 2x - y$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 0)	2.0	0.1	0.2	(1.1, 0.2)
(1.1, 0.2)	2.0	0.1	0.2	(1.2, 0.4)
(1.2, 0.4)	2.0	0.1	0.2	(1.3, 0.6)

$y = 0.6$

55.

(x, y)	$\frac{dy}{dx} = 2 - x$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 1)	0.0	–0.1	0.0	(1.9, 1)
(1.9, 1)	0.1	–0.1	–0.01	(1.8, 0.99)
(1.8, 0.99)	0.2	–0.1	–0.02	(1.7, 0.97)

$y = 0.97$

56.

(x, y)	$\frac{dy}{dx} = 1 + y$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 0)	1.0	–0.1	–0.1	(1.9, –0.1)
(1.9, –0.1)	0.9	–0.1	–0.09	(1.8, –0.19)
(1.8, –0.19)	0.81	–0.1	–0.081	(1.7, –0.271)

$y = -0.271$

57.

(x, y)	$\frac{dy}{dx} = x - y$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 2)	–0.0	–0.1	0	(1.9, 2.0)
(1.9, 2)	–0.1	–0.1	0.01	(1.8, 2.01)
(1.8, 2.01)	–0.21	–0.1	0.021	(1.7, 2.031)

$y = 2.031$

58.

(x, y)	$\frac{dy}{dx} = x - 2y$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 1)	0.0	–0.1	0.0	(1.9, 1.0)
(1.9, 1)	–0.1	–0.1	0.01	(1.8, 1.01)
(1.8, 1.01)	–0.22	–0.1	0.022	(1.7, 1.032)

$y = 1.032$

59. **(a)** Graph (b)

(b) The slope is always positive, so (a) and (c) can be ruled out.

60. **(a)** Graph (b)

(b) The solution should have positive slope when x is negative, zero slope when x is zero and negative slope when x is positive since slope $= \frac{dy}{dx} = -x$.

Graphs (a) and (c) don't show this slope pattern.

61. There are positive slopes in the second quadrant of the slope field. The graph of $y = x^2$ has negative slopes in the second quadrant.

62. The slope of $y = \sin x$ would be +1 at the origin, while the slope field shows a slope of zero at every point on the y-axis.

63.

(x, y)	$\frac{dy}{dx} = 2x+1$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 3)	3.0	0.1	0.3	(1.1, 3.3)
(1.1, 3.3)	3.2	0.1	0.32	(1.2, 3.62)
(1.2, 3.62)	3.4	0.1	0.34	(1.3, 3.96)
(1.3, 3.96)	3.6	0.1	0.36	(1.4, 4.32)

$y = 4.32$

Euler's Method gives an estimate $f(1.4) \approx 4.32$.

The solution to the initial value problem is $f(x) = x^2 + x + 1$, from which we get $f(1.4) = 4.36$. The percentage error is thus $\frac{4.36 - 4.32}{4.36} = 0.9\%$.

64.

(x, y)	$\frac{dy}{dx} = 2x-1$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 3)	3.0	–0.1	–0.3	(1.9, 2.7)
(1.9, 2.7)	2.8	–0.1	–0.28	(1.8, 2.42)
(1.8, 2.42)	2.6	–0.1	–0.26	(1.7, 2.16)
(1.7, 2.16)	2.4	–0.1	–0.24	(1.6, 1.92)

$y = 1.92$

Euler's Method gives an estimate $f(1.6) \approx 1.92$. The solution to the initial value problem is $f(x) = x^2 - x + 1$, from which we get $f(1.6) = 1.96$. The percentage error is thus $\frac{1.96 - 1.92}{1.96} = 2\%$.

65. At every (x, y), $(e^{(x-y)/2})(e^{(y-x)/2}) = -e^0 = -1$, so the slopes are negative reciprocals. The slope lines are therefore perpendicular.

66. Since the slopes must be negative reciprocals, $g(x) = -\frac{1}{\sec x} = -\cos x$.

67. The perpendicular slope field would be produced by $\frac{dy}{dx} = -\sin x$, so $y = \cos x + C$ for any constant C.

68. The perpendicular slope field would be produced by $\frac{dy}{dx} = -x$, so $y = -0.5x^2 + C$ for any constant C.

69. True; they are all lines of the form $y = 5x + C$.

70. False; for example, $f(x) = x^2$ is a solution of $\frac{dy}{dx} = 2x$, but $f^{-1}(x) = \sqrt{x}$ is not a solution of $\frac{dy}{dx} = 2y$.

71. C; for all points with $y = 42$, $m = 42 - 42 = 0$

72. E; $y < 0$, $x^2 > 0$, therefore $\frac{dy}{dx} < 0$.

73. B; $y(0) = e^{0^2} = 1$

$$\frac{dy}{dx} = 2xe^{x^2} = 2xy.$$

74. A

75. **(a)** $\frac{dy}{dx} = x - \frac{1}{x^2}$

$$\int \frac{dy}{dx}\,dx = \int (x - x^{-2})\,dx$$

$$y = \frac{x^2}{2} + x^{-1} + C = \frac{x^2}{2} + \frac{1}{x} + C$$

Initial condition: $y(1) = 2$

$$2 = \frac{1^2}{2} + \frac{1}{1} + C$$

$$2 = \frac{3}{2} + C$$

$$\frac{1}{2} = C$$

Solution: $y = \frac{x^2}{2} + \frac{1}{x} + \frac{1}{2},\ x > 0$

(b) Again, $y = \frac{x^2}{2} + \frac{1}{x} + C$.

Initial condition: $y(-1) = 1$

$$1 = \frac{(-1)^2}{2} + \frac{1}{(-1)} + C$$

$$1 = \frac{-1}{2} + C$$

$$\frac{3}{2} = C$$

Solution: $y = \frac{x^2}{2} + \frac{1}{x} + \frac{3}{2},\ x < 0$

(c) For $x < 0$, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x} + \frac{x^2}{2} + C_1\right)$

$$= -\frac{1}{x^2} + x$$

$$= x - \frac{1}{x^2}.$$

For $x > 0$, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x} + \frac{x^2}{2} + C_2\right)$

$$= -\frac{1}{x^2} + x$$

$$= x - \frac{1}{x^2}.$$

And for $x = 0$, $\frac{dy}{dx}$ is undefined.

(d) Let C_1 be the value from part (b), and let C_2 be the value from part (a). Thus, $C_1 = \frac{3}{2}$ and $C_2 = \frac{1}{2}$.

(e) $y(2) = -1$

$$-1 = \frac{1}{2} + \frac{2^2}{2} + C_2$$

$$-1 = \frac{5}{2} + C_2$$

$$-\frac{7}{2} = C_2$$

$y(-2) = 2$

$$2 = \frac{1}{(-2)} + \frac{(-2)^2}{2} + C_1$$

$$2 = \frac{3}{2} + C_1$$

$$\frac{1}{2} = C_1$$

Thus, $C_1 = \frac{1}{2}$ and $C_2 = -\frac{7}{2}$.

76. **(a)** $\frac{d}{dx}(\ln x + C) = \frac{1}{x}$ for $x > 0$

(b) $\frac{d}{dx}[\ln(-x) + C] = \frac{1}{-x}\frac{d}{dx}(-x)$

$$= \left(\frac{1}{-x}\right)(-1)$$

$$= \frac{1}{x}$$

for $x < 0$

(c) For $x > 0, \ln|x| + C = \ln x + C$, which is a solution to the differential equation, as we showed in part (a). For $x < 0$, $\ln|x| + C = \ln(-x) + C$, which is a solution to the differential equation, as we showed in part (b). Thus, $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all x except 0.

(d) For $x < 0$, we have $y = \ln(-x) + C_1$, which is a solution to the differential equation, as we showed in part (a). For $x > 0$, we have $y = \ln x + C_2$, which is a solution to the differential equation, as we showed part (b). Thus, $\frac{dy}{dx} = \frac{1}{x}$ for all x except 0.

77. **(a)**
$$y' = \int (12x+4)\,dx$$
$$y' = 6x^2 + 4x + C_1$$
$$y = \int (6x^2 + 4x + C_1)\,dx$$
$$y = 2x^3 + 2x^2 + C_1x + C_2$$

(b)
$$y' = \int (e^x + \sin x)\,dx$$
$$y' = e^x - \cos x + C_1$$
$$y = \int (e^x - \cos x + C_1)dx$$
$$y = e^x - \sin x + C_1x + C_2$$

(c)
$$y' = \int (x^3 + x^{-3})\,dx$$
$$y' = \frac{x^4}{4} - \frac{1}{2x^2} + C_1$$
$$y = \int \left(\frac{x^4}{4} - \frac{1}{2x^2} + C_1\right)dx$$
$$y = \frac{x^5}{20} + \frac{1}{2x} + C_1x + C_2$$

78. **(a)**
$$y' = \int (24x^2 - 10)dx$$
$$y' = 8x^3 - 10x + C$$
$$3 = 8(1)^3 - 10(1) + C$$
$$C = 5$$
$$y = \int (8x^3 - 10x + 5)dx$$
$$y = 2x^4 - 5x^2 + 5x + C$$
$$5 = 2(1)^4 - 5(1)^2 + 5(1) + C$$
$$C = 3$$
$$y = 2x^4 - 5x^2 + 5x + 3$$

(b)
$$y' = \int (\cos x - \sin x)dx$$
$$y' = \sin x + \cos x + C$$
$$2 = \sin 0 + \cos 0 + C$$
$$C = 1$$
$$y = \int (\sin x + \cos x + 1)\,dx$$
$$y = -\cos x + \sin x + x + C$$
$$0 = -\cos 0 + \sin 0 + 0 + C$$
$$C = 1$$
$$y = -\cos x + \sin x + x + 1$$

(c)
$$y' = \int (e^x - x)\,dx$$
$$y' = e^x - \frac{x^2}{2} + C$$
$$0 = e^0 - \frac{0^2}{2} + C$$
$$C = -1$$
$$y = \int \left(e^x - \frac{x^2}{2} - 1\right)dx$$
$$y = e^x - \frac{x^3}{6} - x + C$$
$$1 = e^0 - \frac{0^3}{6} - 0 + C$$
$$C = 0$$
$$y = e^x - \frac{x^3}{6} - x$$

79. **(a)**
$$y' = x$$
$$y = \int x\,dx = \frac{x^2}{2} + C$$

(b)
$$y' = -x$$
$$y = \int (-x)\,dx = -\frac{x^2}{2} + C$$

(c)
$$y' = y$$
$$\frac{d}{dx}(Ce^x) = Ce^x$$
$$y = Ce^x$$

(d)
$$y' = -y$$
$$\frac{d}{dx}(Ce^{-x}) = -Ce^{-x}$$
$$y = Ce^{-x}$$

(e) $y' = xy$

$$\frac{d}{dx}\left(Ce^{x^2/2}\right) = Cxe^{x^2/2}$$

$$y = Ce^{x^2/2}$$

80. (a) $y'' = x$

$$y' = \int x\,dx = \frac{x^2}{2} + C_1$$

$$y = \int\left(\frac{x^2}{2} + C_1\right)dx = \frac{x^3}{6} + C_1x + C_2$$

(b) $y'' = -x$

$$y' = \int(-x)\,dx = -\frac{x^2}{2} + C_1$$

$$y = \int\left(-\frac{x^2}{2} + C_1\right)dx = -\frac{x^3}{6} + C_1x + C_2$$

(c) $y'' = -\sin x$

$$y' = \int(-\sin x)dx = \cos x + C_1$$

$$y = \int(\cos x + C_1)\,dx = \sin x + C_1x + C_2$$

(d) $y'' = y$

$$\frac{d}{dx}(C_1e^x + C_2e^{-x}) = C_1e^x - C_2e^{-x} = y'$$

$$\frac{d}{dx}(C_1e^x - C_2e^{-x}) = C_1e^x + C_2e^{-x} = y''$$

$$y = C_1e^x + C_2e^{-x}$$

(e) $y'' = -y$

$$\frac{d}{dx}(C_1\sin x + C_2\cos x)$$
$$= C_1\cos x - C_2\sin x$$
$$= y'$$

$$\frac{d}{dx}\left(C_1\cos x - C_2\sin x\right)$$
$$= -C_1\sin x - C_2\cos x$$

$$y = C_1\sin x + C_2\cos x$$

Section 7.2 Antidifferentiation by Substitution (pp. 340–348)

Exploration 1 Are $\int f(u)\,du$ and $\int f(u)\,dx$ the Same Thing?

1. $\int f(u)\,du = \int u^3du = \frac{u^4}{4} + C$

2. $\frac{u^4}{4} + C = \frac{(x^2)^4}{4} + C = \frac{x^8}{4} + C$

3. $f(u) = u^3 = (x^2)^3 = x^6$

$$\int f(u)\,dx = \int x^6dx = \frac{x^7}{7} + C$$

4. No

Quick Review 7.2

1. $\int_0^2 x^4dx = \frac{1}{5}x^5\Big|_0^2 = \frac{1}{5}(2)^5 - \frac{1}{5}(0)^5 = \frac{32}{5}$

2. $\int_1^5 \sqrt{x-1}\,dx = \int_1^5 (x-1)^{1/2}dx$

$$= \frac{2}{3}(x-1)^{3/2}\Big|_1^5$$
$$= \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2}$$
$$= \frac{2}{3}(8) = \frac{16}{3}$$

3. $\frac{dy}{dx} = 3^x$

4. $\frac{dy}{dx} = 3^x$

5. $\frac{dy}{dx} = 4(x^3 - 2x^2 + 3)^3(3x^2 - 4x)$

6. $\frac{dy}{dx} = 2\sin(4x-5)\cos(4x-5)\cdot 4$

$$= 8\sin(4x-5)\cos(4x-5)$$

7. $\frac{dy}{dx} = \frac{1}{\cos x}\cdot -\sin x = -\tan x$

8. $\frac{dy}{dx} = \frac{1}{\sin x}\cdot\cos x = \cot x$

9. $\frac{dy}{dx} = \frac{1}{\sec x + \tan x}\cdot(\sec x\tan x + \sec^2 x)$

$$= \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$$
$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$
$$= \sec x$$

10. $$\frac{dy}{dx} = \frac{1}{\csc x + \cot x}(-\csc x \cot x - \csc^2 x)$$
$$= -\frac{\csc x \cot x + \csc^2 x}{\csc x + \cot x}$$
$$= -\frac{\csc x(\cot x + \csc x)}{\csc x + \cot x}$$
$$= -\csc x$$

Section 7.2 Exercises

1. $\int (\cos x - 3x^2)\, dx = \sin x - x^3 + C$

2. $\int x^{-2} dx = -x^{-1} + C$

3. $\int \left(t^2 - \frac{1}{t^2}\right) dt = \frac{t^3}{3} + t^{-1} + C$

4. $\int \frac{dt}{t^2 + 1} = \tan^{-1} t + C$

5. $$\int (3x^4 - 2x^{-3} + \sec^2 x)\, dx$$
$$= \frac{3}{5}x^5 + x^{-2} + \tan x + C$$

6. $$\int \left(2e^x + \sec x \tan x - \sqrt{x}\right) dx$$
$$= 2e^x + \sec x - \frac{2}{3}x^{3/2} + C$$

7. $(-\cot u + C)' = -(-\csc^2 u) = \csc^2 u$

8. $(-\csc u + C)' = -(-\csc u \cot u) = \csc u \cot u$

9. $\left(\frac{1}{2}e^{2x} + C\right)' = \frac{1}{2}e^{2x}(2) = e^{2x}$

10. $\left(\frac{1}{\ln 5}5^x + C\right)' = \frac{1}{\ln 5}\, 5^x(\ln 5) = 5^x$

11. $(\tan^{-1} u + C)' = \frac{1}{1 + u^2}$

12. $(\sin^{-1} u + C)' = \frac{1}{\sqrt{1 - u^2}}$

13. $$\int f(u)\, du = \int \sqrt{u}\, du$$
$$= \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{3}x^3 + C$$
$$\int f(u)\, dx = \int \sqrt{u}\, dx$$
$$= \int \sqrt{x^2} dx$$
$$= \int x\, dx$$
$$= \frac{1}{2}x^2 + C$$

14. $$\int f(u)\, du = \int u^2\, du = \frac{1}{3}\, u^3 + C = \frac{1}{3}x^{15} + C$$
$$\int f(u)\, dx = \int u^2\, dx = \int x^{10}\, dx = \frac{1}{11}x^{11} + C$$

15. $$\int f(u)\, du = \int e^u\, du = e^u + C = e^{7x} + C$$
$$\int f(u)\, du = \int e^u dx = \int e^{7x} dx = \frac{1}{7}e^{7x} + C$$

16. $$\int f(u)\, du = \int \sin u\, du$$
$$= -\cos u + C$$
$$= -\cos 4x + C$$
$$\int f(u)\, dx = \int \sin u\, dx$$
$$= \int \sin 4x\, dx$$
$$= -\frac{1}{4}\cos 4x + C$$

17. $$u = 3x$$
$$du = 3\, dx$$
$$\frac{1}{3}du = dx$$
$$\int \sin 3x\, dx = \frac{1}{3}\int \sin u\, du$$
$$= -\frac{1}{3}\cos u + C$$
$$= -\frac{1}{3}\cos 3x + C$$

Check:
$$\frac{d}{dx}\left(-\frac{1}{3}\cos 3x + C\right) = -\frac{1}{3}(-\sin 3x)(3) = \sin 3x$$

18. $u = 2x^2$

$du = 4x\ dx$

$x\ dx = \frac{1}{4} du$

$$\int x\cos(2x^2)\ dx = \frac{1}{4}\int \cos u\ du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(2x^2) + C$$

Check: $\frac{d}{dx}\left(\frac{1}{4}\sin(2x^2) + C\right) = \frac{1}{4}\cos(2x^2)(4x) = x\cos(2x^2)$

19. $u = 2x$

$du = 2\ dx$

$\frac{1}{2} du = dx$

$$\int \sec 2x \tan 2x\ dx = \frac{1}{2}\int \sec u\ \tan u\ du = \frac{1}{2}\sec u + C = \frac{1}{2}\sec 2x + C$$

Check: $\frac{d}{dx}\left(\frac{1}{2}\sec 2x + C\right) = \frac{1}{2}\sec 2x \tan 2x \cdot 2 = \sec 2x\ \tan 2x$

20. $u = 7x - 2$

$du = 7dx$

$\frac{1}{7} du = dx$

$$\int 28(7x-2)^3 dx = \frac{1}{7}\int 28u^3 du = u^4 + C = (7x-2)^4 + C$$

Check: $\frac{d}{dx}\left[(7x-2)^4 + C\right] = 4(7x-2)^3(7) = 28(7x-2)^3$

21. $u = \frac{x}{3}$ $\qquad x = 3u$

$du = \frac{1}{3}dx$ $\qquad x^2 = 9u^2$

$3\ du = dx$

$$\int \frac{dx}{x^2+9} = \int \frac{3du}{9u^2+9} = \frac{3}{9}\int \frac{du}{u^2+1} = \frac{1}{3}\int \frac{du}{u^2+1} = \frac{1}{3}\tan^{-1} u + C = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

Check:

$$\frac{d}{dx}\left(\frac{1}{3}\tan^{-1}\frac{x}{3} + C\right) = \frac{1}{3}\frac{1}{1+\left(\frac{x}{3}\right)^2}\cdot\frac{1}{3} = \frac{1}{9+x^2}$$

22. $u = 1 - r^3$

$du = -3r^2\ dr$

$-\frac{1}{3}du = r^2 dr$

$$\int \frac{9r^2 dr}{\sqrt{1-r^3}} = 9\left(-\frac{1}{3}\right)\int \frac{du}{\sqrt{u}} = -3\int u^{-1/2} du = -3(2)u^{1/2} + C = -6\sqrt{1-r^3} + C$$

Check:

$$\frac{d}{dx}\left(-6\sqrt{1-r^3} + C\right) = -6\left(\frac{1}{2\sqrt{1-r^3}}\right)(-3r^2) = \frac{9r^2}{\sqrt{1-r^3}}$$

23. $u = 1 - \cos\frac{t}{2}$

$du = \frac{1}{2}\sin\frac{t}{2}dt$

$2\ du = \sin\frac{t}{2}dt$

$$\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}dt = 2\int u^2 du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(1-\cos\frac{t}{2}\right)^3 + C$$

Check: $\frac{d}{dx}\left[\frac{2}{3}\left(1-\cos\frac{t}{2}\right)^3+C\right]$

$$=2\left(1-\cos\frac{t}{2}\right)^2\left(\sin\frac{t}{2}\right)\left(\frac{1}{2}\right)$$

$$=\left(1-\cos\frac{t}{2}\right)^2\sin\frac{t}{2}$$

24. $u=y^4+4y^2+1$

$du=(4y^3+8y)\,dy$

$du=4(y^3+2y)\,dy$

$\frac{1}{4}du=(y^3+2y)\,dy$

$\int 8(y^4+4y^2+1)^2(y^3+2y)dy$

$=8\left(\frac{1}{4}\right)\int u^2du$

$=\frac{2}{3}u^3+C$

$=\frac{2}{3}(y^4+4y^2+1)^3+C$

Check: $\frac{d}{dx}\left[\frac{2}{3}(y^4+4y^2+1)^3+C\right]$

$=2(y^4+4y^2+1)^2(4y^3+8y)$

$=8(y^4+4y^2+1)^2(y^3+2y)$

25. Let $u=1-x$

$du=-dx$

$\int\frac{dx}{(1-x)^2}=-\int\frac{du}{u^2}$

$=u^{-1}+C$

$=\frac{1}{1-x}+C$

26. Let $u=x+2$

$du=dx$

$\int\sec^2(x+2)\,dx=\int\sec^2u\,du$

$=\tan u+C$

$=\tan(x+2)+C$

27. Let $u=\tan x$

$du=\sec^2 x\,dx$

$\int\sqrt{\tan x}\sec^2 x\,dx=\int u^{1/2}du$

$=\frac{2}{3}u^{3/2}+C$

$=\frac{2}{3}(\tan x)^{3/2}+C$

28. Let $u=\theta+\frac{\pi}{2}$

$du=d\theta$

$\int\sec\left(\theta+\frac{\pi}{2}\right)\tan\left(\theta+\frac{\pi}{2}\right)d\theta$

$=\int\sec u\tan u\,du$

$=\sec u+C$

$=\sec\left(\theta+\frac{\pi}{2}\right)+C$

29. $\int\tan(4x+2)\,dx$

$u=4x+2$

$du=4\,dx$

$\frac{1}{4}du=dx$

$\frac{1}{4}\int\tan u\,du$

$=-\frac{1}{4}\ln\left|\cos(4x+2)\right|+C$ or

$\frac{1}{4}\ln\left|\sec(4x+2)\right|+C$

30. $\int 3(\sin x)^{-2}dx=3\int\frac{1}{\sin^2 x}dx$

$=3\int\csc^2 x\,dx$

$=-3\cot x+C$

31. Let $u=3z+4$

$du=3\,dz$

$\frac{1}{3}du=dz$

$\int\cos(3z+4)dz=\frac{1}{3}\int\cos u\,du$

$=\frac{1}{3}\sin u+C$

$=\frac{1}{3}\sin(3z+4)+C$

32. Let $u=\cot x$

$du=-\csc^2 x\,dx$

$\int\sqrt{\cot x}\csc^2 x\,dx=-\int u^{1/2}du$

$=-\frac{2}{3}u^{3/2}+C$

$=-\frac{2}{3}(\cot x)^{3/2}+C$

33. Let $u = \ln x$

$$du = \frac{1}{x}dx$$

$$\begin{aligned}\int \frac{\ln^6 x}{x}\,dx &= \int u^6 du \\ &= \frac{1}{7}u^7 + C \\ &= \frac{1}{7}(\ln^7 x) + C\end{aligned}$$

34. Let $u = \tan\left(\frac{x}{2}\right)$

$$du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$$

$$\begin{aligned}\int \tan^7\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx &= 2\int u^7 du \\ &= 2 \bullet \frac{1}{8}u^8 + C \\ &= \frac{1}{4}\tan^8\left(\frac{x}{2}\right) + C\end{aligned}$$

35. Let $u = s^{4/3} - 8$

$$du = \frac{4}{3}s^{1/3}ds$$

$$\frac{3}{4}du = s^{1/3}ds$$

$$\begin{aligned}\int s^{1/3}\cos(s^{4/3} - 8)\,ds &= \frac{3}{4}\int \cos u\,du \\ &= \frac{3}{4}\sin u + C \\ &= \frac{3}{4}\sin(s^{4/3} - 8) + C\end{aligned}$$

36. $\displaystyle\int \frac{dx}{\sin^2 3x} = \int \csc^2 3x\,dx$

Let $u = 3x$

$$du = 3\,dx$$

$$\frac{1}{3}du = dx$$

$$\begin{aligned}\int \csc^2 3x\,dx &= \frac{1}{3}\int \csc^2 u\,du \\ &= -\frac{1}{3}\cot u + C \\ &= -\frac{1}{3}\cot(3x) + C\end{aligned}$$

37. Let $u = \cos(2t + 1)$

$$du = -\sin(2t+1)(2)dt$$

$$-\frac{1}{2}du = \sin(2t+1)dt$$

$$\begin{aligned}\int \frac{\sin(2t+1)}{\cos^2(2t+1)}dt &= -\frac{1}{2}\int u^{-2}du \\ &= \frac{1}{2}u^{-1} + C \\ &= \frac{1}{2\cos(2t+1)} + C \\ &= \frac{1}{2}\sec(2t+1) + C\end{aligned}$$

38. Let $u = 2 + \sin t$

$$du = \cos t\;dt$$

$$\begin{aligned}\int \frac{6\cos t}{(2+\sin t)^2}dt &= 6\int u^{-2}du \\ &= -6u^{-1} + C \\ &= -\frac{6}{2+\sin t} + C\end{aligned}$$

39. $\displaystyle\int \frac{dx}{x\ln x}$

$u = \ln x$

$$du = \frac{dx}{x}$$

$$x\,du = dx$$

$$\int \frac{du}{u} = \ln\,u = \ln(\ln x) + C$$

40. $\displaystyle\int \tan^2 x\sec^2 x\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$$\int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$$

41. $\displaystyle\int \frac{x\,dx}{x^2+1}$

$u = x^2 + 1$

$$du = 2x\,dx$$

$$\frac{1}{2}du = x\,dx$$

$$\frac{1}{2}\int \frac{du}{x^2+1} = \frac{1}{2}\ln\,u + C = \frac{1}{2}\ln(x^2+1) + C$$

42. Let $u = \frac{x}{5}$ $\quad 5u = x$

$du = \frac{1}{5}dx \quad 25u^2 = x^2$

$5du = dx$

$$\int \frac{40\,dx}{x^2+25} = \int \frac{200\,du}{25u^2+5}$$
$$= \frac{200}{25}\int \frac{du}{u^2+1}$$
$$= 8\tan^{-1}u + C$$
$$= 8\tan^{-1}\left(\frac{x}{5}\right) + C$$

43. $\int \frac{dx}{\cot 3x} = \int \frac{\sin 3x}{\cos 3x}dx$

Let $u = \cos 3x$

$du = -3\sin 3x\,dx$

$-\frac{1}{3}du = \sin 3x\,dx$

$$\int \frac{dx}{\cot 3x} = -\frac{1}{3}\int \frac{1}{u}du$$
$$= -\frac{1}{3}\ln|u| + C$$
$$= -\frac{1}{3}\ln|\cos 3x| + C$$

(An equivalent expression is

$\frac{1}{3}\ln\ |\sec 3x| + C$.)

44. Let $u = 5x + 8$

$du = 5\,dx$

$\frac{1}{5}du = dx$

$$\int \frac{dx}{\sqrt{5x+8}} = \frac{1}{5}\int u^{-1/2}\,du$$
$$= \frac{1}{5}\cdot 2u^{1/2} + C$$
$$= \frac{2}{5}\sqrt{5x+8} + C$$

45. $\int \sec x\,dx = \int \sec x \cdot \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

Let $u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x\,dx$

$$\int \sec x\,dx = \int \frac{1}{u}du$$
$$= \ln|u| + C$$
$$= \ln|\sec x + \tan x| + C$$

46. $\int \csc x\,dx = \int \csc x\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right)dx$

$$= \int \frac{\csc^2 x + \csc x\cot x}{\csc x + \cot x}\,dx$$

Let $u = \csc x + \cot x$

$du = -\csc x\cot x - \csc^2 x\,dx$

$$\int \csc x\,dx = -\int \frac{1}{u}du$$
$$= -\ln|u| + C$$
$$= -\ln|\csc x + \cot x| + C$$

47. $\int \sin^3 2x\,dx = \int (\sin^2 2x)\cdot \sin 2x\,dx$

$$= \int (1-\cos^2 2x)\cdot \sin 2x\,dx$$

Let $u = \cos 2x$

$du = -2\sin 2x\,dx$

$-\frac{1}{2}du = \sin 2x\,dx$

$$= -\frac{1}{2}\int (1-u^2)du$$
$$= -\frac{1}{2}\left(u - \frac{u^3}{3}\right) + C$$
$$= -\frac{u}{2} + \frac{u^3}{6} + C$$
$$= -\frac{\cos 2x}{2} + \frac{\cos^3 2x}{6} + C$$

48. $\int \sec^4 x\,dx = \int (\sec^2 x)\sec^2 x\,dx$

$$= \int (1+\tan^2 x)\sec^2 x\,dx$$

Let $u = \tan x$

$du = \sec^2 x\,dx$

$$= \int (1+u^2)du$$
$$= u + \frac{u^3}{3} + C$$
$$= \tan x + \frac{\tan^3 x}{3} + C$$

49. $\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\int 2\sin^2 x\, dx = \int (1 - \cos 2x)\, dx$

Let $u = 2x$

$du = 2\, dx$

$\frac{1}{2} du = dx$

$= \frac{1}{2}\int (1 - \cos u)\, du$

$= \frac{1}{2}(u - \sin u) + C$

$= \frac{1}{2}(2x - \sin 2x) + C$

$= x - \frac{\sin 2x}{2} + C$

50. $\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\int 4\cos^2 x\, dx = \int 2(1 + \cos 2x)\, dx$

Let $u = 2x$

$du = 2\, dx$

$= \int (1 + \cos u)\, du$

$= u + \sin u + C$

$= 2x + \sin 2x + C$

51. $\int \tan^4 x\, dx$

$= \int \tan^2 x \cdot \tan^2 x\, dx$

$= \int \tan^2 x\,(\sec^2 x - 1)\, dx$

$= \int (\tan^2 x \sec^2 x - \tan^2 x)\, dx$

$= \int (\tan^2 x \sec^2 x - \sec^2 x + 1)\, dx$

$= \int (\tan^2 x \sec^2 x - \sec^2 x)\, dx + \int 1\, dx$

$= \int (\tan^2 x - 1)\sec^2 x\, dx + \int dx$

Let $u = \tan x$

$du = \sec^2 x\, dx$

$= \int (u^2 - 1)\, du + \int dx$

$= \frac{1}{3}u^3 - u + x + C$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$

52. $\int (\cos^4 x - \sin^4 x)\, dx$

$= \int (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)\, dx$

$= \int (1)(\cos 2x)\, dx$

$= \frac{1}{2}\sin 2x + C$

53. Let $u = y + 1$ $\quad u(0) = 0 + 1 = 1$

$du = dy$ $\quad u(3) = 3 + 1 = 4$

$\int_0^3 \sqrt{y+1}\, dy = \int_1^4 u^{1/2} du$

$= \frac{2}{3}u^{3/2}\Big|_1^4$

$= \frac{2}{3}(4)^{3/2} - \frac{2}{3}(1)^{3/2}$

$= \frac{2}{8}(8) - \frac{2}{3}$

$= \frac{14}{3}$

54. Let $u = 1 - r^2$ $\quad u(0) = 1 - 0^2 = 1$

$du = -2r\, dr$ $\quad u(1) = 1 - 1^2 = 0$

$-\frac{1}{2} du = r\ dr$

$\int_0^1 r\sqrt{1-r^2}\, dr = -\frac{1}{2}\int_1^0 u^{1/2}\, du = -\frac{1}{3}u^{3/2}\Big|_1^0$

$= -\frac{1}{3}0^{3/2} + \frac{1}{3}\cdot 1^{3/2}$

$= \frac{1}{3}$

55. Let $u = \tan x,\ u\left(-\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right) = -1,$

$u(0) = \tan(0) = 0$

$du = \sec^2 x\, dx$

$\int_{-\pi/4}^{0} \tan x \sec^2 x\, dx = \int_{-1}^{0} u\ du$

$= \frac{1}{2}u^2\Big|_{-1}^{0}$

$= \frac{1}{2}(0) - \frac{1}{2}(-1)^2$

$= -\frac{1}{2}$

56. Let $u = 4 + r^2$ $\quad u(1) = 4 + 1^2 = 5$

$du = 2r\,dr$ $\quad u(-1) = 4 + (-1)^2 = 5$

$\frac{1}{2}du = r\,dr$

$$\int_{-1}^{1} \frac{5r}{(4+r^2)^2}\,dr = \frac{5}{2}\int_{5}^{5} u^{-2}\,du = 0$$

57. Let $u = 1 + \theta^{3/2}$ $\quad u(0) = 1 + 0 = 1$

$du = \frac{3}{2}\theta^{1/2}\,d\theta$ $\quad u(0) = 1 + 1 = 2$

$\frac{2}{3}du = \theta^{1/2}\,d\theta$

$$\begin{aligned}\int_0^1 \frac{10\sqrt{\theta}}{(1+\theta^{3/2})^2}\,d\theta &= \frac{2}{3}(10)\int_1^2 u^{-2}\,du\\ &= -\frac{20}{3}u^{-1}\Big|_1^2\\ &= -\frac{20}{3}\left(\frac{1}{2}-1\right)\\ &= -\frac{20}{3}\left(-\frac{1}{2}\right)\\ &= \frac{10}{3}\end{aligned}$$

58. Let $u = 4 + 3\sin x$

$du = 3\cos x\,dx$

$u(-\pi) = 4 + 3\sin(-\pi) = 4$

$u(\pi) = 4 + 3\sin\pi = 4$

$\frac{1}{3}du = \cos x\;dx$

$$\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx = \frac{1}{3}\int_4^4 u^{-1/2}\,du = 0$$

59. Let $u = t^5 + 2t$ $\quad u(0) = 0 + 0 = 0$

$du = (5t^4 + 2)\,dt$ $\quad u(1) = 1 + 2 = 3$

$$\begin{aligned}\int_0^1 \sqrt{t^5+2t}\;(5t^4+2)\,dt &= \int_0^3 u^{1/2}\,du\\ &= \frac{2}{3}u^{3/2}\Big|_0^3\\ &= \frac{2}{3}(3)^{3/2}\\ &= \frac{2}{3}\sqrt{27}\\ &= 2\sqrt{3}\end{aligned}$$

60. Let $u = \cos 2\theta$ $\quad u(0) = \cos 0 = 1$

$du = -2\sin 2\theta\;d\theta$ $\quad u\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

$-\frac{1}{2}du = \sin 2\theta\;d\theta$

$$\begin{aligned}&\int_0^{\pi/6} \cos^{-3}\,2\theta\sin 2\theta\;d\theta\\ &= -\frac{1}{2}\int_1^{1/2} u^{-3}\,du\\ &= -\frac{1}{2}\cdot\left(-\frac{1}{2}\right)u^{-2}\Big|_1^{1/2}\\ &= \frac{1}{4}\left(\left(\frac{1}{2}\right)^{-2}-1\right)\\ &= \frac{1}{4}(3)\\ &= \frac{3}{4}\end{aligned}$$

61. $\int_0^7 \frac{dx}{x+2}$

$u = x + 2$ $\quad u(0) = 0 + 2 = 2$

$du = dx$ $\quad u(7) = 7 + 2 = 9$

$$\int_2^9 \frac{du}{u} = \ln|u|\Big|_2^9 = \ln\left(\frac{9}{2}\right)$$

62. $\int_2^5 \frac{dx}{2x-3}$

$u = 2x - 3$ $\quad u(2) = 4 - 3 = 1$

$du = 2\,dx$ $\quad u(5) = 10 - 3 = 7$

$\frac{1}{2}du = dx$

$$\frac{1}{2}\int_1^7 \frac{du}{u} = \frac{1}{2}\ln|u|\Big|_1^7 = \frac{1}{2}\ln 7 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 7$$

63. $\int_1^2 \frac{dt}{t-3}$

$u = t - 3$ $\quad u(1) = 1 - 3 = -2$

$du = dt$ $\quad u(2) = 2 - 3 = -1$

$$\begin{aligned}\int_{-2}^{-1} \frac{du}{u} &= \ln|u|\Big|_{-2}^{-1}\\ &= \ln|-1| - \ln|-2|\\ &= \ln 1 - \ln 2\\ &= 0 - \ln 2\\ &= -\ln 2\end{aligned}$$

64. $\int_{\pi/4}^{3\pi/4} \cot x \; dx = \int_{\pi/4}^{3\pi/4} \frac{\cos x \, dx}{\sin x}$

$u = \sin x \qquad u\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$du = \cos x \, dx \qquad u\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u} = 0$

65. $\int_{-1}^{3} \frac{x \, dx}{x^2+1}$

$u = x^2+1 \qquad u(-1) = (-1)^2+1 = 2$

$du = 2x \, dx \qquad u(3) = 3^2+1 = 10$

$\frac{1}{2} du = x \, dx$

$$\begin{aligned}\frac{1}{2}\int_{2}^{10} \frac{du}{u} &= \frac{1}{2} \ln|u|\Big|_2^{10} \\ &= \frac{1}{2}(\ln 10 - \ln 2) \\ &= \frac{1}{2} \ln 5\end{aligned}$$

66. $\int_{0}^{2} \frac{e^x dx}{3+e^x}$

$u = 3+e^x \qquad u(0) = 3+e^0 = 3+1 = 4$

$du = e^x \, dx \qquad u(2) = 3+e^2$

$$\begin{aligned}\int_{4}^{3+e^2} \frac{du}{u} &= \ln|u|\Big|_4^{3+e^2} \\ &= \ln(3+e^2) - \ln 4 \\ &= \ln\frac{3+e^2}{4}\end{aligned}$$

67. Let $u = x^4+9, du = 4x^3 \; dx.$

$u(0) = 0+9 = 9, \qquad u(1) = 1+9 = 10$

(a) $$\begin{aligned}\int_0^1 \frac{x^3 dx}{\sqrt{x^4+9}} &= \int_9^{10} \frac{1}{4} u^{-1/2} \; du \\ &= \frac{1}{2} u^{1/2}\Big|_9^{10} \\ &= \frac{1}{2}\sqrt{10} - \frac{1}{2}\sqrt{9} \\ &= \frac{1}{2}\sqrt{10} - \frac{3}{2}\end{aligned}$$

(b) $$\begin{aligned}\int \frac{x^3}{x^4+9} dx &= \int \frac{1}{4} u^{-1/2} \; du \\ &= \frac{1}{2} u^{1/2} + C \\ &= \frac{1}{2}\sqrt{x^4+9} + C\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{x^3}{x^4+9} \; dx &= \frac{1}{2}\sqrt{x^4+9}\Big|_0^1 \\ &= \frac{1}{2}\sqrt{10} - \frac{1}{2}\sqrt{9} \\ &= \frac{1}{2}\sqrt{10} - \frac{3}{2}\end{aligned}$$

68. Let $u = 1 - \cos 3x$, $du = 3 \sin 3x \, dx$.

$u\left(\frac{\pi}{6}\right) = 1 - \cos\frac{\pi}{2} = 1$

$u\left(\frac{\pi}{3}\right) = 1 - \cos\pi = 2$

(a) $$\begin{aligned}\int_{\pi/6}^{\pi/3} (1-\cos 3x)\sin 3x \, dx &= \int_1^2 \frac{1}{3} u \, du \\ &= \frac{1}{6} u^2\Big|_1^2 \\ &= \frac{1}{6}(2)^2 - \frac{1}{6}(1)^2 \\ &= \frac{1}{2}\end{aligned}$$

(b) $$\begin{aligned}\int (1-\cos 3x)\sin 3x \; dx &= \int \frac{1}{3} u \; du \\ &= \frac{1}{6} u^2 + C \\ &= \frac{1}{6}(1-\cos 3x)^2 + C\end{aligned}$$

$$\begin{aligned}&\int_{\pi/6}^{\pi/3} (1-\cos 3x)\sin 3x \, dx \\ &= \frac{1}{6}(1-\cos 3x)^2\Big|_{\pi/6}^{\pi/3} \\ &= \frac{1}{6}(2)^2 - \frac{1}{6}(1)^2 \\ &= \frac{1}{2}\end{aligned}$$

69. We show that $f'(x) = \tan x$ and $f(3) = 5$, where $f(x) = \ln\left|\frac{\cos 3}{\cos x}\right| + 5.$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\ln\left|\frac{\cos 3}{\cos x}\right| + 5\right) \\ &= \frac{d}{dx}\left(\ln|\cos 3| - \ln|\cos x| + 5\right) \\ &= -\frac{d}{dx}\ln|\cos x| \\ &= -\frac{1}{\cos x}(-\sin x) \\ &= \tan x \end{aligned}$$

$$f(3) = \ln\left|\frac{\cos 3}{\cos 3}\right| + 5 = \ln 1 + 5 = 5$$

70. We show that $f'(x) = \cot x$ and $f(2) = 6$, where

$$f(x) = \ln\left|\frac{\sin x}{\sin 2}\right| + 6 = \ln|\sin x| - \ln|\sin 2| + 6$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\ln|\sin x| - \ln|\sin 2| + 6\right) \\ &= \frac{1}{\sin x}\cdot\cos x - 0 + 0 \\ &= \cot x \end{aligned}$$

$$f(2) = \ln\left|\frac{\sin 2}{\sin 2}\right| + 6 = \ln 1 + 6 = 6$$

71. False; the interval of integration should change from $\left[0, \frac{\pi}{4}\right]$ to [0, 1], resulting in a different numerical answer.

72. True; use the substitution $u = f(x)$, $du = f'(x)\,dx$:

$$\begin{aligned} \int_a^b \frac{f'(x)dx}{f(x)} &= \int_{f(a)}^{f(b)} \frac{du}{u} \\ &= \ln|u|\Big|_{f(a)}^{f(b)} \\ &= \ln|f(b)| - \ln|f(a)| \\ &= \ln f(b) - \ln f(a) \\ &= \ln\left(\frac{f(b)}{f(a)}\right) \end{aligned}$$

73. D

74. E; $\int_0^2 e^{2x}dx = \frac{e^{2x}}{2}\Big|_0^2 = \frac{e^4 - 1}{2}$

75. B;
$$\begin{aligned} \int_3^5 f(x-a)\,dx &= F(x-a)\Big|_3^5 \\ &= F(5-a) - F(3-a) \\ &= 7 \end{aligned}$$

$$\begin{aligned} \int_{3-a}^{5-a} f(x)\,dx &= F(x)\Big|_{3-a}^{5-a} \\ &= F(5-a) - F(3-a) \\ &= 7 \end{aligned}$$

76. A; $\frac{d}{dx}\sin x = \cos x$

$$\cos\left(-\frac{\pi}{2}\right) = 0$$
$$\cos(0) = 1$$
$$\cos\left(\frac{\pi}{2}\right) = 0$$

77. **(a)** Let $u = x + 1$

$du = dx$

$$\begin{aligned} \int \sqrt{x+1}\,dx &= \int u^{1/2}du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}(x+1)^{3/2} + C \end{aligned}$$

Alternatively,

$$\frac{d}{dx}\left(\frac{2}{3}(x+1)^{3/2} + C\right) = \sqrt{x+1}.$$

(b) By the Antiderivative Part of the Fundamental Theorem of Calculus, $\frac{dy_1}{dx} = \sqrt{x+1}$ and $\frac{dy_2}{dx} = \sqrt{x+1}$, so both are antiderivatives of $\sqrt{x+1}$.

(c) Using NINT to find the values of y_1 and y_2, we have:

x	0	1	2	3	4
y_1	0	1.219	2.797	4.667	6.787
y_2	–4.667	–3.448	–1.869	0	2.120
$y_1 - y_2$	4.667	4.667	4.667	4.667	4.667

$C = 4\frac{2}{3}$

(d)
$$\begin{aligned} C &= y_1 - y_2 \\ &= \int_0^x \sqrt{x+1}\ dx - \int_3^x \sqrt{x+1}\ dx \\ &= \int_0^x \sqrt{x+1}\ dx + \int_x^3 \sqrt{x+1}\ dx \\ &= \int_0^3 \sqrt{x+1}\ dx \end{aligned}$$

78. **(a)** $\frac{d}{dx}[F(x)+C]$ should equal $f(x)$.

(b) The slope field should help you visualize the solution curve $y = F(x)$.

(c) The graphs of $y_1 = F(x)$ and $y_2 = \int_0^x f(t)\,dt$ should differ only by a vertical shift C.

(d) A table of values for $y_1 - y_2$ should show that $y_1 - y_2 = C$ for any value of x in the appropriate domain.

(e) The graph of f should be the same as the graph of NDER of $F(x)$.

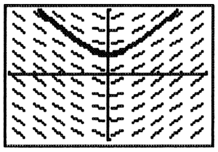
(f) First, we need to find $F(x)$. Let $u = x^2 + 1,\ du = 2x\ dx$.

$$\begin{aligned} \int \frac{x}{\sqrt{x^2+1}}\ dx &= \int \frac{1}{2} u^{-1/2}\ du \\ &= u^{1/2} \\ &= \sqrt{x^2+1} + C \end{aligned}$$

Therefore, we may let $F(x) = \sqrt{x^2+1}$.

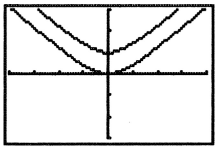
(a)
$$\begin{aligned} \frac{d}{dx}(\sqrt{x^2+1}+C) &= \frac{1}{2\sqrt{x^2+1}}(2x) \\ &= \frac{x}{\sqrt{x^2+1}} \\ &= f(x) \end{aligned}$$

(b)



[−4, 4] by [−3, 3]

(c)

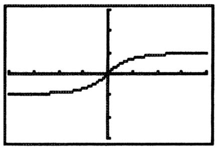


[−4, 4] by [−3, 3]

(d)

x	0	1	2	3	4
y_1	1.000	1.414	2.236	3.162	4.123
y_2	0.000	0.414	1.236	2.162	3.123
$y_1 - y_2$	1	1	1	1	1

(e)



[−4, 4] by [−3, 3]

79. **(a)**
$$\begin{aligned}\int 2\sin x\cos x\,dx &= \int 2u\,du\\ &= u^2+C\\ &= \sin^2 x+C\end{aligned}$$

(b)
$$\begin{aligned}\int 2\sin x\cos x\,dx &= -\int 2u\,du\\ &= -u^2+C\\ &= -\cos^2 x+C\end{aligned}$$

(c) Since $\sin^2 x-(-\cos^2 x)=1,$ the two answers differ by a constant (accounted for in the constant of integration).

80. **(a)**
$$\begin{aligned}\int 2\sec^2 x\tan x\,dx &= \int 2u\,du\\ &= u^2+C\\ &= \tan^2 x+C\end{aligned}$$

(b)
$$\begin{aligned}\int 2\sec^2 x\tan x\,dx &= \int 2u\,du\\ &= u^2+C\\ &= \sec^2 x+C\end{aligned}$$

(c) Since $\sec^2 x - \tan^2 x = 1,$ the two answers differ by a constant (accounted for in the constant of integration).

81. **(a)**
$$\begin{aligned}\int \frac{dx}{\sqrt{1-x^2}} &= \int \frac{\cos u\, du}{\sqrt{1-\sin^2 u}}\\ &= \frac{\cos u\, du}{\sqrt{\cos^2 u}}\\ &= \int 1 du.\end{aligned}$$

(Note cos $u > 0$, so $\sqrt{\cos^2 u} = |\cos u| = \cos u.$)

(b)
$$\begin{aligned}\int \frac{dx}{\sqrt{1-x^2}} &= \int 1 du\\ &= u + C\\ &= \sin^{-1} x + C\end{aligned}$$

82. **(a)**
$$\begin{aligned}\int \frac{dx}{1+x^2} &= \int \frac{\sec^2 u\, du}{1+\tan^2 u}\\ &= \int \frac{\sec^2 u\, du}{\sec^2 u}\\ &= \int 1 du\end{aligned}$$

(b)
$$\begin{aligned}\int \frac{dx}{1+x^2} &= \int 1 du\\ &= u + C\\ &= \tan^{-1} x + C\end{aligned}$$

83. **(a)**
$$\begin{aligned}&\int_0^{1/2} \frac{\sqrt{x}dx}{\sqrt{1-x}}\\ &= \int_{\sin^{-1}\sqrt{0}}^{\sin^{-1}\sqrt{1/2}} \frac{\sin y \cdot 2\sin y \cos y\, dy}{\sqrt{1-\sin^2 y}}\\ &= \int_0^{\pi/4} \frac{2\sin^2 y \cos y\, dy}{\cos y}\\ &= \int_0^{\pi/4} 2\sin^2 y\, dy\end{aligned}$$

(b)
$$\begin{aligned}\int_0^{1/2} \frac{\sqrt{x}dx}{\sqrt{1-x}} &= \int_0^{\pi/4} 2\sin^2 y\, dy\\ &= \int_0^{\pi/4} (1-\cos 2y) dy\\ &= [y - (1/2)\sin 2y]_0^{\pi/4}\\ &= \left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right] - [0-0]\\ &= \frac{(\pi-2)}{4}\end{aligned}$$

84. **(a)**
$$\begin{aligned}\int_0^{\sqrt{3}} \frac{dx}{\sqrt{1+x^2}} &= \int_{\tan^{-1} 0}^{\tan^{-1}\sqrt{3}} \frac{\sec^2 u\, du}{\sqrt{1+\tan^2 u}}\\ &= \int_0^{\pi/3} \frac{\sec^2 u\, du}{\sec u}\\ &= \int_0^{\pi/3} \sec u\, du\end{aligned}$$

(b)
$$\begin{aligned}\int_0^{\sqrt{3}} \frac{dx}{\sqrt{1+x^2}} &= \int_0^{\pi/3} \sec u\, du\\ &= \left[\ln|\sec u + \tan u|\right]_0^{\pi/3}\\ &= \ln\left(2+\sqrt{3}\right) - \ln\,(1+0)\\ &= \ln\left(2+\sqrt{3}\right)\end{aligned}$$

Section 7.3 Antidifferentiation by Parts (pp. 349–357)

Exploration 1 Choosing the Right *u* and *dv*

1. $u = 1 \qquad du = 0$

$dv = x\cos x \qquad v = \int x\cos x\, dx$

Using 1 for *u* is never a good idea because it places us back where we started.

2. $u = x\cos x \qquad du = \cos x - x\sin x$

$dv = dx \qquad v = \int dx = 1$

The selection of *u* = *x* cos *x* will place a more difficult integral into $\int v\,du.$

3. $u = \cos x \qquad du = -\sin x$

$dv = x\,dx \qquad v = \int x\,dx = x^2$

The selection of $dv = x\,dx$ will place a more difficult integral into $\int v\,du.$

4. *u* = *x* and *dv* = cos *x dx* are good choices because the integral is simplified.

Quick Review 7.3

1.
$$\begin{aligned}\frac{dy}{dx} &= (x^3)(\cos 2x)(2) + (\sin 2x)(3x^2)\\ &= 2x^3 \cos 2x + 3x^2 \sin 2x\end{aligned}$$

2.
$$\begin{aligned}\frac{dy}{dx} &= (e^{2x})\left(\frac{3}{3x+1}\right) + \ln\,(3x+1)(2e^{2x})\\ &= \frac{3e^{2x}}{3x+1} + 2e^{2x}\ln\,(3x+1)\end{aligned}$$

3. $\dfrac{dy}{dx}=\dfrac{1}{1+(2x)^2}\cdot 2=\dfrac{2}{1+4x^2}$

4. $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-(x+3)^2}}$

5.
$$\begin{aligned} y&=\tan^{-1}3x\\ \tan y&=3x\\ x&=\frac{1}{3}\tan y \end{aligned}$$

6.
$$\begin{aligned} y&=\cos^{-1}(x+1)\\ \cos y&=x+1\\ x&=\cos y-1 \end{aligned}$$

7.
$$\begin{aligned} \int_0^1 \sin \pi x\,dx&=-\frac{1}{\pi}\cos \pi x\Big|_0^1\\ &=-\frac{1}{\pi}\cos \pi+\frac{1}{\pi}\cos 0\\ &=-\frac{1}{\pi}(-1)+\frac{1}{\pi}\\ &=\frac{2}{\pi} \end{aligned}$$

8. $\dfrac{dy}{dx}=e^{2x}$

$dy=e^{2x}dx$

Integrate both sides.

$\int dy=\int e^{2x}dx$

$y=\dfrac{1}{2}e^{2x}+C$

9. $\dfrac{dy}{dx}=x+\sin x$

$dy=(x+\sin x)dx$

Integrate both sides.

$\int dy=\int (x+\sin x)dx$

$y=\dfrac{1}{2}x^2-\cos x+C$

$y(0) = -1 + C = 2, C = 3$

$y=\dfrac{1}{2}x^2-\cos x+3$

10.
$$\begin{aligned} &\frac{d}{dx}\left(\frac{1}{2}e^x(\sin x-\cos x)\right)\\ &=\frac{1}{2}e^x(\cos x+\sin x)+(\sin x-\cos x)\frac{1}{2}e^x\\ &=\frac{1}{2}e^x\cos x+\frac{1}{2}e^x\sin x+\frac{1}{2}e^x\sin x-\frac{1}{2}e^x\cos x\\ &=e^x\sin x \end{aligned}$$

Section 7.3 Exercises

1. $\int x\sin x\,dx$

$dv=\sin x\,dx \qquad v=\int \sin x\,dx=-\cos x$

$u=x \qquad du=dx$

$-x\cos x-\int -\cos x\,dx=-x\cos x+\sin x+C$

2. $\int xe^x dx$

$dv=e^x dx \qquad v=\int e^x dx=e^x$

$u=x \qquad du=dx$

$xe^x-\int e^x dx=xe^x-e^x+C$

3. $\int 3\,t\,e^{2t}dt$

$dv=e^{2t}dt \qquad v=\int e^{2t}dt=\dfrac{e^{2t}}{2}$

$u=3t \qquad du=3\,dt$

$3t\,\dfrac{e^{2t}}{2}-\int 3\,\dfrac{e^{2t}}{2}dt=\dfrac{3}{2}te^{2t}-\dfrac{3}{4}e^{2t}+C$

4. $\int 2\,t\cos(3t)\,dt$

$dv=\cos 3t\,dt \quad v=\int \cos(3t)\,dt=\dfrac{\sin 3t}{3}$

$u=2t \qquad du=2\,dt$

$2t\,\dfrac{\sin 3t}{3}-\int 2\dfrac{\sin 3t}{3}dt$

$=\dfrac{2}{3}t\sin 3t-\dfrac{2}{9}\cos(3t)+C$

5. $\int x^2\cos x\,dx$

$dv=\cos x\,dx \qquad v=\int \cos x\,dx=\sin x$

$u=x^2 \qquad du=2x\,dx$

$x^2\sin x-\int 2x\sin x\,dx$

$dv=\sin x\,dx \qquad v=\int \sin x\,dx=-\cos x$

$u=2x \qquad du=2dx$

$x^2\sin x+2x\cos x-\int 2\cos x dx$

$=x^2\sin x+2x\cos x-2\sin x+C$

6. $\int x^2 e^{-x} dx$

$dv = e^{-x}dx \qquad v = \int e^{-x}dx = -e^{-x}$

$u = x^2 \qquad du = 2x\, dx$

$-x^2e^{-x} - \int -2x\, e^{-x}dx = -x^2e^{-x} + \int 2xe^{-x} + C$

$dv = e^{-x} \qquad v = \int e^{-x}dx = -e^{-x}$

$u = 2x \qquad du = 2\, dx$

$-x^2e^{-x} - 2xe^{-x} - \int -2e^{-x}dx$

$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

7. $\int y \ln y\, dy$

$dv = y\, dy \qquad v = \int y\, dy = \frac{y^2}{2}$

$u = \ln y \qquad du = \frac{1}{y}\, dy$

$\frac{1}{2}y^2 \ln y - \int \frac{y^2}{2}\frac{1}{y}\, dy = \frac{1}{2}y^2 \ln y - \frac{y^2}{4} + C$

8. $\int t^2 \ln t\, dt$

$dv = t^2\, dt \qquad v = \int t^2 dt = \frac{t^3}{3}$

$u = \ln t \qquad du = \frac{1}{t} dt$

$\frac{1}{3}t^3 \ln t - \int \frac{t^3}{3}\frac{1}{t} dt = \frac{1}{3}t^3 \ln t - \frac{t^3}{9} + C$

9. $\int \log_2 x\, dx = \frac{1}{\ln 2}\int \ln x\, dx$

$dv = dx \qquad v = x$

$u = \ln x \qquad du = \frac{1}{x} dx$

$\frac{1}{\ln 2}\left(x \ln x - \int x \cdot \frac{1}{x} dx \right) = \frac{1}{\ln 2}(x \ln x - x) + C$

10. $\int \tan^{-1} x\, dx$

$dv = dx \qquad v = x$

$u = \tan^{-1} x \qquad du = \frac{1}{1+x^2} dx$

$x \tan^{-1} x - \int \frac{x}{1+x^2} dx$

$u = 1 + x^2 \qquad du = 2x\, dx$

$x \tan^{-1} x - \frac{1}{2}\int \frac{du}{u} = x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$

11. $\int dy = \int \big((x+2)\sin x\big) dx$

$dv = \sin x\, dx \qquad v = \int \sin x\, dx = -\cos x$

$u = x + 2 \qquad du = dx$

$-(x+2)\cos x - \int (-\cos x)\, dx$

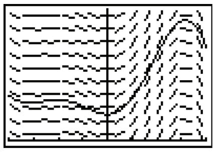
$= -(x+2)\cos x + \sin x + C$

$2 = -(0+2)\cos(0) + \sin(0) + C$

$2 = -2 + C$

$C = 4$

$y = -(x+2)\cos x + \sin x + 4$



[–4, 4] by [0, 10]

12. $\int dy = \int 2xe^{-x} dx$

$dv = e^{-x}dx \qquad v = \int e^{-x}dx = -e^{-x}$

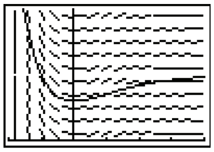
$u = 2x \qquad du = 2\, dx$

$-2xe^{-x} - \int -2e^{-x}dx = -2xe^{-x} - 2e^{-x} + C$

$3 = -2(0)\, e^{(-0)} - 2e^{(-0)} + C$

$5 = C$

$y = -2xe^{-x} - 2e^{-x} + 5$



[–2, 4] by [0, 10]

13. $\int du = \int x \sec^2 x\, dx$

$dv = \sec^2 x\, dx \qquad v = \int \sec^2 x\, dx = \tan x$

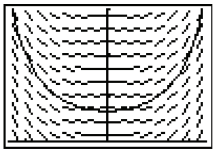
$w = x \qquad dw = dx$

$x \tan x - \int \tan x\, dx = x \tan x + \ln|\cos x| + C$

$1 = 0 \tan(0) + \ln|\cos(0)| + C$

$C = 1$

$u = x \tan(x) + \ln|\cos(x)| + 1$



[–1.2, 1.2] by [0, 3]

14. $\int dz = x^3 \ln x \, dx$

$dv = x^3 dx \qquad v = \int x^3 dx = \frac{x^4}{4}$

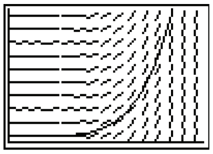
$u = \ln x \qquad du = \frac{1}{x} dx$

$\frac{x^4}{4} \ln x - \int \frac{x^4}{4} \frac{1}{x} dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + C$

$5 = \frac{(1)^4}{4} \ln (1) - \frac{(1)^4}{16} + C$

$C = \frac{81}{16}$

$z = \frac{x^4}{4} \ln x - \frac{x^4}{16} + \frac{81}{16}$



[0, 5] by [0, 100]

15. $\int dy = \int x\sqrt{x-1} \, dx$

$dv = (x-1)^{1/2} dx \qquad v = \int (x-1)^{1/2} dx$

$= \frac{2}{3}(x-1)^{3/2}$

$u = x \qquad du = dx$

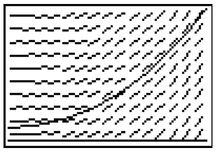
$\frac{2}{3} x (x-1)^{3/2} - \int \frac{2}{3} (x-1)^{3/2} dx$

$= \frac{2}{3} x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C$

$2 = \frac{2}{3} (1) (1-1)^{3/2} - \frac{4}{15}(1-1)^{5/2} + C$

$C = 2$

$y = \frac{2}{3} x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + 2$



[1, 5] by [0, 20]

16. $\int dy = \int 2x \sqrt{x+2} \, dx$

$dv = (x+2)^{1/2} dx \qquad v = \int (x+2)^{1/2} \, dx$

$= \frac{2}{3} (x+2)^{3/2}$

$u = 2x \qquad du = 2dx$

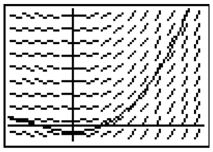
$\frac{4}{3}x(x+2)^{3/2} - \int \frac{4}{3}(x+2)^{3/2}\, dx$

$= \frac{4}{3}x(x+2)^{3/2} - \frac{8}{15}(x+2)^{5/2} + C$

$0 = \frac{4}{3}(-1)(-1+2)^{3/2} - \frac{8}{15}(-1+2)^{5/2} + C$

$C = \frac{28}{15}$

$y = \frac{4}{3}x(x+2)^{3/2} - \frac{8}{15}(x+2)^{5/2} + \frac{28}{15}$



[–2, 4] by [–3, 25]

17. $\int e^x \sin x\, dx$

$dv = e^x\, dx \qquad v = \int e^x\, dx = e^x$

$u = \sin x \qquad du = \cos x\, dx$

$e^x \sin x - \int e^x \cos x\, dx$

$dv = e^x\, dx \qquad v = \int e^x\, dx = e^x$

$u = \cos x \qquad du = -\sin x\, dx$

$\int e^x \sin x\, dx = e^x \sin x - \left[e^x \cos x - \int -e^x \sin x\, dx\right]$

$\int e^x \sin x\, dx = \frac{e^x}{2}(\sin x - \cos x) + C$

18. $\int e^{-x} \cos\ x\, dx$

$dv = \cos x\, dx \qquad v = \int \cos x\, dx = \sin x$

$u = e^{-x} \qquad du = -e^{-x}\, dx$

$e^{-x} \sin x - \int -e^{-x} \sin x\, dx$

$dv = \sin x\, dx \qquad v = \int \sin x\, dx = -\cos x$

$u = e^{-x} \qquad du = -e^{-x}\, dx$

$\int e^{-x} \cos x\, dx = e^{-x} \sin x - \left[e^{-x} \cos x - \int -e^{-x} \cos x\, dx\right]$

$\int e^{-x} \cos x\, dx = \frac{e^{-x}}{2}(\sin x - \cos x) + C$

19. $\int e^x \cos\ 2x\ dx$

$dv = \cos 2x\ dx \quad v = \int \cos 2x\ dx = \frac{1}{2}\sin 2x$

$u = e^x \qquad du = e^x dx$

$\frac{1}{2}e^x \sin 2x - \int \frac{1}{2}\sin 2x\ e^x dx$

$$dv = \frac{1}{2}\sin 2x\,dx \qquad v = \int \frac{1}{2}\sin 2x\;dx = -\frac{1}{4}\cos 2x$$

$$u = e^x \qquad du = e^x dx$$

$$\int e^x \cos 2x\;dx = \frac{1}{2}e^x \sin 2x - \left[-\frac{1}{4}e^x \cos 2x - \int -\frac{1}{4}\cos 2x\;e^x dx\right]$$

$$\int e^x \cos 2x\;dx = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}\int e^x \cos 2x\,dx$$

$$\frac{5}{4}\int e^x \cos 2x\,dx = \frac{1}{4}e^x(2\sin 2x + \cos 2x)$$

$$\int e^x \cos 2x\,dx = \frac{e^x}{5}(2\sin 2x + \cos 2x) + C$$

20. $\int e^{-x}\sin 2x\;dx$

$$dv = \sin 2x\;dx \quad v = \int \sin 2x\;dx = -\frac{1}{2}\cos 2x$$

$$u = e^{-x} \qquad du = -e^{-x}dx$$

$$-\frac{1}{2}e^{-x}\cos 2x - \int \frac{1}{2}\cos 2x\;e^{-x}dx$$

$$dv = \frac{1}{2}\cos 2x\;dx \quad v = \int \frac{1}{2}\cos 2x\;dx = \frac{1}{4}\sin 2x$$

$$u = e^{-x} \qquad du = -e^{-x}dx$$

$$\int e^{-x}\sin 2x\;dx = -\frac{1}{2}e^{-x}\cos 2x - \left[\frac{1}{4}e^{-x}\sin 2x - \int -\frac{1}{4}e^{-x}\sin 2x\;dx\right]$$

$$\int e^{-x}\sin 2x\;dx = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x - \frac{1}{4}\int e^{-x}\sin 2x\;dx$$

$$\frac{5}{4}\int e^{-x}\sin 2x\;dx = -\frac{1}{4}e^{-x}(2\cos 2x + \sin 2x)$$

$$\int e^{-x}\sin 2x\;dx = -\frac{e^{-x}}{5}(2\cos 2x + \sin 2x) + C$$

21. Use tabular integration with $f(x) = x^4$ and $g(x) = e^{-x}$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^4	(+)	e^{-x}
$4x^3$	(−)	$-e^{-x}$
$12x^2$	(+)	e^{-x}
$24x$	(−)	$-e^{-x}$
24	(+)	e^{-x}
0		$-e^{-x}$

$$\int x^4 e^{-x}\;dx = -x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} - 24xe^{-x} - 24e^{-x} + C$$

$$= -(x^4 + 4x^3 + 12x^2 + 24x + 24)e^{-x} + C$$

22. Use tabular integration with $f(x)=x^2-5x$ and $g(x)=e^x$

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2-5x	(+)	e^x
$2x-5$	(–)	e^x
2	(+)	e^x
0		e^x

$$\int (x^2-5x)e^x\,dx=(x^2-5x)e^x-(2x-5)e^x+2e^x$$
$$=(x^2-7x+7)e^x+C$$

23. Use tabular integration with $f(x)=x^3$ and $g(x)=e^{-2x}$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	e^{-2x}
$3x^2$	(–)	$-\frac{1}{2}e^{-2x}$
$6x$	(+)	$\frac{1}{4}e^{-2x}$
6	(–)	$-\frac{1}{8}e^{-2x}$
0		$\frac{1}{16}e^{-2x}$

$$\int x^3e^{-2x}\,dx=-\frac{1}{2}x^3e^{-2x}-\frac{3}{4}x^2e^{-2x}-\frac{3}{4}xe^{-2x}-\frac{3}{8}e^{-2x}+C$$
$$=-\left(\frac{x^3}{2}+\frac{3x^2}{4}+\frac{3x}{4}+\frac{3}{8}\right)e^{-2x}+C$$

24. Use tabular integration with $f(x)=x^3$ and $g(x)=\cos 2x$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	$\cos 2x$
$3x^2$	(–)	$\frac{1}{2}\sin 2x$
$6x$	(+)	$-\frac{1}{4}\cos 2x$
6	(–)	$-\frac{1}{8}\sin 2x$
0		$\frac{1}{16}\cos 2x$

$$\frac{x^3}{2}\sin 2x+\frac{3x^2}{4}\cos 2x-\frac{3x}{4}\sin 2x-\frac{3}{8}\cos 2x+C$$

25. Use tabular integration with $f(x) = x^2$ and $g(x) = \sin 2x$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	(+)	$\sin 2x$
$2x$	(−)	$-\frac{1}{2}\cos 2x$
2	(+)	$-\frac{1}{4}\sin 2x$
0		$\frac{1}{8}\cos 2x$

$$\int x^2 \sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$

$$= \left(\frac{1-2x^2}{4}\right)\cos\ 2x + \frac{x}{2}\sin\ 2x + C$$

$$\int_0^{\pi/2} x^2 \sin 2x\,dx = \left[\left(\frac{1-2x^2}{4}\right)\cos\ 2x + \frac{x}{2}\sin\ 2x\right]_0^{\pi/2}$$

$$= \left(\frac{1-2\left(\frac{\pi}{2}\right)^2}{4}\right)(-1) + 0 - \left(\frac{1}{4}\right)(1) - 0$$

$$= \frac{\pi^2}{8} - \frac{1}{2}$$

26. Use tabular integration with $f(x) = x^3$ and $g(x) = \cos 2x$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	$\cos 2x$
$3x^2$	(−)	$\frac{1}{2}\sin 2x$
$6x$	(+)	$-\frac{1}{4}\cos 2x$
6	(−)	$-\frac{1}{8}\sin 2x$
0		$\frac{1}{16}\cos 2x$

$$\int x^3 \cos 2x\,dx = \frac{1}{2}x^3\sin 2x + \frac{3}{4}x^2\cos 2x - \frac{3}{4}x$$

$$\sin 2x - \frac{3}{8}\cos 2x = \left(\frac{x^3}{2} - \frac{3x}{4}\right)\sin 2x + \left(\frac{3x^2}{4} - \frac{3}{8}\right)\cos 2x + C$$

$$\int_0^{\pi/2} x^3 \cos 2x\,dx = \left[\left(\frac{x^3}{2} - \frac{3x}{4}\right)\sin 2x + \left(\frac{3x^2}{4} - \frac{3}{8}\right)\cos 2x\right]_0^{\pi/2}$$

$$= 0 + \left(\frac{3\pi^2}{16} - \frac{3}{8}\right)(-1) - 0 - \left(-\frac{3}{8}\right)(1)$$

$$= \frac{3}{4} - \frac{3\pi^2}{16}$$

27. Let $u = e^{2x} \qquad dv = \cos 3x\, dx$

$du = 2e^{2x}\, dx \qquad v = \frac{1}{3}\sin 3x$

$$\int e^{2x} \cos 3x\, dx = (e^{2x})\left(\frac{1}{3}\sin 3x\right) - \int \left(\frac{1}{3}\sin 3x\right)(2e^{2x}\, dx)$$

$$= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x} \sin 3x\, dx$$

Let $u = e^{2x} \qquad dv = \sin 3x\, dx$

$du = 2e^{2x}\, dx \qquad v = -\frac{1}{3}\cos 3x$

$$\int e^{2x} \cos 3x\, dx = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left[(e^{2x})\left(-\frac{1}{3}\cos 3x\right) - \int\left(-\frac{1}{3}\cos 3x\right)(2e^{2x}\, dx)\right]$$

$$= \frac{1}{9}e^{2x}(3\sin 3x + 2\cos 3x) - \frac{4}{9}\int e^{2x}\cos 3x\, dx$$

$$\frac{13}{9}\int e^{2x}\cos 3x\, dx = \frac{1}{9}e^{2x}(3\sin 3x + 2\cos 3x)$$

$$\int e^{2x}\cos 3x\, dx = \frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x)$$

$$\int_{-2}^{3} e^{2x}\cos 3x\, dx = \left[\frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x)\right]_{-2}^{3}$$

$$= \frac{1}{13}[e^{6}(3\sin 9 + 2\cos 9) - e^{-4}(3\sin(-6) + 2\cos(-6)]$$

$$= \frac{1}{13}[e^{6}(2\cos 9 + 3\sin 9) - e^{-4}(2\cos 6 - 3\sin 6)]$$

28. Let $u = e^{-2x} \qquad dv = \sin 2x\, dx$

$du = -2e^{-2x}\, dx \qquad v = -\frac{1}{2}\cos 2x$

$$\int e^{-2x}\sin 2x\, dx = (e^{-2x})\left(-\frac{1}{2}\cos 2x\right) - \int\left(-\frac{1}{2}\cos 2x\right)(-2e^{-2x}\, dx)$$

$$= -\frac{1}{2}e^{-2x}\cos 2x - \int e^{-2x}\cos 2x\, dx$$

Let $u = e^{-2x} \qquad dv = \cos 2x\, dx$

$du = -2e^{-2x}dx \qquad v = \frac{1}{2}\sin 2x$

$$\int e^{-2x}\sin 2x\,dx = -\frac{1}{2}e^{-2x}\cos\,2x - \left[(e^{-2x})\left(\frac{1}{2}\sin 2x\right) - \int\left(\frac{1}{2}\sin 2x\right)(-2e^{-2x}\;dx)\right]$$

$$= -\frac{1}{2}e^{-2x}(\cos 2x + \sin 2x)\; - \int e^{-2x}\sin 2x\,dx$$

$$2\int e^{-2x}\sin 2x\,dx = -\frac{1}{2}e^{-2x}(\cos 2x + \sin 2x) + C$$

$$\int e^{-2x}\sin 2x\,dx = -\frac{e^{-2x}}{4}(\cos 2x + \sin 2x) + C$$

$$\int_{-3}^{2} e^{-2x}\sin 2x\,dx = \left[-\frac{e^{-2x}}{4}(\cos 2x + \sin 2x)\right]_{-3}^{2}$$

$$= -\frac{e^{-4}}{4}(\cos\;4 + \sin\;4) + \frac{e^{6}}{4}[\cos(-6) + \sin\,(-6)]$$

$$= -\frac{e^{-4}}{4}(\cos\;4 + \sin\;4) + \frac{e^{6}}{4}(\cos\;6 - \sin\;6)$$

29. $y = \int x^2 e^{4x}\;dx$

Let $u = x^2$ $\qquad dv = e^{4x}dx$

$du = 2x\;dx \qquad v = \frac{1}{4}e^{4x}$

$$y = (x^2)\left(\frac{1}{4}e^{4x}\right) - \int\left(\frac{1}{4}e^{4x}\right)(2x\;dx)$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{2}\int xe^{4x}dx$$

Let $u = x$ $\qquad dv = e^{4x}\;dx$

$du = dx \qquad v = \frac{1}{4}e^{4x}$

$$y = \frac{1}{4}x^2e^{4x} - \frac{1}{2}\left[(x)\left(\frac{1}{4}e^{4x}\right) - \int\left(\frac{1}{4}e^{4x}\right)dx\right]$$

$$y = \frac{1}{4}x^2\mathrm{e}^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C$$

$$y = \left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right)e^{4x} + C$$

30. $y = \int x^2\;\ln x\;dx$

Let $u = \ln x$ $\qquad dv = x^2\;dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{3}x^3$

$$y = (\ln x)\left(\frac{1}{3}x^3\right) - \int\left(\frac{1}{3}x^3\right)\left(\frac{1}{x}dx\right)$$

$$y = \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2dx$$

$$y = \frac{1}{3}x^3\;\ln x - \frac{1}{9}x^3 + C$$

31. $y = \int \theta \sec^{-1} \theta \, d\theta$

Let $u = \sec^{-1} \theta \qquad dv = \theta \, d\theta$

$du = \dfrac{1}{\theta\sqrt{\theta^2 - 1}} du \qquad v = \dfrac{1}{2}\theta^2$

Note that we are told $\theta > 1$, so no absolute value is needed in the expression for *du*.

$$y = (\sec^{-1}\theta)\left(\frac{1}{2}\theta^2\right) - \int \left(\frac{1}{2}\theta^2\right)\left(\frac{1}{\theta\sqrt{\theta^2 - 1}} d\theta\right)$$

$$y = \frac{\theta^2}{2}\sec^{-1} \theta - \frac{1}{2}\int \frac{\theta \, d\theta}{\sqrt{\theta^2 - 1}}$$

Let $w = \theta^2 - 1$, $dw = 2\theta \, d\theta$

$$y = \frac{\theta^2}{2}\sec^{-1} \theta - \frac{1}{4}\int w^{-1/2} dw$$

$$y = \frac{\theta^2}{2}\sec^{-1}\theta - \frac{1}{2}w^{1/2} + C$$

$$y = \frac{\theta^2}{2}\sec^{-1}\theta - \frac{1}{2}\sqrt{\theta^2 - 1} + C$$

32. $y = \int \theta \sec\theta \tan\theta \, d\theta$

Let $u = \theta \qquad dv = \sec\theta \tan\theta \, d\theta$

$du = d\theta \qquad v = \sec\theta$

$$y = \theta \sec\theta - \int \sec \theta \, d\theta$$

$$y = \theta \sec\theta - \ln|\sec\theta + \tan\theta| + C$$

Note : In the last step, we used the result of Exercise 45 in Section 7.2.

33. Let $u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx$$
$$= -x \cos x + \sin x + C$$

(a) $\int_0^\pi |x \sin x| \, dx = \int_0^\pi x \sin x \, dx$

$$= [-x \cos x + \sin x]_0^\pi$$
$$= -\pi(-1) + 0 + 0(1) - 0$$
$$= \pi$$

(b) $\int_\pi^{2\pi} |x \sin x| \, dx = -\int_\pi^{2\pi} x \sin x \, dx$

$$= [x \cos x - \sin x]_\pi^{2\pi}$$
$$= 2\pi(1) - 0 - \pi(-1) + 0$$
$$= 3\pi$$

(c) $\int_0^{2\pi} |x \sin x| \, dx$

$$= \int_0^\pi |x \sin x| \, dx + \int_\pi^{2\pi} |x \sin x| \, dx$$
$$= \pi + 3\pi = 4\pi$$

34. We begin by evaluating $\int (x^2 + x + 1) e^{-x} \, dx.$

Let $u = x^2 + x + 1 \qquad dv = e^{-x} \, dx$

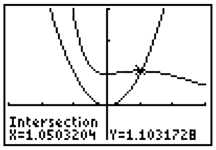
$du = (2x + 1) \, dx \qquad v = -e^{-x}$

$$\int (x^2 + x + 1)e^{-x} \, dx$$
$$= -(x^2 + x + 1)e^{-x} + \int (2x + 1)e^{-x} \, dx$$

Let $u = 2x + 1 \qquad dv = e^{-x} \, dx$

$du = 2 \, dx \qquad v = -e^{-x}$

$$\int (x^2 + x + 1)e^{-x} \, dx$$
$$= -(x^2 + x + 1)e^{-x} - (2x + 1)e^{-x} + \int 2e^{-x} \, dx$$
$$= -(x^2 + x + 1)e^{-x} - (2x + 1)e^{-x} - 2e^{-x} + C$$
$$= -(x^2 + 3x + 4)e^{-x} + C$$



[−3, 3] by [−3, 3]

The graph shows that the two curves intersect at $x = k$, where $k \approx 1.050$. The area we seek is

$$\int_0^k (x^2 + x + 1)e^{-x} - \int_0^k x^2 \, dx$$
$$= \left[-(x^2 + 3x + 4)e^{-x}\right]_0^k - \left[\frac{1}{3}x^3\right]_0^k$$
$$\approx (-2.888 + 4) - (0.386 - 0)$$
$$\approx 0.726$$

35. First, we evaluate $\int e^{-t} \cos t \, dt.$

Let $u = e^{-t} \qquad dv = \cos t \, dt$

$du = -e^{-t} \, dt \qquad v = \sin t$

$$\int e^{-t} \cos t \, dt = e^{-t}\sin t + \int \sin t \, e^{-t} \, dt$$

Let $u = e^{-t} \qquad dv = \sin t \, dt$

$du = -e^{-t} \, dt \qquad v = -\cos t$

$$\int e^{-t}\cos t \, dt = e^{-t}\sin t - e^{-t}\cos t - \int e^{-t}\cos t \, dt$$
$$2\int e^{-t}\cos t \, dt = e^{-t}(\sin t - \cos t) + C$$
$$\int e^{-t}\cos t \, dt = \frac{1}{2}e^{-t}(\sin t - \cos t) + C$$

Now we find the average value of $y = 2e^{-t}\cos t$ for $0 \le t \le 2\pi$.

$$\begin{aligned}\text{Average value } &= \frac{1}{2\pi}\int_0^{2\pi} 2e^{-t}\cos t\,dt \\ &= \frac{1}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt \\ &= \frac{1}{2\pi}e^{-t}(\sin t - \cos t)\Big|_0^{2\pi} \\ &= \frac{1}{2\pi}[e^{-2\pi}(-1) - e^0(-1)] \\ &= \frac{1-e^{-2\pi}}{2\pi} \approx 0.159\end{aligned}$$

36. True; use parts, letting $u = x$, $dv = g(x)dx$, and $v = f(x)$.

37. True; use parts, letting $u = x^2$, $dv = g(x)dx$, and $v = f(x)$.

38. B; $\int x^2 \cos x\,dx$

$= x^2 \sin x + 2x\cos x - 2\sin x + C$

See problem 5.

$\int 2x \sin x\,dx = -2x\cos x + 2\sin x + C$

See problem 1.

$h(x) = x^2 \sin x + C$

39. B; $\int x \sin(5x)\,dx$

$dv = \sin(5x)dx \qquad v = \int \sin(5x)\,dx = -\frac{1}{5}\cos 5x$

$u = x \qquad du = dx$

$-\frac{1}{5}x\cos(5x) - \int -\frac{1}{5}\cos(5x)\,dx$

$= -\frac{1}{5}x\cos(5x) + \frac{1}{25}\sin(5x) + C$

40. C; $\int x \csc^2 x\,dx$

$dv = \csc^2 x\,dx \qquad v = \int \csc^2 x\,dx = -\cot x$

$u = x \qquad du = dx$

$-x\cot x - \int -\cot x\,dx$

$= -x\cot x + \ln|\sin x| + C$

41. C; $\int dy = \int 4x \ln x\,dx$

$dv = 4x\,dx \qquad v = \int 4x\,dx = 2x^2$

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$2x^2 \ln x - \int 2x^2 \frac{1}{x}\,dx = 2x^2 \ln x - x^2 + C$

42. **(a)** Let $u = x \qquad dv = e^x\ dx$

$du = dx \qquad v = e^x$

$$\begin{aligned}\int xe^x\ dx &= xe^x - \int e^x\ dx\\ &= xe^x - e^x + C\\ &= (x-1)e^x + C\end{aligned}$$

(b) Using the result from part (a):

Let $u = x^2 \qquad dv = e^x\ dx$

$du = 2x\ dx \qquad v = e^x$

$$\begin{aligned}\int x^2 e^x\ dx &= x^2 e^x - \int 2xe^x\ dx\\ &= x^2e^x - 2(x-1)e^x + C\\ &= (x^2 - 2x + 2)e^x + C\end{aligned}$$

(c) Using the result from part (b):

Let $u = x^3 \qquad dv = e^x\ dx$

$du = 3x^2dx \qquad v = e^x$

$$\begin{aligned}\int x^3 e^x\ dx &= x^3\ e^x - \int 3x^2\ e^x\ dx\\ &= x^3e^x - 3(x^2 - 2x + 2)e^x + C\\ &= (x^3 - 3x^2 + 6x - 6)e^x + C\end{aligned}$$

(d) $\left[x^n - \dfrac{d}{dx}x^n + \dfrac{d}{dx^2}x^n - \cdots + (-1)^n \dfrac{d^n}{dx^n}x^n\right]e^x + C$ or

$\left[x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^{n-1}(n)!x + (-1)^n(n!)\right]e^x + C$

(e) Use mathematical induction or argue based on tabular integration. Alternately, show that the derivative of the answer to part (d) is $x^n e^x$:

$$\frac{d}{dx}\left[\left(x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^{n-1}(n!)x + (-1)^n n!\right)e^x + C\right]$$

$$= [x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^{n-1}(n!)x + (-1)^n n!]e^x$$

$$+ e^x \frac{d}{dx}\left[x^n - nx^{n-1} + n(n-1)n^{n-2} - \cdots \qquad + (-1)^{n-1}(n!)x + (-1)^n n!\right]$$

$$= [x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^{n-1} + (n!)x + (-1)^n n!]e^x$$

$$+\left[nx^{n-1} - n(n-1)x^{n-2} + n(n-1)(n-2)x^{n-3} - \cdots + (-1)^{n-1}n!\right]e^x$$

$$= x^n e^x$$

43. Let $w = \sqrt{x}$. Then $dw = \dfrac{dx}{2\sqrt{x}}$, so

$$\begin{aligned} dx &= 2\sqrt{x}\ dw = 2w\,dw.\\ \int \sin\sqrt{x}\ dx &= \int (\sin w)(2w\ dw)\\ &= 2\int w \sin w\,dw \end{aligned}$$

Let $u = w \qquad dv = \sin w\ dw$

$du = dw \qquad v = -\cos w$

$$\begin{aligned} \int w \sin w\,dw &= -w\cos w + \int \cos w\,dw\\ &= -w\cos w + \sin w + C\\ \int \sin\sqrt{x}\ dx &= 2\int w\sin w\ dw\\ &= -2w\cos w + 2\sin w + C\\ &= -2\sqrt{x}\ \cos\sqrt{x} + 2\sin\sqrt{x} + C \end{aligned}$$

44. Let $w = \sqrt{3x+9}$. Then $dw = \dfrac{1}{2\sqrt{3x+9}}(3)\ dx$,

so $dx = \dfrac{2}{3}\sqrt{3x+9}\ dw = \dfrac{2}{3}w\,dw.$

$$\int e^{\sqrt{3x+9}}\ dw = \int (e^w)\left(\frac{2}{3}w\,dw\right) = \frac{2}{3}\int w\,e^w\ dw$$

Let $u = w \qquad dv = e^w\ dw$

$du = dw \qquad v = e^w$

$$\begin{aligned} \int w\,e^w\ dw &= w\,e^w - \int e^w\ dw\\ &= w\,e^w - e^w\\ &= (w-1)e^w\\ \int e^{\sqrt{3x+9}}dx &= \frac{2}{3}\int w\,e^w\ dw\\ &= \frac{2}{3}(w-1)e^w\\ &= \frac{2}{3}\left(\sqrt{3x+9}-1\right)e^{\sqrt{3x+9}} + C \end{aligned}$$

45. Let $w = x^2$. Then $dw = 2x\,dx$.

$$\int x^7 e^{x^2} dx = \int (x^2)^3 e^{x^2}\, x\ dx = \frac{1}{2}\int w^3\ e^w\ dw.$$

Use tabular integration with $f(x) = w^3$ and $g(w) = e^w$.

$f(w)$ and its derivatives		$g(w)$ and its integrals
w^3	(+)	e^w
$3w^2$	(−)	e^w
$6w$	(+)	e^w
6	(−)	e^w
0		e^w

$$\begin{aligned} &\int w^3\, e^w\ dw\\ &= w^3\, e^w - 3w^2 e^w + 6w\,e^w - 6e^w + C\\ &= (w^3 - 3w^2 + 6w - 6)\,e^w + C \end{aligned}$$

$$\begin{aligned} \int x^7 e^{x^2}\ d &= \frac{1}{2}\int w^3\ e^w\ dw\\ &= \frac{1}{2}(w^3 - 3w^2 + 6w - 6)e^w + C\\ &= \frac{(x^6 - 3x^4 + 6x^2 - 6)e^{x^2}}{2} + C \end{aligned}$$

46. Let $y = \ln r$. Then $dy = \dfrac{1}{r}dr$, and so $dr = r\ dy = e^y\ dy$. Using the result of Exercise 17, we have:

$$\begin{aligned} &\int \sin(\ln r)\ dr\\ &= \int (\sin y)e^y\,dy\\ &= \frac{1}{2}e^y(\sin y - \cos y) + C\\ &= \frac{1}{2}e^{\ln r}[\sin(\ln r) - \cos(\ln r)] + C\\ &= \frac{r}{2}[\sin(\ln r) - \cos(\ln r)] + C \end{aligned}$$

47. Let $u = x^n \qquad dv = \cos x\ dx$

$du = nx^{n-1}dx \qquad v = \sin x$

$$\begin{aligned} \int x^n \cos x\ dx &= x^n \sin x - \int (\sin x)(nx^{n-1}dx)\\ &= x^n \sin x - n\int x^{n-1}\sin x\ dx \end{aligned}$$

48. Let $u = x^n \qquad dv = \sin x\ dx$

$du = nx^{n-1}dx \qquad v = -\cos x$

$$\begin{aligned} &\int n^x \sin x\ dx\\ &= (x^n)(-\cos x) - \int (-\cos x)(nx^{n-1})\ dx\\ &= -x^n \cos x + n\int x^{n-1}\cos x\ dx \end{aligned}$$

49. Let $u = x^n \qquad dv = e^{ax}\,dx$

$$du = nx^{n-1}\,dx \qquad v = \frac{1}{a}e^{ax}$$

$$\int x^n e^{ax}\,dx$$
$$= (x^n)\left(\frac{1}{a}e^{ax}\right) - \int\left(\frac{1}{a}e^{ax}\right)(nx^{n-1}\,dx)$$
$$= \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx,\ a \neq 0$$

50. Let $u = (\ln x)^n \qquad dv = dx$

$$du = \frac{n(\ln x)^{n-1}}{x}\,dx \qquad v = x$$

$$\int (\ln x)^n\,dx = (\ln x)^n(x) - \int x\left[\frac{n(\ln x)^{n-1}}{x}\right]dx$$
$$= x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

51. **(a)** Let $y = f^{-1}(x)$. Then $x = f(y)$, so $dx = f'(y)\,dy$. Hence,

$$\int f^{-1}(x)\,dx = \int (y)[f'(y)\,dy]$$
$$= \int y\,f'(y)\,dy$$

(b) Let $u = y \qquad dv = f'(y)dy$

$$du = dy \qquad v = f(y)$$

$$\int y\,f'(y)\,dy = y\,f(y) - \int f(y)\,dy$$
$$= f^{-1}(x)(x) - \int f(y)\,dy$$

Hence,

$$\int f^{-1}(x)\,dx = \int y\,f'(y)\,dy$$
$$= x\,f^{-1}(x) - \int f(y)\,dy.$$

52. Let $u = f^{-1}(x) \qquad dv = dx$

$$du = \left(\frac{d}{dx}f^{-1}(x)\right)dx \qquad v = x$$

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx$$

53. **(a)** Using $y = f^{-1}(x) = \sin^{-1} x$ and $f(y) = \sin y,\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have:

$$\int \sin^{-1}x\,dx = x\sin^{-1} x - \int \sin y\,dy$$
$$= x\sin^{-1} x + \cos y + C$$
$$= x\sin^{-1} x + \cos(\sin^{-1}x) + C$$

(b) $\int \sin^{-1} x\,dx$

$$= x\sin^{-1} x - \int x\left(\frac{d}{dx}\sin^{-1} x\right)dx$$
$$= x\sin^{-1} x - \int x\frac{1}{\sqrt{1-x^2}}\,dx$$
$$u = 1 - x^2,\ du = -2x\,dx$$
$$= x\sin^{-1} x + \frac{1}{2}\int u^{-1/2}du$$
$$= x\sin^{-1} x + u^{1/2} + C$$
$$= x\sin^{-1} x + \sqrt{1-x^2} + C$$

(c) $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

54. **(a)** Using $y = f^{-1}(x) = \tan^{-1}$ and $f(y) = \tan y,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$, we have:

$$\int \tan^{-1}x\,dx$$
$$= x\tan^{-1} x - \int \tan y\,dy$$
$$= x\tan^{-1} x - \ln|\sec y| + C$$
$$= x\tan^{-1} x + \ln|\cos y| + C$$
$$= x\tan^{-1} x + \ln\left|\cos\ (\tan^{-1} x)\right| + C$$

(b) $\int \tan^{-1} x\,dx$

$$= x\tan^{-1} x - \int x\left(\frac{d}{dx}\tan^{-1} x\right)dx$$
$$= x\tan^{-1} x - \int x\left(\frac{1}{1+x^2}\right)dx$$
$$u = 1 + x^2,\ du = 2x\,dx$$
$$= x\tan^{-1} x - \frac{1}{2}\int u^{-1}\,du$$
$$= x\tan^{-1} x - \frac{1}{2}\ln|u| + C$$
$$= x\tan^{-1} x - \frac{1}{2}\ln\ (1+x^2) + C$$

(c) $\ln\left|\cos(\tan^{-1}x)\right| = \ln\left|\frac{1}{\sqrt{1+x^2}}\right|$

$$= -\frac{1}{2}\ln(1+x^2)$$

55. **(a)** Using $y = f^{-1}(x) = \cos^{-1} x$ and $f(y) = \cos x$, $0 \le x \le \pi$, we have:

$$\int \cos^{-1} x\, dx = x\cos^{-1} x - \int \cos y\, dy$$
$$= x\cos^{-1} x - \sin y + C$$
$$= x\cos^{-1} x - \sin\,(\cos^{-1} x) + C$$

(b) $\int \cos^{-1} x\, dx$

$$= x\cos^{-1} x - \int x\left(\frac{d}{dx}\cos^{-1} x\right) dx$$
$$= x\cos^{-1} x - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right) dx$$

$u = 1 - x^2$, $du = -2x\, dx$

$$= x\cos^{-1} x - \frac{1}{2}\int u^{-1/2}\, du$$
$$= x\cos^{-1} x - u^{1/2} + C$$
$$= x\cos^{-1} x - \sqrt{1-x^2} + C$$

(c) $\sin\,(\cos^{-1} x) = \sqrt{1-x^2}$

56. **(a)** Using $y = f^{-1}(x) = \log_2 x$ and $f(y) = 2^y$, we have

$$\int \log_2 x\, dx = x\log_2 x - \int 2^y\, dy$$
$$= x\log_2 x - \frac{2^y}{\ln\,2} + C$$
$$= x\log_2 x - \frac{1}{\ln\,2}2^{\log_2 x}$$

(b) $$\int \log_2 x\, dx = x\log_2 x - \int x\left(\frac{d}{dx}\log_2 x\right) dx$$
$$= x\log_2 x - \int x\left(\frac{1}{x\ln 2}\right) dx$$
$$= x\log_2 x - \int \frac{dx}{\ln\,2}$$
$$= x\log_2 x - \left(\frac{1}{\ln\,2}\right)x + C$$

(c) $2^{\log_2 x} = x$

57. Let $u = \sec x$ $\quad dv = \sec^2 x\, dx$

$du = \sec x \tan x\, dx$ $\quad v = \tan x$

$$\int \sec^3 x\, dx$$
$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$
$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)dx$$
$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$\int \sec^3 x\, dx$$
$$= \sec x \tan x - \int \sec^3 x\, dx + \ln|\sec x + \tan x|$$

Add $\int \sec^3 x\, dx$ to both sides.

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x|$$

Multiply both sides by $\frac{1}{2}$.

$$\int \sec^3 x\, dx$$
$$= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$$

58. Let $u = \csc x$ $\quad dv = \csc^2 x\, dx$

$du = -\csc x \cot x\, dx$ $\quad v = -\cot x$

$$\int \csc^2 x\, dx$$
$$= -\csc x \cot x - \int \csc x \cot^2 x\, dx$$
$$= -\csc x \cot x - \int \csc x(\csc^2 x - 1)dx$$
$$= -\csc x \cot x - \int \csc^3 x\, dx + \int \csc x\, dx$$

$$\int \csc^3 x\, dx$$
$$= -\csc x \cot x - \int \csc^3 x\, dx - \ln|\csc x + \cot x|$$

Add $\int \csc^3 x\, dx$ to both sides.

$$2\int \csc^3 x\, dx = -\csc x \cot x - \ln|\csc x + \cot x|$$

Multiply both sides by $\frac{1}{2}$.

$$\int \csc^3 x\, dx$$
$$= -\frac{1}{2}(\csc x \cot x + \ln|\csc x + \cot x|) + C$$

Quick Quiz Sections 7.1–7.3

1. E

2. C; $\sqrt{x} = \sin y$; $x = \sin^2 y$; $dx = 2\sin y \cos y\, dy$

$$x = 0 \Rightarrow \sin y = \sqrt{0} \Rightarrow y = 0$$

$$x = \frac{1}{2} \Rightarrow \sin y = \frac{\sqrt{2}}{2} \Rightarrow y = \frac{\pi}{4}$$

$$\int_0^{1/2} \frac{\sqrt{x}}{\sqrt{1-x}}\, dx$$

$$= 2\int_0^{\pi/4} \frac{\sin y}{\sqrt{1-\sin^2 y}} \sin y \cos y\, dy$$

$$= 2\int_0^{\pi/4} \sin^2 y\, dy$$

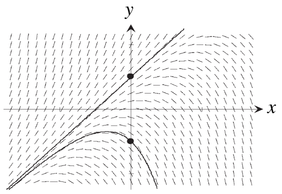
3. A; $\int xe^{2x}\, dx$

$$dv = e^{2x}dx \qquad v = \int e^{2x}dx = \frac{e^{2x}}{2}$$

$$u = x \qquad du = dx$$

$$\frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

4. (a)



(b) Let $\frac{dy}{dx} = 2$ and $y = 2x + b$ in the differential equation:

$$2 = 2(2x + b) - 4x$$
$$2 = 2b$$
$$b = 1$$

(c) First, note that $\frac{dy}{dx} = 2(0) - 4(0) = 0$ at the point (0, 0).

Also, $\frac{d^2y}{dx^2} = \frac{d}{dx}(2y - 4x) = 2\frac{dy}{dx} - 4$, which is –4 at the point (0, 0).
By the Second Derivative test, g has a local maximum at (0, 0).

Section 7.4 Exponential Growth and Decay (pp. 358–368)

Exploration 1 Choosing a Convenient Base

1. $2y_0 = y_0 2^{h\cdot 5}$

$$2 = 2^{5h}$$
$$5h = 1$$

$h = \frac{1}{5}$, h is the reciprocal of the doubling period.

2.

$$3 = 2^{\frac{1}{5}t}$$
$$\log 3 = \frac{1}{5}t\log 2$$
$$\frac{5\log 3}{\log 2} = t = 7.925 \text{ years.}$$

3. $3y_0 = y_0 3^{h\cdot 10}$

$$3^1 = 3^{10h}$$

$h = \frac{1}{10}$,

h is the reciprocal of the tripling period.

4.

$$2 = 3^{\frac{1}{10}t}$$
$$\log 2 = \frac{1}{10}t\log 3$$
$$\frac{10\log 2}{\log 3} = t = 6.3093 \text{ years.}$$

5. $\frac{1}{2}y_0 = y_0\left(\frac{1}{2}\right)^{h\cdot 15}$

$$\left(\frac{1}{2}\right)^1 = \left(\frac{1}{2}\right)^{15h}$$

$h = \frac{1}{15}$;

h is the reciprocal of the half-life.

6.

$$.10 = \left(\frac{1}{2}\right)^{\frac{1}{15}t}$$
$$\log(0.10) = \frac{1}{15}t\log\left(\frac{1}{2}\right)$$
$$\frac{15\log(0.10)}{\log\left(\frac{1}{2}\right)} = t = 49.83 \text{ years.}$$

Quick Review 7.4

1. $a = e^b$

2. $c = \ln d$

3.
$$\begin{aligned} \ln(x+3) &= 2 \\ x+3 &= e^2 \\ x &= e^2 - 3 \end{aligned}$$

4.
$$\begin{aligned} 100e^{2x} &= 600 \\ e^{2x} &= 6 \\ 2x &= \ln 6 \\ x &= \frac{1}{2}\ln 6 \end{aligned}$$

5.
$$\begin{aligned} 0.85^x &= 2.5 \\ \ln 0.85^x &= \ln 2.5 \\ x \ln 0.85 &= \ln 2.5 \\ x &= \frac{\ln 2.5}{\ln 0.85} \end{aligned}$$

6.
$$\begin{aligned} 2^{k+1} &= 3^k \\ \ln\ 2^{k+1} &= \ln\ 3^k \\ (k+1)\ln 2 &= k\ln 3 \\ \ln 2 &= k(\ln 3 - \ln 2) \\ k &= \frac{\ln 2}{\ln 3 - \ln 2} \end{aligned}$$

7.
$$\begin{aligned} 1.1^t &= 10 \\ \ln 1.1^t &= \ln 10 \\ t\ln 1.1 &= \ln 10 \\ t &= \frac{\ln 10}{\ln 1.1} = \frac{1}{\log 1.1} \end{aligned}$$

8.
$$\begin{aligned} e^{-2t} &= \frac{1}{4} \\ -2t &= \ln\left(\frac{1}{4}\right) \\ t &= -\frac{1}{2}\ln\left(\frac{1}{4}\right) = \frac{1}{2}\ln 4 = \ln 2 \end{aligned}$$

9.
$$\begin{aligned} \ln(y+1) &= 2x - 3 \\ y+1 &= e^{2x-3} \\ y &= -1 + e^{2x-3} \end{aligned}$$

10.
$$\begin{aligned} \ln|y+2| &= 3t - 1 \\ |y+2| &= e^{3t-1} \\ y+2 &= \pm e^{3t-1} \\ y &= -2 \pm e^{3t-1} \end{aligned}$$

Section 7.4 Exercises

1.
$$\begin{aligned} &\int y\,dy = \int x\,dx \\ &\frac{y^2}{2} = \frac{x^2}{2} + C \\ &(2)^2 = (1)^2 + C \\ &C = 3 \\ &y = \sqrt{x^2+3}, \text{ valid for all real numbers} \end{aligned}$$

2.
$$\begin{aligned} &\int y\,dy = -\int x\,dx \\ &\frac{y^2}{2} = -\frac{x^2}{2} + C \\ &(3)^2 = -(4)^2 + C \\ &C = 25 \\ &y = \sqrt{25 - x^2}, \text{ valid on the interval } (-5, 5) \end{aligned}$$

3.
$$\begin{aligned} &\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx \\ &\ln|y| = \ln|x| + C \\ &|y| = |x| + C \\ &2 = 2 + C \\ &C = 0 \\ &y = x, \text{ valid on the interval } (0, \infty) \end{aligned}$$

4.
$$\begin{aligned} &\int \frac{1}{y}\,dy = \int 2x\,dx \\ &\ln\ y = x^2 + C \\ &|y| = e^{x^2+C} = e^C e^{x^2} \\ &y = Ae^{x^2} \\ &3 = Ae^{0^2} \\ &3 = A \\ &y = 3e^{x^2}, \text{ valid for all real numbers} \end{aligned}$$

5.
$$\begin{aligned} &\int \frac{dy}{y+5} = \int (x+2)\,dx \\ &\ln|y+5| = \frac{x^2}{2} + 2x + C \\ &|y+5| = e^{x^2/2+2x+C} = e^C e^{x^2/2+2x} \\ &y+5 = \pm e^C e^{x^2/2+2x} = Ae^{x^2/2+2x} \\ &y = Ae^{x^2/2+2x} - 5 \\ &y = 6e^{x^2/2+2x} - 5, \end{aligned}$$

valid for all real numbers.

6. $\int \frac{dy}{\cos^2 y} = \int dx$

$\tan y = x + C$

$\tan(0) = 0 + C$

$C = 0$

$y = \tan^{-1} x,$ valid for all real numbers.

7. $\frac{dy}{dx} = \cos\, x\, e^{y} e^{\sin\, x}$

$\int e^{-y}\, dy = \int \cos\, x\, e^{\sin\, x}\, dx$

$-e^{-y} = e^{\sin\, x} + C$

$-e^{0} = e^{\sin\, 0} + C$

$C = -2$

$y = -\ln\,(2e - e^{\sin\, x}),$ valid for all real numbers.

8. $\frac{dy}{dx} = e^{-y} e^{x}$

$\int e^{y}\, dy = \int e^{x}\, dx$

$e^{y} = e^{x} + C$

$C = e^{2} - e^{0} = e^{2} - 1$

$y = \ln\,(e^{x} + e^{2} - 1),$ valid for all real numbers.

9. $\int \frac{1}{y^2}\, dy = \int -2x\, dx$

$-y^{-1} = -x^{2} + C$

$\frac{1}{.25} = 1 + C$

$C = 3$

$y = \frac{1}{x^{2} + 3},$ valid for all real numbers.

10. $\frac{dy}{dx} = \frac{4\sqrt{y}\, \ln\, x}{x}$

$\int \frac{dy}{\sqrt{y}} = \int \frac{4 \ln x}{x}\, dx$

$u = \ln\, x$

$du = \frac{1}{x}\, dx$

$2\sqrt{y} = \int 4u\, du$

$2\sqrt{y} = 2u^{2} + C$

$y = (\ln\, x)^{4} + C$

$1 = (\ln\, e) + C$

$C = 0$

$y = (\ln\, x)^{4},$ valid on the interval $(0, \infty)$.

11. $y(t) = y_0 e^{kt}$

$y(t) = 100e^{1.5t}$

12. $y(t) = y_0 e^{kt}$

$y(t) = 200e^{-0.5t}$

13. $y(t) = y_0 e^{kt}$

$y(t) = 50e^{kt}$

$y(5) = 100 = 50e^{5k}$

$2 = e^{5k}$

$\ln 2 = 5k$

$k = 0.2 \ln 2$

Solution: $y(t) = 50e^{(0.2\, \ln 2)t}$ or

$y(t) = 50 \cdot 2^{0.2t}$

14. $y(t) = y_0 e^{kt}$

$y(t) = 60e^{kt}$

$y(10) = 30 = 60e^{10k}$

$\frac{1}{2} = e^{10k}$

$\ln \frac{1}{2} = 10k$

$k = 0.1 \ln \frac{1}{2} = -0.1 \ln 2$

Solution: $y(t) = 60e^{-(0.1\, \ln 2)t}$ or

$y(t) = 60 \cdot 2^{-t/10}$

15. Doubling time:

$A(t) = A_0 e^{rt}$

$2000 = 1000e^{0.086t}$

$2 = e^{0.086t}$

$\ln 2 = 0.086t$

$t = \frac{\ln 2}{0.086} \approx 8.06$ yr

Amount in 30 years:

$A = 1000e^{(0.086)(30)} \approx \$13{,}197.14$

16. Annual rate:

$A(t) = A_0 e^{rt}$

$4000 = 2000e^{(r)(15)}$

$2 = e^{15r}$

$\ln 2 = 15r$

$r = \frac{\ln 2}{15} \approx 0.0462 = 4.62\%$

Amount in 30 years:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ A &= 2000e^{[(\ln 2)/15](30)} \\ &= 2000e^{2\ln 2} \\ &= 2000\cdot 2^2 \\ &= \$8000 \end{aligned}$$

17. Initial deposit:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 2898.44 &= A_0 e^{(0.0525)(30)} \\ A_0 &= \frac{2898.44}{e^{1.575}} \approx \$600.00 \end{aligned}$$

Doubling time:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 1200 &= 600e^{0.0525t} \\ 2 &= e^{0.0525t} \\ \ln 2 &= 0.0525t \\ t &= \frac{\ln 2}{0.0525} \approx 13.2 \text{ years} \end{aligned}$$

18. Annual rate:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 10{,}405.37 &= 1200e^{(r)(30)} \\ \frac{104.0537}{12} &= e^{30r} \\ \ln\frac{104.0537}{12} &= 30r \\ r &= \frac{1}{30}\ln\frac{104.0537}{12} \approx 0.072 = 7.2\% \end{aligned}$$

Doubling time:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 2400 &= 1200e^{0.072t} \\ 2 &= e^{0.072t} \\ \ln 2 &= 0.072t \\ t &= \frac{\ln 2}{0.072} \approx 9.63 \text{ years} \end{aligned}$$

19. **(a)** Annually:

$$\begin{aligned} 2 &= 1.0475^t \\ \ln 2 &= t\ln 1.0475 \\ t &= \frac{\ln 2}{\ln 1.0475} \approx 14.94 \text{ years} \end{aligned}$$

(b) Monthly:

$$\begin{aligned} 2 &= \left(1+\frac{0.0475}{12}\right)^{12t} \\ \ln 2 &= 12t\ln\left(1+\frac{0.0475}{12}\right) \\ t &= \frac{\ln 2}{12\ln\left(1+\frac{0.0475}{12}\right)} \approx 14.62 \text{ years} \end{aligned}$$

(c) Quarterly:

$$\begin{aligned} 2 &= \left(1+\frac{0.0475}{4}\right)^{4t} \\ \ln 2 &= 4t\ln 1.011875 \\ t &= \frac{\ln 2}{4\ln 1.011875} \approx 14.68 \text{ years} \end{aligned}$$

(d) Continuously:

$$\begin{aligned} 2 &= e^{0.0475t} \\ \ln 2 &= 0.0475t \\ t &= \frac{\ln 2}{0.0475} \approx 14.59 \text{ years} \end{aligned}$$

20. **(a)** Annually:

$$\begin{aligned} 2 &= 1.0825^t \\ \ln 2 &= t\ln 1.0825 \\ t &= \frac{\ln 2}{\ln 1.0825} \approx 8.74 \text{ years} \end{aligned}$$

(b) Monthly:

$$\begin{aligned} 2 &= \left(1+\frac{0.0825}{12}\right)^{12t} \\ \ln 2 &= 12t\ln\left(1+\frac{0.0825}{12}\right) \\ t &= \frac{\ln 2}{12\ln\left(1+\frac{0.0825}{12}\right)} \approx 8.43 \text{ years} \end{aligned}$$

(c) Quarterly:

$$\begin{aligned} 2 &= \left(1+\frac{0.0825}{4}\right)^{4t} \\ \ln 2 &= 4t\ln 1.020625 \\ t &= \frac{\ln 2}{4\ln 1.020625} \approx 8.49 \text{ years} \end{aligned}$$

(d) Continuously:

$$\begin{aligned} 2 &= e^{0.0825t} \\ \ln 2 &= 0.0825t \\ t &= \frac{\ln 2}{0.0825} \approx 8.40 \text{ years} \end{aligned}$$

21. $\frac{dy}{dt} = -0.0077y$

$$\int \frac{1}{y}\,dy = \int -0.0077\,dt$$
$$\ln\,y = -0.0077t$$
$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.0077} = 90 \text{ years}$$

22. $\frac{dy}{dt} = -ky$

$$\int \frac{1}{y}\,dy = \int (-k)\,dt$$
$$\ln\,y = -kt$$
$$-\frac{\ln\left(\frac{1}{2}\right)}{65} = k$$
$$k = 0.01067$$

23. **(a)** Since there are 48 half-hour doubling times in 24 hours, there will be $2^{48} \approx 2.8\times10^{14}$ bacteria.

(b) The bacteria reproduce fast enough that even if many are destroyed there are still enough left to make the person sick.

24. Using $y = y_0e^{kt}$, we have $10{,}000 = y_0e^{3k}$ and $40{,}000 = y_0e^{5k}$. Hence $\frac{40{,}000}{10{,}000} = \frac{y_0e^{5k}}{y_0e^{3k}}$, which gives $e^{2k} = 4$, or $k = \ln\;2$. Solving $10{,}000 = y_0e^{3\ln\,2}$, we have $y_0 = 1250$. There were 1250 bacteria initially. We could solve this more quickly by noticing that the population increased by a factor of 4, i.e., doubled twice, in 2 hrs, so the doubling time is 1 hr. Thus in 3 hrs the population would have doubled 3 times, so the initial population was $\frac{10{,}000}{2^3} = 1250.$

25.
$$0.9 = e^{-0.18t}$$
$$\ln 0.9 = -0.18t$$
$$t = -\frac{\ln 0.9}{0.18} \approx 0.585 \text{ day}$$

26. **(a)** Half-life $= \frac{\ln\;2}{k} = \frac{\ln 2}{0.005} \approx 138.6$ days

(b)
$$0.05 = e^{-0.005t}$$
$$\ln 0.05 = -0.005t$$
$$t = -\frac{\ln 0.05}{0.005} \approx 599.15 \text{ days}$$

The sample will be useful for about 599 days.

27. Since $y_0 = y(0) = 2$, we have:

$$y = 2e^{kt}$$
$$5 = 2e^{(k)(2)}$$
$$\ln 5 = \ln 2 + 2k$$
$$k = \frac{\ln 5 - \ln 2}{2} = 0.5\ln 2.5$$

Function: $y = 2e^{(0.5\ln 2.5)t}$ or $y \approx 2e^{0.4581t}$

28. Since $y_0 = y(0) = 1.1$, we have:

$$y = 1.1e^{kt}$$
$$3 = 1.1e^{(k)(-3)}$$
$$\ln 3 = \ln 1.1 - 3k$$
$$k = \frac{1}{3}(\ln 1.1 - \ln 3)$$

Function: $y = 1.1e^{(\ln 1.1 - \ln 3)t/3}$ or

$y \approx 1.1e^{-0.3344t}$

29. At time $t = \frac{3}{k}$, the amount remaining is $y_0e^{-kt} = y_0e^{-k(3/k)} = y_0e^{-3} \approx 0.0498y_0$. This is less than 5% of the original amount, which means that over 95% has decayed already.

30.
$$T - T_s = (T_0 - T_s)\,e^{-kt}$$
$$35 - 65 = (T_0 - 65)e^{-(k)(10)}$$
$$50 - 65 = (T_0 - 65)e^{-(k)(20)}$$

Dividing the first equation by the second, we have:

$$2 = e^{10k}$$
$$k = \frac{1}{10}\ln 2$$

Substituting back into the first equation, we have:

$$-30 = (T_0 - 65)e^{-[(\ln 2)/10](10)}$$
$$-30 = (T_0 - 65)\left(\frac{1}{2}\right)$$
$$-60 = T_0 - 65$$
$$5 = T_0$$

The beam's initial temperature is 5°F.

31. **(a)** First, we find the value of k.

$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$60 - 20 = (90 - 20)e^{-(k)(10)}$$
$$\frac{4}{7} = e^{-10k}$$
$$k = -\frac{1}{10}\ln\frac{4}{7}$$

When the soup cools to 35°, we have:

$$35 - 20 = (90 - 20)e^{[(1/10)\ln(4/7)]t}$$
$$15 = 70e^{[(1/10)\ln(4/7)]t}$$
$$\ln\frac{3}{14} = \left(\frac{1}{10}\ln\frac{4}{7}\right)t$$
$$t = \frac{10\ln\left(\frac{3}{14}\right)}{\ln\left(\frac{4}{7}\right)} \approx 27.53 \text{ min}$$

It takes a total of about 27.53 minutes, which is an additional 17.53 minutes after the first 10 minutes.

(b) Using the same value of k as in part (a), we have:

$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$35 - (-15) = [90 - (-15)]e^{[(1/10)\ln(4/7)]t}$$
$$50 = 105e^{[(1/10)\ln(4/7)]t}$$
$$\ln\frac{10}{21} = \left(\frac{1}{10}\ln\frac{4}{7}\right)t$$
$$t = \frac{10\ln\left(\frac{10}{21}\right)}{\ln\left(\frac{4}{7}\right)} \approx 13.26$$

It takes about 13.26 minute

32. First, we find the value of k. Taking "right now" as $t = 0$, 60° above room temperature means $T_0 - T_s = 60$. Thus, we have

$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$70 = 60e^{(-k)(-20)}$$
$$\frac{7}{6} = e^{20k}$$
$$k = \frac{1}{20}\ln\frac{7}{6}$$

(a) $T - T_s = (T_0 - T_s)\, e^{-kt}$

$$= 60e^{(-(1/20)\ln(7/6))(15)} \approx 53.45$$

It will be about 53.45°C above room temperature.

(b) $T - T_s = (T_0 - T_s)\, e^{-kt}$

$$= 60e^{(-(1/20)\ln(7/6))(120)} \approx 23.79$$

It will be about 23.79° above room temperature.

(c) $T - T_s = (T_0 - T_s)e^{-kt}$

$$10 = 60e^{(-(1/20)\ln(7/6))t}$$
$$\ln\frac{1}{6} = \left(-\frac{1}{20}\ln\frac{7}{6}\right)t$$
$$t = -\frac{20\ln\left(\frac{1}{6}\right)}{\ln\left(\frac{7}{6}\right)} \approx 232.47 \text{ min}$$

It will take about 232.47 min or 3.9 hr.

33. **(a)** The bird gains weight faster at $B = 40$ then at $B = 70$, since $\frac{dB}{dt} = 0.2(100 - B)$ takes on smaller values for larger values of B.

(b) $$\frac{d^2B}{dt^2} = \frac{d}{dt}\left(\frac{dB}{dt}\right) = 0.2(-1)\frac{dB}{dt}$$
$$= -0.04(100 - B)$$

Thus $\frac{d^2B}{dt^2} < 0$ for all B between 0 and 100. The graph of the function must be concave down throughout the interval shown, so it cannot resemble the given graph.

(c) $$\frac{dB}{dt} = 0.2(100 - B)$$
$$\frac{dB}{100 - B} = 0.2\,dt$$
$$\int\frac{1}{100 - B}dB = \int 0.2\,dt$$
$$-\ln|100 - B| = 0.2t + C$$
$$100 - B = e^{-0.2t + C}$$
$$100 - B = Ae^{-0.2t}$$
$$B(t) = 100 - Ae^{-0.2t}$$
$$B(0) = 20 \Rightarrow 20 = 100 - Ae^0 \Rightarrow A = 80$$

The function is $B(t) = 100 - 80e^{-0.2t}$.

34. **(a)** At (0, 1200),

$$\frac{dW}{dt} = 0.04(1200 - 300) = 36.$$

The line is $W - 1200 = 36(t - 0)$ or $W = 36t + 1200$. After 3 months, the approximate amount is

$W\left(\frac{1}{4}\right)=36\left(\frac{1}{4}\right)+1200=1209$ tons of solid waste.

(b) $$\frac{d^2W}{dt^2}=\frac{d}{dt}[0.04(W-300)]$$
$$=0.04\frac{dW}{dt}$$
$$=0.0016(W-300)$$

$\frac{d^2W}{dt}>0$ for $W>300$ so the graph of W is concave up for $t\geq 0$. Thus the tangent line approximation is an underestimate.

(c) $$\frac{dW}{dt}=0.04(W-300)$$
$$\frac{1}{W-300}dW=0.04\,dt$$
$$\int\frac{1}{W-300}dW=\int 0.04\,dt$$
$$\ln|W-300|=0.04t+C$$
$$|W-300|=e^{0.04t+C}$$
$$W-300=Ae^{0.04t}$$
$$W(t)=Ae^{0.04t}+300$$

$W(0)=1200\Rightarrow$

$1200=Ae^0+300\Rightarrow A=900$

The function is $W(t)=900e^{0.04t}+300$.

35. Use $k=\frac{\ln\ 2}{5700}$ (see Example 5).

$$e^{-kt}=0.445$$
$$-kt=\ln 0.445$$
$$t=-\frac{\ln 0.445}{k}$$
$$=-\frac{5700\ln 0.445}{\ln 2}\approx 6658\text{ years}$$

Crater Lake is about 6658 years old.

36. Use $k=\frac{\ln\ 2}{5700}$ (see Example 5).

(a) $$e^{-kt}=0.17$$
$$-kt=\ln 0.17$$
$$t=-\frac{\ln 0.17}{k}$$
$$=-\frac{5700\ln 0.17}{\ln 2}\approx 14{,}571\text{ years}$$

The animal died about 14,571 years before A.D. 2000, in 12,571 B.C.E.

(b) $$e^{-kt}=0.18$$
$$-kt=\ln 0.18$$
$$t=-\frac{\ln 0.18}{k}$$
$$=-\frac{5700\ln 0.18}{\ln 2}\approx 14{,}101\text{ years}$$

The animal died about 14,101 years before C.E. 2000, in 12,101 B.C.E.

(c) $$e^{-kt}=0.16$$
$$-kt=\ln 0.16$$
$$t=-\frac{\ln 0.16}{k}$$
$$=-\frac{5700\ln 0.16}{\ln 2}\approx 15{,}070\text{ years}$$

The animal died about 15,070 years before C.E. 2000, in 13,070 B.C.E.

37. $$\frac{1}{3}=e^{-kt}$$
$$k=\frac{\ln\left(\frac{1}{3}\right)}{5}=0.22$$
$$-\frac{\ln\left(\frac{1}{3}\right)}{5}=-0.22$$
$$\frac{1}{2}=e^{-0.22t}$$
$$t=\frac{\ln\left(\frac{1}{2}\right)}{-0.22}=3.15\text{ years}$$

38. $$3=e^{rt}$$
$$r=\frac{\ln(3)}{10}=0.11$$
$$4=e^{0.11t}$$
$$t=\frac{\ln(4)}{0.11}=12.60\text{ years}$$

39. $$y=y_0e^{-kt}$$
$$800=1000e^{-(k)(10)}$$
$$0.8=e^{-10k}$$
$$k=-\frac{\ln 0.8}{10}$$

At $t=10+14=24$ h:

$$y=1000e^{-(-\ln 0.8/10)24}$$
$$=1000e^{2.4\ln 0.8}\approx 585.4\text{ kg}$$

About 585.4 kg will remain.

40. $$0.2 = e^{-0.1t}$$
$$\ln 0.2 = -0.1t$$
$$t = -10\ln 0.2 \approx 16.09 \text{ yr}$$
It will take about 16.09 years.

41. (a) $$\frac{dp}{dh} = kp$$
$$\frac{dp}{p} = k\ dh$$
$$\int \frac{dp}{p} = \int k\ dh$$
$$\ln|p| = kh + C$$
$$e^{\ln|p|} = e^{kh+C}$$
$$|p| = e^C e^{kh}$$
$$p = Ae^{kh}$$
Initial condition: $p = p_0$ when $h = 0$
$$p_0 = Ae^0$$
$$A = p_0$$
Solution: $p = p_0 e^{kh}$

Using the given altitude-pressure data, we have $p_0 = 1013$ millibars, so:
$$p = 1013e^{kh}$$
$$90 = 1013e^{(k)(20)}$$
$$\frac{90}{1013} = e^{20k}$$
$$k = \frac{1}{20}\ln\frac{90}{1013} \approx -0.121 \text{ km}^{-1}$$
Thus, we have $p \approx 1013e^{-0.121h}$.

(b) At 50 km, the pressure is $1013e^{((1/20)\ln(90/1013))(50)}$ ≈ 2.383 millibars.

(c) $$900 = 1013e^{kh}$$
$$\frac{900}{1013} = e^{kh}$$
$$h = \frac{1}{k}\ln\frac{900}{1013}$$
$$= \frac{20\ln\left(\frac{900}{1013}\right)}{\ln\left(\frac{90}{1013}\right)} \approx 0.977 \text{ km}$$
The pressure is 900 millibars at an altitude of about 0.977 km.

42. By the Law of Exponential Change, $y = 100e^{-0.6t}$. At $t = 1$ hour, the amount remaining will be $100e^{-0.6(1)} \approx 54.88$ grams.

43. (a) By the Law of Exponential Change, the solution is $V = V_0 e^{-(1/40)t}$.

(b) $$0.1 = e^{-(1/40)t}$$
$$\ln 0.1 = -\frac{t}{40}$$
$$t = -40\ln 0.1 \approx 92.1 \text{ sec}$$
It will take about 92.1 seconds.

44. (a) $A(t) = Pe^t$

It grows by a factor of e each year.

(b) $$3 = e^t$$
$$\ln 3 = t$$
It will take $\ln 3 \approx 1.1$ yr.

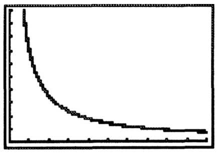
(c) In one year your account grows from A_0 to A_0e, so you can earn $A_0e - A_0$, or $(e - 1)$ times your initial amount. This represents an increase of about 172%.

45. (a) $$90 = e^{(r)(100)}$$
$$\ln 90 = 100r$$
$$r = \frac{\ln 90}{100} \approx 0.045 \text{ or } 4.5\%$$

(b) $$131 = e^{(r)(100)}$$
$$\ln 131 = 100r$$
$$r = \frac{\ln 131}{100} \approx 0.049 \text{ or } 4.9\%$$

46. (a) $$2y_0 = y_0 e^{rt}$$
$$2 = e^{rt}$$
$$\ln 2 = rt$$
$$t = \frac{\ln 2}{r}$$

(b)



[0, 0.1] by [0, 100]

(c) $\ln 2 \approx 0.69$, so the doubling time is $\frac{0.69}{r}$ which is almost the same as the rules.

(d) $\frac{70}{5} = 14$ years or $\frac{72}{5} = 14.4$ years

(e)
$$\begin{aligned} 3y_0 &= y_0e^{rt} \\ 3 &= e^{rt} \\ \ln 3 &= rt \\ t &= \frac{\ln 3}{r} \end{aligned}$$

Since ln 3 ≈ 1.099, a suitable rule is $\frac{108}{100r}$ or $\frac{108}{i}$.

(We choose 108 instead of 110 because 108 has more factors.)

47. False; the correct solution is $|y| = e^{kx+C}$, which can be written (with a new C) as $y = Ce^{kx}$.

48. True; the differential equation is solved by an exponential equation that can be written in any base. Note that $Ce^{2t} = C(3^{kt})$ when $k = \frac{2}{\ln 3}$.

49. D; $\mathrm{A}(t) = \mathrm{A}_0e^{rt}$
$$\begin{aligned} 2 &= 1e^{7r} \\ r &= \frac{\ln(2)}{7} = 0.099 \\ t &= \frac{\ln(3)}{0.099} = 11.1 \end{aligned}$$

50. C;
$$\begin{aligned} A &= A_0\left(\frac{1}{2}\right)^{t/r} \\ 1 &= 100\left(\frac{1}{2}\right)^{199/r} \\ \ln(.01) &= \frac{199}{r}\ln(0.5) \\ r &= \frac{199\ln(0.5)}{\ln(0.01)} = 30 \end{aligned}$$

51. D

52. E;
$$\begin{aligned} T - 68 &= (425 - 68)e^{-kt} \\ 195 - 68 &= 357e^{-30k} \\ e^{-30k} &= \frac{127}{357} = 0.356 \\ k &= \frac{\ln(0.356)}{-30} = .0344 \\ 100 &= 68 + 357e^{(-0.0344)t} \\ t &= \frac{\ln\left(\frac{100-68}{357}\right)}{-0.0344} = 70 \text{ min} \end{aligned}$$
$$70 - 30 = 40$$

53. (a) Since acceleration is $\frac{dv}{dt}$, we have

$$\text{Force} = m\frac{dv}{dt} = -kv.$$

(b) From $m\frac{dv}{dt} = -kv$ we get $\frac{dv}{dt} = -\frac{k}{m}v$, which is the differential equation for exponential growth modeled by $v = Ce^{-(k/m)t}$. Since $v = v_0$ at $t = 0$, it follows that $C = v_0$.

(c) In each case, we would solve $2 = e^{-(k/m)t}$. If k is constant, an increase in m would require an increase in t. The object of larger mass takes longer to slow down. Alternatively, one can consider the equation $\frac{dv}{dt} = -\frac{k}{m}v$ to see that v changes more slowly for larger values of m.

54. (a) $s(t) = \int v_0e^{-(k/m)t}dt = -\frac{v_0m}{k}e^{-(k/m)t} + C$

Initial condition: $s(0) = 0$
$$\begin{aligned} 0 &= -\frac{v_0m}{k} + C \\ \frac{v_0m}{k} &= C \\ s(t) &= -\frac{v_0m}{k}e^{-(k/m)t} + \frac{v_0m}{k} \\ &= \frac{v_0m}{k}(1 - e^{-(k/m)t}) \end{aligned}$$

(b) $\lim\limits_{t\to\infty} s(t) = \lim\limits_{t\to\infty} \frac{v_0m}{k}(1 - e^{-(k/m)t}) = \frac{v_0m}{k}$

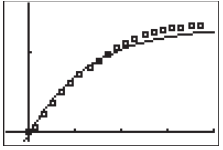
55.
$$\begin{aligned} \frac{v_0m}{k} &= \text{coasting distance} \\ \frac{(1.0)(49.90)}{k} &= 1.32 \\ k &= \frac{4990}{132} \end{aligned}$$

We know that $\frac{v_0m}{k} = 1.32$ and $\frac{k}{m} = \frac{4990}{132(49.9)} = \frac{25}{33}$.

We have:

$$s(t)=\frac{v_0m}{k}(1-e^{-(k/m)t})$$
$$=1.32(1-e^{-25t/33})$$
$$\approx 1.32(1-e^{-0.758t})$$

A graph of the model is shown superimposed on a graph of the data.



[–0.5, 4] by [–0.2, 1.6]

56. (a)

x	$\left(1+\frac{1}{x}\right)^x$
10	2.5937
100	2.7048
1000	2.7169
10,000	2.7181
100,000	2.7183

$e \approx 2.7183$

(b) $r=2$

x	$\left(1+\frac{2}{x}\right)^x$
10	6.1917
100	7.2446
1000	7.3743
10,000	7.3876
100,000	7.3889

$e^2 \approx 7.389$

$r=0.5$

x	$\left(1+\frac{0.5}{x}\right)^x$
10	1.6289
100	1.6467
1000	1.6485
10,000	1.6487
100,000	1.6487

$e^{0.5} \approx 1.6487$

(c) As we compound more times, the increment of time between compounding approaches 0. Continuous compounding is based on an instantaneous rate of change which is a limit of average rates as the increment in time approaches 0.

57. (a) To simplify calculations somewhat, we may write:

$$v(t)=\sqrt{\frac{mg}{k}}\frac{(e^{at}-e^{-at})e^{at}}{(e^{at}+e^{-at})e^{at}}$$
$$=\sqrt{\frac{mg}{k}}\frac{e^{2at}-1}{e^{2at}+1}$$
$$=\sqrt{\frac{mg}{k}}\frac{(e^{2at}+1)-2}{e^{2at}+1}$$
$$=\sqrt{\frac{mg}{k}}\left(1-\frac{2}{e^{2at}+1}\right)$$

The left side of the differential equation is:

$$m\frac{dv}{dt}=m\sqrt{\frac{mg}{k}}(2)(e^{2at}+1)^{-2}(2ae^{2at})$$
$$=4ma\sqrt{\frac{mg}{k}}(e^{2at}+1)^{-2}(e^{2at})$$
$$=4m\sqrt{\frac{gk}{m}}\sqrt{\frac{mg}{k}}(e^{2at}+1)^{-2}(e^{2at})$$
$$=\frac{4mge^{2at}}{(e^{2at}+1)^2}$$

The right side of the differential equation is:

$$mg-kv^2$$
$$=mg-k\left(\frac{mg}{k}\right)\left(1-\frac{2}{e^{2at}+1}\right)^2$$
$$=mg\left[1-\left(1-\frac{2}{e^{2at}+1}\right)^2\right]$$
$$=mg\left(1-1+\frac{4}{e^{2at}+1}-\frac{4}{(e^{2at}+1)^2}\right)$$
$$=mg\frac{4(e^{2at}+1)-4}{(e^{2at}+1)^2}$$
$$=\frac{4\,mg\,e^{2at}}{(e^{2at}+1)^2}$$

Since the left and right sides are equal, the differential equation is satisfied.

And $v(0)=\sqrt{\frac{mg}{k}}\frac{e^0-e^0}{e^0+e^0}=0,$ so the initial condition is also satisfied.

(b) $$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(\sqrt{\frac{mg}{k}}\frac{e^{at}-e^{-at}}{e^{at}+e^{-at}}\bullet\frac{e^{-at}}{e^{-at}}\right)$$
$$= \lim_{t\to\infty}\left(\sqrt{\frac{mg}{k}}\frac{1-e^{-2at}}{1+e^{-2at}}\right)$$
$$= \sqrt{\frac{mg}{k}}\left(\frac{1-0}{1+0}\right)$$
$$= \sqrt{\frac{mg}{k}}$$

The limiting velocity is $\sqrt{\frac{mg}{k}}$.

(c) $\sqrt{\frac{mg}{k}} = \sqrt{\frac{160}{0.005}} \approx 179$ ft/sec
The limiting velocity is about 179 ft/sec, or about 122 mi/hr.

Section 7.5 Logistic Growth (pp. 369–378)

Exploration 1 Exponential Growth Revisited

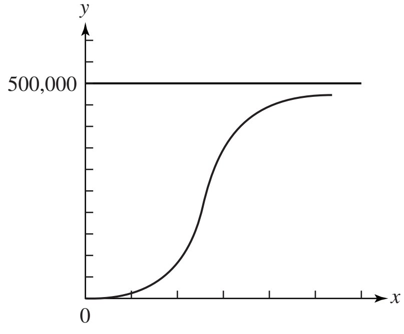
1. $100(2)^{12} = 409{,}600$

2. $100(2)^{(12\cdot 24)} = 4.97\times 10^{88}$

3. No; this number is much larger than the estimated number of atoms.

4. $500{,}000 = 100(2)^x$
$\frac{\log 5000}{\log 2} = x = 12.29$ hours

5.



Exploration 2 Learning From the Differential Equation

1. $\frac{dP}{dt}$ will be close to zero when P is close to 0 and when P is close to M.

2. P is half the value of M at its vertex.

3. When $P = \frac{M}{2}$, $\frac{dP}{dt} = kP(M-P)$ is at its maximum.

4. When the initial population is less than M, the initial growth rate is positive.

5. When the initial population is more than M, the initial growth rate is negative.

6. When the initial population is equal to M, the growth rate is 0.

7. $\lim_{t\to\infty} P(t) = M$, regardless of the initial population. The limit depends only on M.

Quick Review 7.5

1. $$\begin{array}{r} x+1 \\ x-1\overline{)\,x^2} \\ \underline{x^2-x} \\ x \\ \underline{x-1} \\ 1 \end{array}$$

$x+1+\frac{1}{x-1}$

2. $$\begin{array}{r} 1 \\ x^2-4\overline{)\,x^2} \\ \underline{x^2-4} \\ 4 \end{array}$$

$1+\frac{4}{x^2-4}$

3. $$\begin{array}{r} 1 \\ x^2+x-2\overline{)\,x^2+x+1} \\ \underline{x^2+x-2} \\ 3 \end{array}$$

$1+\frac{3}{x^2+x-2}$

4. $$\begin{array}{r} x \\ x^2-1\overline{)\,x^3\quad -5} \\ \underline{x^3-x} \\ x-5 \end{array}$$

$x+\frac{x-5}{x^2-1}$

5. $(-\infty, \infty)$

6. $$\lim_{x\to\infty}\frac{60}{1+5e^{-0.1x}}=\frac{60}{1+5(0)}=60$$

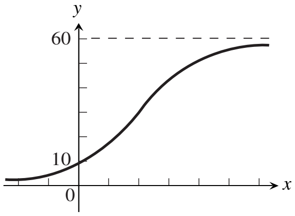
7. As $x \to -\infty$, $-0.1x \to \infty$, and $e^{-0.1x} \to \infty$, so

$$\lim_{x\to-\infty}\frac{60}{1+5e^{-0.1x}}=0.$$

8. $$y(0)=\frac{60}{1+5\ e^{(-0.1(0))}}=10$$

9. From problems 6 and 7, the two horizontal asymptotes are $y = 0$ and $y = 60$.

10.



Section 7.5 Exercises

1. $A(x-4)+B(x)=x-12$

$$\begin{aligned} x=4,\quad 4B&=-8\\ B&=-2\\ x=1,\ A(1-4)+(-2)(1)&=1-12\\ A&=3\end{aligned}$$

2. $A(x-2)+B(x+3)=2x+16$

$$\begin{aligned} x=2,\ B(2+3)&=2(2)+16\\ 5B&=20\\ B&=4\\ x=-3,\ A(-3-2)&=2(-3)+16\\ -5A&=10\\ A&=-2\end{aligned}$$

3. $A(x+5)+B(x-2)=16-x$

$$\begin{aligned} x=-5,\ B(-5-2)&=16-(-5)\\ -7B&=21\\ B&=-3\\ x=2,\ A(2+5)&=16-2\\ 7A&=14\\ A&=2\end{aligned}$$

4. $A(x+3)+B(x-3)=3$

$$\begin{aligned} x=-3,\ B(-3-3)&=3\\ -6B&=3\\ B&=-\frac{1}{2}\\ x=3,\ A(3+3)&=3\\ 6A&=3\\ A&=\frac{1}{2}\end{aligned}$$

5. See problem 1.

$$\begin{aligned}\int\frac{x-12}{x^2-4}dx&=\int\left(\frac{3}{x}+\frac{-2}{x-4}\right)dx\\ &=3\ \ln|x|-2\ \ln|x-4|+C\\ &=\ln\left(\frac{|x|^3}{(x-4)^2}\right)+C\end{aligned}$$

6. See problem 2.

$$\begin{aligned}&\int\frac{2x+16}{x^2+x-6}dx\\ &=\int\left(\frac{-2}{x+3}+\frac{4}{x-2}\right)dx\\ &=-2\ln|x+3|+4\ln|x-2|+C\\ &=\ln\left(\frac{(x-2)^4}{(x+3)^2}\right)+C\end{aligned}$$

7. $$\begin{array}{r} 2x \\ x^2-4\overline{)2x^3} \\ \underline{2x^3-8x} \\ 8x \end{array}$$

$$\int\left(2x+\frac{8x}{x^2-4}\right)dx$$

$u=x^2-4$

$du=2x\ dx$

$$\begin{aligned}x^2+4\int\frac{du}{u}&=x^2+4\ln|u|+C\\ &=x^2+\ln(x^2-4)^4+C\end{aligned}$$

8. $$\begin{array}{r} 1 \\ x^2-9\overline{)x^2-6} \\ \underline{x^2-9} \\ 3 \end{array}$$

$$\int 1+\frac{3}{x^2-9}dx=x+\int\frac{A}{x+3}+\frac{B}{x-3}dx$$

$A(x-3)+B(x+3)=3$

$$x = 3,\ B(3+3) = 3$$
$$B = \frac{1}{2}$$
$$x = -3,\ A(-3-3) = 3$$
$$A = -\frac{1}{2}$$
$$x + \int \frac{-\frac{1}{2}}{x+3} + \frac{\frac{1}{2}}{x-3}\,dx = x + \ln\sqrt{\left|\frac{x-3}{x+3}\right|} + C$$

9. $2\int \frac{dx}{x^2+1} = 2\tan^{-1} x + C$

10. $3\int \frac{dx}{x^2+9} = \tan^{-1}\left(\frac{x}{3}\right) + C$

11. $\int \frac{7}{2x^2-5x-3}\,dx$

$$\frac{A}{2x+1} + \frac{B}{x-3} = \frac{7}{(2x+1)(x-3)}$$
$$A(x-3) + B(2x+1) = 7$$
$$x = 3,\ B(2(3)+1) = 7$$
$$B = 1$$
$$x = -\frac{1}{2},\ A\left(-\frac{1}{2}-3\right) = 7$$
$$A = -2$$
$$\int\left(\frac{-2}{2x+1} + \frac{1}{x-3}\right)dx = \ln\left|\frac{x-3}{2x+1}\right| + C$$

12. $\int \frac{1-3x}{3x^2-5x-3}\,dx$

$$\frac{A}{3x-2} + \frac{B}{x-1} = \frac{1-3x}{(3x-2)(x-1)}$$
$$A(x-1) + B(3x-2) = 1-3x$$
$$x = 1,\ B(3(1)-2) = 1-3(1)$$
$$B = -2$$
$$x = \frac{2}{3},$$
$$A = \left(\frac{2}{3}-1\right) = 1-3\left(\frac{2}{3}\right)$$
$$-\frac{1}{3}A = -1$$
$$A = 3$$
$$\int\left(\frac{3}{3x-2} + \frac{-2}{x-1}\right)dx$$
$$= \ln|3x-2| - 2\ln|x-1| + C$$
$$= \ln\left|\frac{3x-2}{(x-1)^2}\right| + C$$

13. $\int \frac{8x-7}{2x^2-x-3}$

$$\frac{A}{x+1} + \frac{B}{2x-3} = \frac{8x-7}{(x+1)(2x-3)}$$
$$A(2x-3) + B(x+1) = 8x-7$$
$$x = \frac{3}{2},\ B\left(\frac{3}{2}+1\right) = 8\left(\frac{3}{2}\right) - 7$$
$$\frac{5}{2}B = 5$$
$$B = 2$$
$$x = -1,\ A(2x-3) = 8x-7$$
$$A(-2-3) = 8-7$$
$$-5A = -15$$
$$A = 3$$
$$\int\left(\frac{3}{x+1} + \frac{2}{2x-3}\right)dx$$
$$= 3\ln|x+1| + \ln|2x-3| + C$$
$$= \ln\left(|x+1|^3\,|2x-3|\right) + C$$

14. $\int \frac{5x+14}{x^2+7x}\,dx$

$$\frac{A}{x} + \frac{B}{x+7} = \frac{5x+14}{x(x+7)}$$
$$A(x+7) + Bx = 5x+14$$
$$x = -7,\ -7B = 5(-7) + 14$$
$$-7B = -21$$
$$B = 3$$
$$x = 0,\ A(0+7) = 5(0) + 14$$
$$7A = 14$$
$$A = 2$$
$$\int\left(\frac{2}{x} + \frac{3}{x+7}\right)dx = 2\ln|x| + 3\ln|x+7| + C$$
$$= \ln\left(x^2\,|x+7|^3\right) + C$$

15. $\int dy = \int \frac{2x-6}{x^2-2x}\,dx$

$$\frac{A}{x} + \frac{B}{x-2} = \frac{2x-6}{x(x-2)}$$
$$A(x-2) + Bx = 2x-6$$
$$x = 2,\ 2B = 2(2) - 6$$
$$2B = -2$$
$$B = -1$$

$x = 0,\ A(0-2) = 2(0) - 6$

$-2A = -6$

$A = 3$

$\int \left(\frac{3}{x} + \frac{-1}{x-2} \right) dx$

$y = 3 \ln|x| - \ln|x-2| + C$

$y = \ln\left|\frac{x^3}{x-2}\right| + C$

16. $\int du = \int \frac{2}{x^2 - 1}\, dx$

$\frac{A}{x+1} + \frac{B}{x-1} = \frac{2}{(x+1)(x-1)}$

$A(x-1) + B(x+1) = 2$

$x = 1,\ B(1+1) = 2$

$2B = 2$

$B = 1$

$x = -1,\ A(-1-1) = 2$

$-2A = 2$

$A = -1$

$u = \int \left(\frac{-1}{x+1} + \frac{1}{x-1} \right) dx$

$u = -\ln|x+1| + \ln|x-1| + C$

$u = \ln\left|\frac{x-1}{x+1}\right| + C$

17. $\int F'(x)\, dx = \int \frac{2}{x^3 - x}\, dx$

$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} = \frac{2}{x(x+1)(x-1)}$

$A(x+1)(x-1) + Bx(x-1) + Cx(x+1) = 2$

$x = 1,\ 2C = 2$

$C = 1$

$x = -1,\ 2B = 2$

$B = 1$

$x = 0,\ -A = 2$

$A = -2$

$\int \left(\frac{-2}{x} + \frac{1}{x+1} + \frac{1}{x-1} \right) dx$

$F(x) = -2\ln|x| + \ln|x+1| + \ln|x-1| + C$

$F(x) = \ln\left(\frac{|x^2 - 1|}{x^2} \right) + C$

18. $\int G'(t)\, dt = \int \frac{2t^3}{t^3 - t}\, dt$

$$\begin{array}{r|l} & 2 \\ t^3 - t & 2t^3 \\ & \underline{2t^3 - 2t} \\ & 2t \end{array}$$

$= \int 2 + \frac{2t}{t^3 - t}\, dt$

$= 2t + \int \frac{2}{t^2 - 1}\, dt$

$\frac{A}{t-1} + \frac{B}{t+1} = \frac{2}{(t-1)(t+1)}$

$A(t+1) + B(t-1) = 2$

$t = -1,\ B(-1-1) = 2$

$-2B = 2$

$B = -1$

$t = 1,\ A(1+1) = 2$

$2A = 2$

$A = 1$

$G(t) = 2t + \int \left(\frac{1}{(t-1)} + \frac{-1}{(t+1)} \right) dt$

$= 2t + \ln|t-1| - \ln|t+1| + C$

$= 2t + \ln\left|\frac{t-1}{t-1}\right| + C$

19. $\int \frac{2x}{x^2 - 4}\, dx$

$u = x^2 - 4$

$du = 2x\, dx$

$\int \frac{du}{u} = \ln u + C = \ln|x^2 - 4| + C$

20. $\int \frac{4x - 3}{2x^2 - 3x + 1}\, dx$

$u = 2x^2 - 3x + 1$

$du = (4x - 3)\, dx$

$\int \frac{du}{u} = \ln u + C = \ln|2x^2 - 3x + 1| + C$

21. $\int \frac{x^2+x-1}{x^2-x}\,dx$

$$\begin{array}{r} 1 \\ x^2-x\overline{\smash{)}x^2+x-1} \\ \underline{x^2-x} \\ 2x-1 \end{array}$$

$$\int\left(1+\frac{2x-1}{x^2-x}\right)dx$$

$$u = x^2-x$$
$$du = (2x-1)\,dx$$

$$x+\int\frac{du}{u} = x\ln u + C = x+\ln\left|x^2-x\right|+C$$

22. $\int \frac{2x^3}{x^2-1}\,dx$

$$\begin{array}{r} 2x \\ x^2-1\overline{\smash{)}2x^3} \\ \underline{2x^3-2x} \\ 2x \end{array}$$

$$\int\left(2x+\frac{2x}{x^2-1}\right)dx$$

$$u = x^2-1$$
$$du = 2x\,dx$$

$$x^2+\int\frac{du}{u} = x^2+\ln u + C = x^2+\ln\left|x^2-1\right|+C$$

23. **(a)** 200 individuals

(b) 100 individuals

(c) $\frac{dP(100)}{dt} = 0.006(100)(200-100)$
$= 60$ individuals per year.

24. **(a)** 700 individuals

(b) 350 individuals

(c) $\frac{dP(350)}{dt} = 0.0008(350)(700-350)$
$= 98$ individuals per year.

25. **(a)** 1200 individuals

(b) 600 individuals

(c) $\frac{dP(600)}{dt} = 0.0002(600)(1200-600)$
$= 72$ individuals per year.

26. **(a)** 5000 individuals

(b) 2500 individuals

(c) $\frac{dP(2500)}{dt} = 10^{-5}(2500)(5000-2500)$
$= 62.5$ individuals per year.

27. $\frac{dP}{dt} = 0.006\ P(200-P)$

$$\int\frac{dP}{P(200-P)} = \int 0.006\,dt$$

$$\frac{A}{P}+\frac{B}{200-P} = \frac{1}{P(200-P)}$$
$$A(200-P)+BP = 1$$
$$P = 200,\ 200B = 1$$
$$B = 0.005$$
$$P = 0,\ A(200-0) = 1$$
$$200A = 1$$
$$A = 0.005$$

$$\int\left(\frac{0.005}{P}+\frac{0.005}{200-P}\right)dP = 0.006t$$

$$\int\left(\frac{1}{P}+\frac{1}{200-P}\right)dP = 1.2t$$

$$\ln P - \ln\,(200-P) = 1.2t+C$$

$$\ln\left(\frac{200-P}{P}\right) = -1.2t-C$$

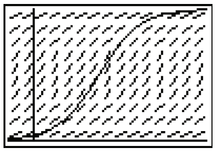
$$\frac{200}{P}-1 = e^{-1.2t}\ e^{-c}$$

$$\frac{200}{P} = 1+e^{-1.2t}\ e^{-c}$$

$$\frac{200}{8} = 1+e^{-1.2(0)}\ e^{-c}$$

$$e^{-c} = 24$$

$$P = \frac{200}{1+24\,e^{-1.2t}}$$



[–1, 7] by [0, 200]

28. $$\frac{dP}{dt}=0.0008\ P(700-P)$$

$$\int\frac{dP}{P(700-P)}=\int 0.0008\,dt$$

$$\frac{A}{P}+\frac{B}{700-P}=\frac{1}{P(700-P)}$$

$$A(700-P)+BP=1$$

$$P=700,\ 700B=1$$

$$B=\frac{1}{700}$$

$$P=0,\ A(700-0)=1$$

$$A=\frac{1}{700}$$

$$\int\left(\frac{\frac{1}{700}}{P}+\frac{\frac{1}{700}}{700-P}\right)dP=0.0008t$$

$$\int\left(\frac{1}{P}+\frac{1}{700-P}\right)dP=0.56t$$

$$\ln P-\ln(700-P)=0.56t+C$$

$$\ln\left(\frac{700-P}{P}\right)=-0.56t-C$$

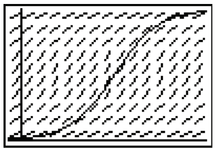
$$\frac{700}{P}-1=e^{-0.56t}\ e^{-c}$$

$$\frac{700}{P}=1+e^{-0.56t}\ e^{-c}$$

$$\frac{700}{10}=1+e^{-0.56(0)}\ e^{-c}$$

$$e^{-c}=69$$

$$P=\frac{700}{1+69e^{-0.56t}}$$



[–1, 15] by [0, 700]

29. $$\frac{dP}{dt}=0.0002P(1200-P)$$

$$\int\frac{dP}{P(1200-P)}=\int 0.0002\,dt$$

$$\frac{A}{P}+\frac{B}{1200-P}=\frac{1}{P(1200-P)}$$

$$A(1200-P)+BP=1$$

$$P=1200,\ 1200B=1$$

$$B=\frac{1}{1200}$$

$$P=0,\ A(1200-0)=1$$

$$1200A=1$$

$$A=\frac{1}{1200}$$

$$\int\left(\frac{\frac{1}{1200}}{P}+\frac{\frac{1}{1200}}{1200-P}\right)dP=0.0002t$$

$$\int\left(\frac{1}{P}+\frac{1}{1200-P}\right)dP=0.24t$$

$$\ln P-\ln(1200-P)=0.24t+C$$

$$\ln\left(\frac{1200-P}{P}\right)=-0.24\,t-C$$

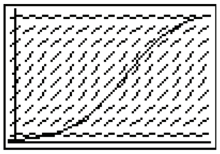
$$\frac{1200}{P}-1=e^{-0.24t}e^{-c}$$

$$\frac{1200}{P}=1+e^{-0.24t}e^{-c}$$

$$\frac{1200}{20}=1+e^{-0.24(0)}e^{-c}$$

$$e^{-c}=59$$

$$P=\frac{1200}{1+59\,e^{-0.24t}}$$



[–1, 30] by [0, 1200]

30. $$\frac{dP}{dt}=10^{-5}\ P(5000-P)$$

$$\int\frac{dP}{P(5000-P)}=\int 10^{-5}\ dt$$

$$\frac{A}{P}+\frac{B}{5000-P}=\frac{1}{P(5000-P)}$$

$$A(5000-P)+BP=1$$

$$P=5000,\ 5000B=1$$

$$B=0.0002$$

$$P=0,\ A(5000-0)=1$$

$$5000A=1$$

$$A=0.0002$$

$$\int\left(\frac{0.0002}{P}+\frac{0.0002}{5000-P}\right)dP=10^{-5}t$$

$$\int\left(\frac{1}{P}+\frac{1}{5000-P}\right)dP=0.05t$$

$$\ln P-\ln\left(5000-P\right)=0.05\,t+C$$

$$\ln\left(\frac{5000-P}{P}\right)=-0.05\,t-C$$

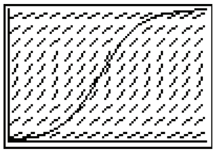
$$\frac{5000}{P}-1=e^{-0.05t}e^{-c}$$

$$\frac{5000}{P}=1+e^{-0.05t}e^{-c}$$

$$\frac{5000}{50}=1+e^{-0.05(0)}e^{-c}$$

$$e^{-c}=99$$

$$P=\frac{5000}{1+99e^{-0.05t}}$$



[–1, 200] by [0, 5000]

31. **(a)** $P(t)=\dfrac{1000}{1+e^{4.8-0.7t}}$

$$=\frac{1000}{1+e^{4.8-0.7t}}$$

$$=\frac{M}{1+Ae^{-Mkt}}$$

This is a logistic growth model with $M=1000$ and $k=\dfrac{0.7}{1000}=0.0007.$

(b) $P(0)=\dfrac{1000}{1+e^{4.8}}\approx 8$

Initially there are 8 rabbits.

32. **(a)** $P(t)=\dfrac{200}{1+e^{5.3-t}}$

$$=\frac{200}{1+e^{5.3}e^{-t}}$$

$$=\frac{M}{1+Ae^{-Mkt}}$$

This is a logistic growth model with $M=200$ and $k=\dfrac{1}{200}=0.05.$

(b) $P(0)=\dfrac{200}{1+e^{5.3}}\approx 1$

Initially 1 student has the measles.

33. **(a)** $\dfrac{dP}{dt}=0.0015P(150-P)$

$$=kP(M-P)$$

Thus, $k = 0.0015$ and $M = 150.$

$$P=\frac{M}{1+Ae^{-Mkt}}=\frac{150}{1+Ae^{-0.225t}}$$

Initial condition: $P(0) = 6$

$$6=\frac{150}{1+Ae^{0}}$$

$$1+A=25$$

$$A=24$$

Formula: $P=\dfrac{150}{1+24e^{-0.225t}}$

(b)

$$100=\frac{150}{1+24e^{-0.225t}}$$

$$1+24e^{-0.225t}=\frac{3}{2}$$

$$24e^{-0.225t}=\frac{1}{2}$$

$$e^{-0.225t}=\frac{1}{48}$$

$$-0.225t=-\ln\ 48$$

$$t=\frac{\ln 48}{0.225}\approx 17.21\text{ weeks}$$

$$125=\frac{150}{1+24e^{-0.225t}}$$

$$1+24e^{-0.225t}=\frac{6}{5}$$

$$24e^{-0.225t}=\frac{1}{5}$$

$$e^{-0.225t}=\frac{1}{120}$$

$$-0.225t=-\ln\ 120$$

$$t=\frac{\ln 120}{0.225}\approx 21.28$$

It will take about 17.21 weeks to reach 100 guppies, and about 21.28 weeks to reach 125 guppies.

34. **(a)** $\dfrac{dP}{dt}=0.0004P(250-P)=kP(M-P)$

Thus, $k=0.0004$ and $M=250.$

$$P=\frac{M}{1+Ae^{-Mkt}}=\frac{250}{1+Ae^{-0.1t}}$$

Initial condition: $P(0)=28,$ where $t=0$ represents the year 1970.

$$28 = \frac{250}{1+Ae^0}$$
$$28(1+A) = 250$$
$$A = \frac{250}{28} - 1 = \frac{111}{14} \approx 7.9286$$

Formula:

$$P(t) = \frac{250}{1+111e^{-0.1t}/14}, \text{ or approximately}$$
$$P(t) = \frac{250}{1+7.9286e^{-0.1t}}$$

(b) The population $P(t)$ will round to 250 when $P(t) \ge 249.5$.

$$249.5 = \frac{250}{1+111e^{-0.1t}/14}$$
$$249.5\left(1+\frac{111e^{-0.1t}}{14}\right) = 250$$
$$\frac{(249.5)(111e^{-0.1t})}{14} = 0.5$$
$$e^{-0.1t} = \frac{14}{55{,}389}$$
$$-0.1t = \ln\frac{14}{55{,}389}$$
$$t = 10(\ln 55{,}389 - \ln 14)$$
$$\approx 82.8$$

It will take about 83 years.

35.

$$\frac{dP}{dt} = kP(M-P)$$
$$\int \frac{dP}{P(M-P)} = \int k\,dt$$
$$\frac{Q}{P} + \frac{R}{M-P} = \frac{1}{P(M-P)}$$
$$Q(M-P)+RP = 1$$
$$P = 0,\ MQ = 1$$
$$Q = \frac{1}{M}$$
$$P = M,\ MR = 1$$
$$R = \frac{1}{M}$$
$$\int\left(\frac{\frac{1}{M}}{P} + \frac{\frac{1}{M}}{(M-P)}\right)dP = kt + C$$
$$\int\left(\frac{1}{P} + \frac{1}{(M-P)}\right)dP = Mkt + C$$
$$\ln\left(\frac{M-P}{P}\right) = -Mkt - C$$
$$\frac{M}{P} - 1 = e^{-Mkt}e^{-C}$$
$$\frac{M}{P} = 1 + e^{-Mkt}A$$
$$P = \frac{M}{1+Ae^{-Mkt}}$$

36. **(a)**

$$\frac{dP}{dt} = k(M-P)$$
$$\int \frac{dP}{M-P} = \int k\,dt$$
$$-\ln(M-P) = kt + C$$
$$M - P = e^{-kt}\ e^{-c}$$

Let $e^{-c} = A$ then $P = M - Ae^{-kt}$.

(b) $\lim_{t\to\infty} P(t) = M - Ae^{-k\infty} = M$

(c) When $t = 0$.

(d) This curve has no inflection point. If the initial population is greater than M, the curve is always concave up and approaches $y = M$ asymptotically from above. If the initial population is smaller than M, the curve is always concave down and approaches $y = M$ asymptotically from below.

37. $\frac{dW}{dt} = 0.0008W(216-W)$ is the form

$\frac{dW}{dt} = kW(M-W)$ with $k = 0.0008$ and $M =$ 216 so $Mk = 0.1728$.

$$W(t) = \frac{216}{1+Ae^{-0.1728t}}$$

$W(0) = 24 \Rightarrow 24 = \frac{216}{1+Ae^0}$

$\Rightarrow 1 + A = \frac{216}{24} = 9 \Rightarrow A = 8$

The model is $W(t) = \frac{216}{1+8e^{-0.1728t}}$.

38. $\frac{dW}{dt} = 0.0015W(147 - W)$ is in the form

$\frac{dW}{dt} = kW(M - W)$ with $K = 0.0015$ and

$m = 147$, so $kM = 0.2205$.

$$W(t) = \frac{147}{1 + Ae^{-0.2205t}}$$

$$W(0) = 24.5 \Rightarrow 24.5 = \frac{147}{1 + Ae^{0}}$$

$$\Rightarrow 1 + A = \frac{147}{24.5} = 6 \Rightarrow A = 5$$

The model is $W(t) = \frac{147}{1 + 5e^{-0.2205t}}$.

39. False; it does look exponential, but it resembles the solution to

$\frac{dP}{dt} = kP(100 - 10) = (90k)P$.

40. True; the graph will be a logistic curve with $\lim_{t \to \infty} P(t) = 100$ and $\lim_{t \to -\infty} P(t) = 0$.

41. D; $\frac{600}{2} = 300$.

42. B; $M = 0.9$, so at most 90% of the population will be infected. The remaining 10% will not be infected.

43. D; $\int_2^3 \frac{3}{(x-1)(x+2)}\,dx$

$$\frac{A}{x-1} + \frac{B}{x+2} = \frac{3}{(x-1)(x+2)}$$

$$A(x+2) + B(x-1) = 3$$

$$x = -2,\ B(-2-1) = 3$$
$$-3B = 3$$
$$B = -1$$
$$x = 1,\ A(1+2) = 3$$
$$3A = 3$$
$$A = 1$$

$$\int\left(\frac{1}{x-1} + \frac{-1}{x+2}\right)dx = \ln\left(\frac{x-1}{x+2}\right)\Bigg|_2^3 = \ln\left(\frac{8}{5}\right)$$

44. B

45. **(a)** Note that $k > 0$ and $M > 0$, so the sign of $\frac{dP}{dt}$ is the same as the sign of $(M - P)(P - m)$. For $m < P < M$, both $M - P$ and $P - m$ are positive, so the product is positive. For $P < m$ or $P > M$, the expressions $M - P$ and $P - m$ have opposite signs, so the product is negative.

(b) $\frac{dP}{dt} = \frac{k}{M}(M - P)(P - m)$

$$\frac{dP}{dt} = \frac{k}{1200}(1200 - P)(P - 100)$$

$$\frac{1200}{(1200 - P)(P - 100)}\frac{dP}{dt} = k$$

$$\frac{1100}{(1200 - P)(P - 100)}\frac{dP}{dt} = \frac{11}{12}k$$

$$\frac{(P - 100) + (1200 - P)}{(1200 - P)(P - 100)}\frac{dP}{dt} = \frac{11}{12}k$$

$$\left(\frac{1}{1200 - P} + \frac{1}{P - 100}\right)\frac{dP}{dt} = \frac{11}{12}k$$

$$\int\left(\frac{1}{1200 - P} + \frac{1}{P - 100}\right)dP = \int\frac{11}{12}k\,dt$$

$$-\ln|1200 - P| + \ln|P - 100| = \frac{11}{12}kt + C$$

$$\ln\left|\frac{P - 100}{1200 - P}\right| = \frac{11}{12}kt + C$$

$$\frac{P - 100}{1200 - P} = \pm e^{C}e^{11kt/12}$$

$$\frac{P - 100}{1200 - P} = Ae^{11kt/12}$$

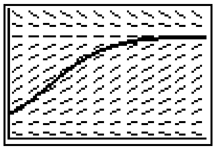
$$P - 100 = 1200Ae^{11kt/12} - APe^{11kt/12}$$

$$P(1 + Ae^{11kt/12}) = 1200Ae^{11kt/12} + 100$$

$$P = \frac{1200Ae^{11kt/12} + 100}{1 + Ae^{11kt/12}}$$

(c)
$$300 = \frac{1200Ae^{0} + 100}{1 + Ae^{0}}$$
$$300(1 + A) = 1200A + 100$$
$$300 - 100 = 1200A - 300A$$
$$200 = 900A$$
$$A = \frac{2}{9}$$
$$P(t) = \frac{1200\left(\frac{2}{9}\right)e^{11kt/12} + 100}{1 + \left(\frac{2}{9}\right)e^{11kt/12}}$$
$$P(t) = \frac{1200(2)e^{11kt/12} + 100(9)}{9 + 2e^{11kt/12}}$$
$$P(t) = \frac{300(8e^{11kt/12} + 3)}{9 + 2e^{11kt/12}}$$

(d)



[0, 75] by [0, 1500]

Note that the slope field is given by
$$\frac{dP}{dt} = \frac{0.1}{1200}(1200 - P)(P - 100).$$

(e)
$$\frac{dP}{dt} = \frac{k}{M}(M - P)(P - m)$$
$$\frac{M}{(M - P)(P - m)}\frac{dP}{dt} = k$$
$$\frac{M}{M - m}\frac{M - m}{(M - P)(P - m)}\frac{dP}{dt} = k$$
$$\frac{(P - m) + (M - P)}{(M - P)(P - m)}\frac{dP}{dt} = \frac{M - m}{M}k$$
$$\left(\frac{1}{M - P} + \frac{1}{P - m}\right)\frac{dP}{dt} = \frac{M - m}{M}k$$
$$\int\left(\frac{1}{M - P} + \frac{1}{P - m}\right)dP = \int\frac{M - m}{M}k\,dt$$
$$-\ln|M - P| + \ln|P - m| = \frac{M - m}{M}kt + C$$
$$\ln\left|\frac{P - m}{M - P}\right| = \frac{M - m}{M}kt + C$$
$$\frac{P - m}{M - P} = \pm e^{C}e^{(M - m)kt/M}$$
$$\frac{P - m}{M - P} = Ae^{(M - m)kt/M}$$
$$P - m = (M - P)Ae^{(M - m)kt/M}$$
$$P(1 + Ae^{(M - m)kt/M}) = AMe^{(M - m)kt/M} + m$$
$$P = \frac{AMe^{(M - m)kt/M} + m}{1 + Ae^{(M - m)kt/M}}$$

$$P(0)=\frac{AMe^0+m}{1+Ae^0}=\frac{AM+m}{1+A}$$

$$P(0)(1+A)=AM+m$$

$$A(P(0)-M)=m-P(0)$$

$$A=\frac{m-P(0)}{P(0)-M}=\frac{P(0)-m}{M-P(0)}$$

Therefore, the solution to the differential equation is

$P=\dfrac{AMe^{(M-m)kt/M}}{1+Ae^{(M-m)kt/M}}$ where

$A=\dfrac{P(0)-m}{M-P(0)}$.

46. **(a)** Let $u=\frac{x}{a}$; then $x = au$, $dx = a\,du$

$$\int\frac{dx}{a^2+x^2}=\int\frac{a\,du}{a^2+a^2u^2}$$

$$=\frac{a}{a^2}\int\frac{du}{1+u^2}$$

$$=\frac{1}{a}\tan^{-1}(u)+C$$

$$=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

(b) Let $u=\frac{x}{a}$; then $x = au$, $dx = a\,du$

$$\int\frac{dx}{a^2-x^2}=\int\frac{a\,du}{a^2-a^2u^2}$$

$$=\frac{a}{a^2}\int\frac{du}{1-u^2}$$

$$=-\frac{1}{a}\int\frac{du}{u^2-1}$$

$$=-\frac{1}{a}\int\frac{1}{(u+1)(u-1)}du$$

$$\frac{A}{u+1}+\frac{B}{u-1}=\frac{1}{(u+1)(u-1)}$$

$$A(u-1)+B(u+1)=1$$

$$u=-1,\ A(-2)=1$$

$$A=-\frac{1}{2}$$

$$u=1,\ B(2)=1$$

$$B=\frac{1}{2}$$

$$-\frac{1}{a}\int\left(\frac{-\frac{1}{2}}{u+1}+\frac{\frac{1}{2}}{u-1}\right)du$$

$$=\frac{1}{2a}\int\left(\frac{1}{u+1}-\frac{1}{u-1}\right)du$$

$$=\frac{1}{2a}\ln\left|\frac{u+1}{u-1}\right|+C$$

$$=\frac{1}{2a}\ln\left|\frac{\frac{x}{a}+1}{\frac{x}{a}-1}\right|+C$$

$$=\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+C$$

(c) Let $u = a + x$, $du = dx$

$$\int\frac{dx}{(a+x)^2}=\int\frac{du}{u^2}=-\frac{1}{u}+C=-\frac{1}{x+a}+C$$

47. **(a)** $\displaystyle\int\frac{5x}{(x+3)^2}dx$

$$\frac{A}{x+3}+\frac{B}{(x+3)^2}=\frac{5x}{(x+3)^2}$$

$$A(x+3)+B=5x$$

$$x=3,\ B=-15$$

$$x=0,\ A(x+3)-15=5x$$

$$A(3)-15=0$$

$$A=5$$

$$\int\left(\frac{5}{x+3}-\frac{15}{(x+3)^2}\right)dx$$

$$=5\ln|x+3|+\frac{15}{x+3}+C$$

(b) $\displaystyle\int\frac{5x}{(x+3)^3}dx$

$$\frac{A}{(x+3)}+\frac{B}{(x+3)^2}+\frac{C}{(x+3)^3}=\frac{5x}{(x+3)^3}$$

$$A(x+3)^2+B(x+3)+C=5x$$

$$x=-3,\ C=15$$

$$A(x+3)^2+B(x+3)-15=5x$$

$x = 0$, $9A + 3B - 15 = 0$, $B = 5 + 3A$

$x = 1$, $16A + 4B - 15 = 5$, $B = 5 + 4A$

$5 + 3A + 5 + 4A$

$A = 0$

$$B = 5 + 3(0) = 5$$

$$\int\left(\frac{5}{(x+3)^2}-\frac{15}{(x+3)^3}\right)dx$$

$$= -\frac{5}{x+3}+\frac{15}{2(x+3)^2}+C$$

48. **(a)** This is true since

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}$$

$$= \frac{A(x-1)^2+B(x-1)+C}{(x-1)^3}$$

(b) $A(x-1)^2+B(x-1)+C = x^2+3x+5$

$x = 0,\ A-B+C = 5$

$x = 1,\ C = 9$

$x = 2,\ A+B+C = 15$

Then

$$\begin{array}{l} A-B=-4 \\ \underline{A+B=6} \\ \quad 2A=2 \\ \qquad A=1 \end{array}$$

$B = 5$

(c) $$\int\left(\frac{1}{x-1}+\frac{5}{(x-1)^2}+\frac{9}{(x-1)^3}\right)dx$$

$$= \ln|x-1|-\frac{5}{x-1}-\frac{9}{2(x-1)^2}+C$$

Quick Quiz Sections 7.4 and 7.5

1. C; $y = y_0e^{kt}$

$t = 1,\ 2 = y_0e^{k}$

$t = 5,\ \ 3 = y_0e^{5k}$

$$\frac{3}{2} = \frac{y_0e^{5k}}{y_0e^{k}} = e^{4k}$$

$$k = \frac{\ln\left(\frac{3}{2}\right)}{4}$$

$$y_0 = 2e^{-\frac{\ln(3/2)}{4}} = 1.807$$

$$t = 8,\ y = 1.807e^{\left(\frac{\ln(3/2)}{4}\cdot 8\right)} = 4.066$$

2. C; $F(x) = \int_a^x \cos(t^2)dt$

$F(1) = 0 \ \Rightarrow\ a = 1$

$F(x) = \int_1^x \cos(t^2)dt$

$F(5) = \int_1^5 \cos(t^2)dt = -0.293$

[Use NINT $(\cos(x^2), x, 1, 5)$ to evaluate the integral.]

3. A; $\int\frac{dx}{(x-1)(x+3)}$

$$\frac{A}{x-1}+\frac{B}{x+3} = \frac{1}{(x-1)(x+3)}$$

$A(x+3)+B(x-1) = 1$

$x = -3,\ -4B = 1$

$$B = -\frac{1}{4}$$

$x = 1,\ 4A = 1$

$$A = \frac{1}{4}$$

$$\int\left(\frac{\frac{1}{4}}{x-1}+\frac{-\frac{1}{4}}{x+3}\right)dx = \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right|+C$$

4. $$\frac{dP}{dt} = \frac{P}{5}\left(\frac{10-P}{10}\right)$$

$$\int\frac{dP}{P(10-P)} = \int\frac{1}{50}dt$$

$$\frac{A}{P}+\frac{B}{10-P} = \frac{1}{P(10-P)}$$

$A(10-P)+BP = 1$

$P = 10,\ 10B = 1$

$B = 0.1$

$P = 0,\ 10A = 1$

$A = 0.1$

$$\int\left(\frac{0.1}{P}+\frac{0.1}{10-P}\right)dP = \frac{1}{50}t+C$$

$$\ln\left|\frac{10-P}{P}\right| = -\frac{1}{5}t-C$$

$$P = \frac{10}{1+e^{-1/5t}e^{-C}}$$

(a) $$P(0) = 3 = \frac{10}{1+Ae^{-1/5(0)}}$$

$A = 2.33$

$$\lim_{t\to\infty} P(t) = \frac{10}{1+2.33e^{-1/5(t)}} = 10$$

(b) $P(0) = 20 = \dfrac{10}{1 + Ae^{-1/5(0)}}$

$$A = -0.5$$

$$\lim_{t\to\infty} P(t) = \frac{10}{1 + -0.5e^{-1/5(t)}} = 10$$

(c) Separate the variables.

$$\frac{dY}{Y} = \frac{1}{5}\left(1 - \frac{t}{10}\right)dt$$

$$\ln Y = \frac{t}{5} - \frac{t^2}{100} + C_1$$

$$Y = Ce^{t/5 - t^2/100} \quad \text{where } C = e^{C_1}$$

$$3 = Ce^0 \Rightarrow C = 3$$

$$Y = 3e^{t/5 - t^2/100}$$

(d) $\displaystyle\lim_{t\to\infty} 3e^{t/5 - t^2/100} = \lim_{t\to\infty} \frac{3e^{t/5}}{e^{t^2/100}}$

$$= \lim_{t\to\infty} \frac{3e^{t/5}}{(e^{t/5})^{t/20}}$$

$$= 0$$

Chapter 7 Review Exercises (pp. 379–382)

1. $\displaystyle\int_0^{\pi/3} \sec^2\theta\, d\theta = \tan\theta\Big|_0^{\pi/3}$

$$= \tan\frac{\pi}{3} - \tan 0$$

$$= \sqrt{3}$$

2. $\displaystyle\int_1^2 \left(x + \frac{1}{x^2}\right)dx = \left[\frac{1}{2}x^2 - x^{-1}\right]_1^2$

$$= \left(\frac{1}{2}(4) - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right)$$

$$= \frac{3}{2} + \frac{1}{2}$$

$$= \frac{4}{2}$$

$$= 2$$

3. Let $u = 2x + 1$, $du = 2\,dx$, $\frac{1}{2}du = dx$

$$\int_0^1 \frac{36}{(2x+1)^3}dx = 18\int_1^3 \frac{1}{u^3}du$$

$$= 18\left(-\frac{1}{2}\right)u^{-2}\Big|_1^3$$

$$= -9\left(\frac{1}{9} - 1\right)$$

$$= -9\left(-\frac{8}{9}\right)$$

$$= 8$$

4. Let $u = 1 - x^2,\ du = -2x\,dx,\ -du = 2x\,dx$

$$\int_{-1}^1 2x\sin(1 - x^2)\,dx = -\int_0^0 \sin u\,du = 0$$

5. Let $u = \sin x$, $du = \cos x\,dx$

$$\int_0^{\pi/2} 5\sin^{3/2} x\cos x\,dx$$

$$= \int_0^1 5u^{3/2}\,du$$

$$= 5\cdot\frac{2}{5}u^{5/2}\Big|_0^1$$

$$= 2(1 - 0)$$

$$= 2$$

6. $\displaystyle\int_{1/2}^4 \frac{x^2 + 3x}{x}dx = \int_{1/2}^4 (x + 3)\,dx\,(x \neq 0)$

$$= \left(\frac{1}{2}x^2 + 3x\right)\Big|_{1/2}^4$$

$$= \left(\frac{1}{2}(16) + 3(4)\right) - \left(\frac{1}{2}\left(\frac{1}{4}\right) + \frac{3}{2}\right)$$

$$= 20 - \left(\frac{1}{8} + \frac{12}{8}\right)$$

$$= 20 - \frac{13}{8}$$

$$= \frac{147}{8}$$

7. Let $u = \tan x,\ du = \sec^2 x\,dx$

$$\int_0^{\pi/4} e^{\tan x}\sec^2 x\,dx = \int_0^1 e^u\,du$$

$$= e^u\Big|_0^1$$

$$= e^1 - e^0$$

$$= e - 1$$

8. Let $u = \ln r,\ du = \frac{1}{r}dr$

$$\begin{aligned}\int_1^e \frac{\sqrt{\ln r}}{r}dr &= \int_0^1 u^{1/2}\,du \\ &= \frac{2}{3}u^{3/2}\Big|_0^1 \\ &= \frac{2}{3}(1-0) \\ &= \frac{2}{3}\end{aligned}$$

9. $\int_0^1 \frac{x}{x^2+5x+6}dx$

$$\frac{x}{(x+3)(x+2)}$$

$$\frac{A}{x+3}+\frac{B}{x+2}=\frac{x}{(x+3)(x+2)}$$

$$A(x+2)+B(x+3)=x$$

$$\begin{aligned}x=-2,\ B(-2+3)&=-2 \\ B&=-2 \\ x=-3,\ A(-3+2)&=-3 \\ -A&=-3 \\ A&=3\end{aligned}$$

$$\begin{aligned}\int \frac{3}{x+3}+\frac{-2}{x+2}dx &= \ln(x+3)^3-\ln(x+2)^2\Big|_0^1 \\ &= \ln\left(\frac{256}{243}\right)\end{aligned}$$

10. $\int_1^2 \frac{2x+6}{x^2-3x}dx$

$$\frac{2x+6}{x(x-3)}$$

$$\frac{A}{x}+\frac{B}{x-3}=\frac{2x+6}{x(x-3)}$$

$$A(x-3)+Bx=2x+6$$

$$\begin{aligned}x=3,\ 3B&=2(3)+6 \\ B&=4 \\ x=0,\ A(0-3)&=2(0)+6 \\ -3A&=6 \\ A&=-2\end{aligned}$$

$$\begin{aligned}\int_1^2 \frac{-2}{x}+\frac{4}{x-3}dx &= -2\ln x+4\ln(x-3)\Big|_1^2 \\ &= -6\ln 2\end{aligned}$$

11. Let $u = 2-\sin x$, $du = -\cos x\,dx$,
$-du = \cos x\,dx$

$$\begin{aligned}\int \frac{\cos x}{2-\sin x}dx &= -\int \frac{1}{u}du \\ &= -\ln|u|+C \\ &= -\ln|2-\sin x|+C\end{aligned}$$

12. Let $u = 3x+4$, $\quad du = 3dx$

$$\frac{1}{3}du = dx$$

$$\begin{aligned}\int \frac{dx}{\sqrt[3]{3x+4}} &= \frac{1}{3}\int u^{-1/3}\,du \\ &= \frac{1}{3}\cdot\frac{3}{2}u^{2/3}+C \\ &= \frac{1}{2}(3x+4)^{2/3}+C\end{aligned}$$

13. Let $u = t^2+5$, $\quad du = 2t\,dt$

$$\frac{1}{2}du = t\,dt$$

$$\begin{aligned}\int \frac{t\,dt}{t^2+5} &= \frac{1}{2}\int \frac{1}{u}du \\ &= \frac{1}{2}\ln|u|+C \\ &= \frac{1}{2}\ln\left|t^2+5\right|+C \\ &= \frac{1}{2}\ln(t^2+5)+C\end{aligned}$$

14. Let $u = \frac{1}{\theta},\ du = -\frac{1}{\theta^2}d\theta$

$$\begin{aligned}\int \frac{1}{\theta^2}\sec\frac{1}{\theta}\tan\frac{1}{\theta}d\theta &= -\int \sec u\tan u\,du \\ &= -\sec u+C \\ &= -\sec\frac{1}{\theta}+C\end{aligned}$$

15. Let $u = \ln y,\ du = \frac{1}{y}dy$

$$\begin{aligned}\int \frac{\tan(\ln y)}{y}dy &= \int \tan u\,du \\ &= \int \frac{\sin u}{\cos u}du\end{aligned}$$

Let $w = \cos u$

$dw = -\sin u\, du$

$$= -\int \frac{1}{w}\, dw$$
$$= -\ln|w| + C$$
$$= -\ln|\cos u| + C$$
$$= -\ln|\cos(\ln y)| + C$$

16. Let $u = e^x$, $du = e^x dx$

$$\int e^x \sec(e^x)\, dx = \int \sec u\, du$$
$$= \ln|\sec u + \tan u| + C$$
$$= \ln\left|\sec(e^x) + \tan(e^x)\right| + C$$

17. Let $u = \ln x$, $du = \frac{1}{x} dx$

$$\int \frac{dx}{x \ln x} = \int \frac{1}{u}\, du$$
$$= \ln|u| + C$$
$$= \ln|\ln x| + C$$

18. $\int \frac{dt}{t\sqrt{t}} = \int \frac{dt}{t^{3/2}}$

$$= \int t^{-3/2}\, dt$$
$$= -2t^{-1/2} + C$$
$$= -\frac{2}{\sqrt{t}} + C$$

19. Use tabular integration with $f(x) = x^3$ and $g(x) = \cos x$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	$\cos x$
$3x^2$	(–)	$\sin\ x$
$6x$	(+)	$-\cos x$
6	(–)	$-\sin x$
0		$\cos x$

$$\int x^3 \cos x\, dx$$
$$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + C$$

20. Let $u = \ln x \qquad dv = x^4\, dx$

$du = \frac{1}{x} dx \qquad v = \frac{1}{5} x^5$

$$\int x^4 \ln x\, dx = \frac{1}{5} x^5 \ln x - \int \frac{1}{5} x^5 \left(\frac{1}{x}\right) dx$$
$$= \frac{1}{5} x^5 \ln x - \frac{1}{5} \int x^4\, dx$$
$$= \frac{1}{5} x^5 \ln x - \frac{1}{25} x^5 + C$$

21. Let $u = e^{3x} \qquad dv = \sin x\, dx$

$du = 3e^{3x}\, dx \qquad v = -\cos x$

$$\int e^{3x} \sin x\, dx = -e^{3x} \cos x + \int 3\cos x\, e^{3x}\, dx$$

Integrate by parts again

Let $u = 3e^{3x} \qquad dv = \cos x\, dx$

$du = 9e^{3x}\, dx \qquad v = \sin x$

$$\int e^{3x} \sin x\, dx$$
$$= -e^{3x} \cos x + 3e^{3x} \sin x - \int 9e^{3x} \sin x\, dx$$
$$10 \int e^{3x} \sin x\, dx = -e^{3x} \cos x + 3e^{3x} \sin x + C$$
$$\int e^{3x} \sin x\, dx$$
$$= \frac{1}{10}[-e^{3x} \cos x + 3e^{3x} \sin x] + C$$
$$= \left(\frac{3\sin x}{10} - \frac{\cos x}{10}\right) e^{3x} + C$$

22. Let $u = x^2 \qquad dv = e^{-3x}\, dx$

$du = 2x\, dx \qquad v = -\frac{1}{3} e^{-3x}$

$$\int x^2 e^{-3x}\, dx$$
$$= -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \int e^{-3x} x\, dx$$

Let $u = x \qquad dv = e^{-3x}\, dx$

$du = dx \qquad v = -\frac{1}{3} e^{-3x}$

$$= -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x}\, dx\right]$$
$$= -\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} + \frac{2}{9} \int e^{-3x}\, dx$$
$$= -\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} - \frac{2}{27} e^{-3x} + C$$
$$= \left(-\frac{x^2}{3} - \frac{2x}{9} - \frac{2}{27}\right) e^{-3x} + C$$

23. $\int \frac{25}{x^2-25}dx = \int \frac{25}{(x+5)(x-5)}\ dx$

$\frac{A}{x+5}+\frac{B}{x-5}=\frac{25}{(x+5)(x-5)}$

$A(x-5)+B(x+5)=25$

$x=5,\ B(5+5)=25$

$10B=25$

$B=\frac{5}{2}$

$x=-5,\ A(-5-5)=25$

$-10A=25$

$A=-\frac{5}{2}$

$\int\left(\frac{-\frac{5}{2}}{x+5}+\frac{\frac{5}{2}}{x-5}\right)dx=\frac{5}{2}\ln\left|\frac{x-5}{x+5}\right|+C$

24. $\int \frac{5x+2}{2x^2+x-1}dx = \int \frac{5x+2}{(2x-1)(x+1)}dx$

$\frac{A}{2x-1}+\frac{B}{x+1}=\frac{5x+2}{(2x-1)(x+1)}$

$A(x+1)+B(2x-1)=5x+2$

$x=-1,\ B(2(-1)-1)=5(-1)+2$

$-3B=-3$

$B=1$

$x=\frac{1}{2},\ \ A\left(\frac{1}{2}+1\right)=5\left(\frac{1}{2}\right)+2$

$\frac{3}{2}A=\frac{9}{2}$

$A=3$

$\int\left(\frac{3}{2x-1}+\frac{1}{x+1}\right)dx$

$=\frac{3}{2}\ln|2x-1|+\ln|x+1|$

$=\frac{1}{2}\ln\left|(2x-1)^3(x+1)^2\right|+C$

25. $\frac{dy}{dx}=1+x+\frac{x^2}{2}$

$dy=\left(1+x+\frac{x^2}{2}\right)dx$

$\int dy=\int\left(1+x+\frac{x^2}{2}\right)dx$

$y=x+\frac{1}{2}x^2+\frac{1}{6}x^3+C$

$y(0)=C=1$

$y=\frac{x^3}{6}+\frac{x^2}{2}+x+1$

26. $\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^2$

$dy=\left(x+\frac{1}{x}\right)^2\ dx$

$\int dy=\int\left(x+\frac{1}{x}\right)^2\ dx$

$y=\int\left(x^2+2+\frac{1}{x^2}\right)dx$

$y=\frac{1}{3}x^3+2x-x^{-1}+C$

$y(1)=\frac{1}{3}+2-1+C=1$

$\frac{4}{3}+C=1$

$C=-\frac{1}{3}$

$y=\frac{x^3}{3}+2x-\frac{1}{x}-\frac{1}{3}$

27. $\frac{dy}{dt}=\frac{1}{t+4}$

$dy=\frac{1}{t+4}dt$

$\int dy=\int\frac{1}{t+4}dt$

$y=\ln|t+4|+C$

$y(-3)=\ln(1)+C=2$

$C=2$

$y=\ln(t+4)+2$

28. $\frac{dy}{d\theta}=\csc 2\theta\cot 2\theta$

$dy=\csc 2\theta\cot 2\theta\, d\theta$

$\int dy=\int\csc 2\theta\cot 2\theta\, d\theta$

$y=-\frac{1}{2}\csc 2\theta+C$

$y\left(\frac{\pi}{4}\right)=-\frac{1}{2}+C=1$

$C=\frac{3}{2}$

$y=-\frac{1}{2}\csc 2\theta+\frac{3}{2}$

29. $$\frac{d(y')}{dx} = 2x - \frac{1}{x^2}$$
$$d(y') = \left(2x - \frac{1}{x^2}\right)dx$$
$$\int d(y') = \int\left(2x - \frac{1}{x^2}\right)dx$$
$$y' = x^2 + x^{-1} + C$$
$$y'(1) = 2 + C = 1$$
$$C = -1$$
$$y' = x^2 + x^{-1} - 1$$
$$\int dy = \int (x^2 + x^{-1} - 1)\,dx$$
$$y = \frac{1}{3}x^3 + \ln x - x + C$$
$$y' = x^2 + x^{-1} - 1$$
$$\int dy = \int (x^2 + x^{-1} - 1)\,dx$$
$$y = \frac{1}{3}x^3 + \ln x - x + C$$
$$y(1) = \frac{1}{3} + 0 - 1 + C = 0$$
$$-\frac{2}{3} + C = 0$$
$$C = \frac{2}{3}$$
$$y = \frac{x^3}{3} + \ln x - x + \frac{2}{3}$$

30. $$\frac{d(r'')}{dt} = -\cos t$$
$$d(r'') = -\cos t\,dt$$
$$\int d(r'') = \int -\cos t\,dt$$
$$r'' = -\sin t + C$$
$$r''(0) = C = -1$$
$$r'' = -\sin t - 1$$
$$\int d(r') = \int (-\sin t - 1)\,dt$$
$$r' = \cos t - t + C$$
$$r'(0) = 1 + C = -1$$
$$C = -2$$
$$r' = \cos t - t - 2$$
$$\int dr = \int (\cos t - t - 2)\,dt$$
$$r = \sin t - \frac{t^2}{2} - 2t + C$$
$$r(0) = C = -1$$
$$r = \sin t - \frac{t^2}{2} - 2t - 1$$

31. $$\frac{dy}{dx} = y + 2$$
$$\frac{dy}{y+2} = dx$$
$$\int \frac{dy}{y+2} = \int dx$$
$$\ln|y+2| = x + C$$
$$y + 2 = Ce^x$$
$$y = Ce^x - 2$$
$$y(0) = C - 2 = 2$$
$$C = 4$$
$$y = 4e^x - 2$$

32. $$\frac{dy}{dx} = (2x+1)(y+1)$$
$$\frac{dy}{y+1} = (2x+1)\,dx$$
$$\int \frac{dy}{y+1} = \int (2x+1)\,dx$$
$$\ln|y+1| = x^2 + x + C$$
$$y + 1 = Ce^{x^2+x}$$
$$y = Ce^{x^2+x} - 1$$
$$y(-1) = C - 1 = 1$$
$$C = 2$$
$$y = 2e^{x^2+x} - 1$$

33. $$\frac{dy}{dt} = y(1-y)$$
$$\frac{dy}{y(1-y)} = dt$$
$$\frac{A}{y} + \frac{B}{1-y} = \frac{1}{y(1-y)}$$
$$A(1-y) + By = 1$$
$$y = 1, B = 1$$
$$y = 0, A = 1$$
$$\int\left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \int dt$$
$$\ln|y| - \ln|1-y| = t + C$$
$$\ln\left|\frac{1-y}{y}\right| = -t - C$$
$$\frac{1-y}{y} = e^{-t}e^{-c}$$
$$\frac{1}{y} - 1 = e^{-t}e^{-c}$$
$$y = \frac{1}{1 + Ae^{-t}}$$

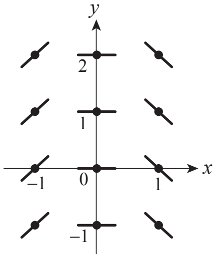
$$y(0) = 0.1 = \frac{1}{1 + Ae^{-(0)}}$$
$$A = 9$$
$$y = \frac{1}{1 + 9\,e^{-t}}$$

34. $$\frac{dy}{dx} = 0.001y\,(100 - y)$$
$$\frac{dy}{0.001y\,(100 - y)} = dx$$
$$\frac{A}{0.001\,y} + \frac{B}{100 - y} = \frac{1}{0.001y(100 - y)}$$
$$A(100 - y) + B(0.001y) = 1$$
$$y = 100,\ B(0.1) = 1$$
$$B = 10$$
$$y = 0,\ 100A = 1$$
$$A = 0.01$$
$$\int\left(\frac{0.01}{0.001\,y} + \frac{10}{100 - y}\right) dy = x + C$$
$$\int\left(\frac{0.001}{0.001\,y} + \frac{1}{100 - y}\right) dy = 0.1x + C$$
$$\ln y - \ln|100 - y| = 0.1x + C$$
$$\ln\left|\frac{100 - y}{y}\right| = -0.1x - C$$
$$\frac{100}{y} - 1 = e^{-0.1x}e^{-c}$$
$$y = \frac{100}{1 + Ae^{-0.1\,x}}$$
$$y(0) = 5 = \frac{100}{1 + Ae^{-0.1(0)}}$$
$$A = 19$$
$$y = \frac{100}{1 + 19e^{-0.1x}}$$

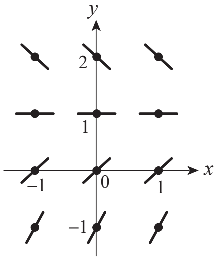
35. $y = \int_4^x \sin^3 t\ dt + 5$

36. $y = \int_1^x \sqrt{1 + t^4}\ dt + 2$

37.



38.



39. Graph (b). Slope lines are vertical for points on the line $y = -x$.

40. Graph (d). Slope lines are vertical for points on the line $y = x$.

41. Graph (c). Slope lines are horizontal for points on the x- and y-axes. Slopes are positive in Quadrants I and III. Slopes are negative in Quadrants II and IV.

42. Graph (a). Slope lines are horizontal for points on the x- and y-axes. Slopes are positive in Quadrants II and IV; negative in Quadrants I and III.

43.

(x, y)	$\frac{dy}{dx} = x + y - 1$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 1)	1.0	0.1	0.1	(1.1, 1.1)
(1.1, 1.1)	1.2	0.1	0.12	(1.2, 1.22)
(1.2, 1.22)	1.42	0.1	0.142	(1.3, 1.362

$y = 1.362$

44.

(x, y)	$\frac{dy}{dx} = x - y$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 2)	−1.0	−0.1	0.1	(0.9, 2.1)
(0.9, 2.1)	−1.2	−0.1	0.12	(0.8, 2.22)
(0.8, 2.22)	−1.42	−0.1	0.142	(0.7, 2.362

$y = 2.362$

45. We seek the graph of a function whose derivative is $\frac{\sin x}{x}$. Graph (b) is increasing on $[-\pi, \pi]$, where $\frac{\sin x}{x}$ is positive, and oscillates slightly outside of this interval. This is the correct choice, and this can be verified by graphing NINT $\left(\frac{\sin x}{x}, x, 0, x\right)$.

46. We seek the graph of a function whose derivative is e^{-x^2}. Since $e^{-x^2} > 0$ for all x, the desired graph is increasing for all x. Thus, the only possibility is graph (d), and we may verify that this is correct by graphing NINT $(e^{-x^2}, x, 0, x)$.

47. (iv) The given graph looks the graph of $y = x^2$, which satisfies $\frac{dy}{dx} = 2x$ and $y(1) = 1$.

48. Yes, $\frac{d^2y}{dx^2} = 0,$ so $\frac{dy}{dx} = C.$ Since $y'(0) = 1,$

$\frac{dy}{dx} = 1.$ Then $y = \int 1\,dx = x + C.$ Since

$y(0) = 0, C = 0.$

$y = x$ is a solution.

49. (a) $\frac{dv}{dt} = 2 + 6t$

$$\int dv = \int (2 + 6t)\,dt$$

$$v = 2t + 3t^2 + C$$

Initial condition: $v = 4$ when $t = 0$

$4 = 0 + C$

$4 = C$

$v = 2t + 3t^2 + 4$

(b) $\int_0^1 v(t)\,dt = \int_0^1 (2t + 3t^2 + 4)\,dt$

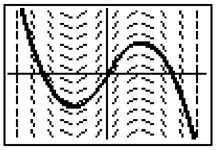
$$= \Big[t^2 + t^3 + 4t\Big]_0^1$$

$$= 6 - 0$$

$$= 6$$

The particle moves 6 m.

50.



[–10, 10] by [–10, 10]

51. (a) Half-life $= \frac{\ln 2}{k}$

$$2.645 = \frac{\ln 2}{k}$$

$$k = \frac{\ln 2}{2.645}$$

$$\approx 0.262059$$

(b) Mean life $= \frac{1}{k} \approx 3.81593$ years

52. $T - T_s = (T_0 - T_s)e^{-kt}$

$T - 40 = (220 - 40)e^{-kt}$

Use the fact that $T = 180$ and $t = 15$ to find k.

$$180 - 40 = (220 - 40)e^{-(k)(15)}$$

$$e^{15k} = \frac{180}{140} = \frac{9}{7}$$

$$k = \frac{1}{15}\ln\frac{9}{7}$$

$$T - 40 = (220 - 40)e^{-((1/15)\ln(9/7))t}$$

$$70 - 40 = (220 - 40)e^{-((1/15)\ln(9/7))t}$$

$$e^{((1/15)\ln(9/7))t} = \frac{180}{30} = 6$$

$$\left(\frac{1}{15}\ln\frac{9}{7}\right)t = \ln 6$$

$$t = \frac{15\ln 6}{\ln(9/7)} \approx 107 \text{ min}$$

It took a total of about 107 minutes to cool from 220°F to 70°F. Therefore, the time to cool from 180°F to 70°F was about 92 minutes.

53. $T - T_s = (T_0 - T_s)e^{-kt}$

We have the system:

$$\begin{cases} 39 - T_s = (46 - T_s)e^{-10k} \\ 33 - T_s = (46 - T_s)e^{-20k} \end{cases}$$

Thus, $\frac{39 - T_s}{46 - T_s} = 10^{-10k}$ and $\frac{33 - T_s}{46 - T_s} = e^{-20k}$

Since $(e^{-10k})^2 = e^{-20k}$, this means:

$$\left(\frac{39 - T_s}{46 - T_s}\right)^2 = \frac{33 - T_s}{46 - T_s}$$

$$(39 - T_s)^2 = (33 - T_s)(46 - T_s)$$

$$1521 - 78T_s + T_s^2 = 1518 - 79T_s + T_s^2$$

$$T_s = -3$$

The refrigerator temperature was –3°C.

54. See Examples 3 and 5 in Section 7.4. Use the fact that the half-life of C-14 is 5700 years to find k:

$$\frac{1}{2} = e^{-k(5700)}$$

$$\ln\left(\frac{1}{2}\right) = -5700\,k$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{-5700} = \frac{\ln 2}{5700}$$

The painting contains 99.5% of its original Carbon-14.

$$0.995 = e^{\left(-\frac{\ln 2}{5700}t\right)}$$

$$\ln(0.995) = -\frac{\ln 2}{5700}t$$

$$t = -\frac{5700}{\ln 2}\ln(0.995) \approx 41.2$$

The painting is about 41.2 years old.

55. Since 90% of the Carbon-14 has decayed, 10% remains. We showed in Problem 54 that, for Carbon-14, $k = \frac{\ln 2}{5700}$.

$$0.10 = e^{\left(-\frac{\ln 2}{5700}t\right)}$$
$$\ln(0.10) = -\frac{\ln 2}{5700}t$$
$$t = -\frac{5700}{\ln 2}\ln(0.10) \approx 18{,}935$$

The sample is about 18,935 years old.

56. Use t = 1988 − 1924 = 64 years.

$$250\, e^{r\cdot 64} = 7500$$
$$e^{64r} = 30$$
$$64r = \ln 30$$
$$r = \frac{\ln 30}{64} \approx 0.053$$

The rate of appreciation is about 0.053, or 5.3%.

57. $L = L_0 e^{-kx}$ where x represents the depth in feet and L_0 is the surface intensity.

When x = 18 ft, $L = \frac{1}{2}L_0$, so

$$\frac{1}{2}L_0 = L_0 e^{-k\cdot 18}$$
$$\frac{1}{2} = e^{-k\cdot 18}$$
$$\ln\left(\frac{1}{2}\right) = -18k$$
$$k = \frac{\ln\left(\frac{1}{2}\right)}{-18} = \frac{\ln 2}{18}$$

We want to know the depth at which

$$L = \frac{1}{10}L_0$$
$$\frac{1}{10}L_0 = L_0 e^{\left(-\frac{\ln 2}{18}x\right)}$$
$$0.1 = e^{\left(-\frac{\ln 2}{18}x\right)}$$
$$\ln(0.1) = -\frac{\ln 2}{18}x$$
$$x = -\frac{18}{\ln 2}\ln(0.1) \approx 59.8.$$

You can work without artificial light to a depth of about 59.8 feet.

58. (a)

$$\frac{dy}{dt} = \frac{kA}{V}(c-y)$$
$$\int \frac{dy}{c-y} = \int \frac{kA}{V}dt$$
$$-\ln|c-y| = \frac{kA}{V}t + C$$
$$\ln|c-y| = -\frac{kA}{V}t - C$$
$$|c-y| = e^{-(kA/V)t-C}$$
$$c-y = \pm e^{-(kA/V)t-C}$$
$$y = c \pm e^{-(kA/V)t-C}$$
$$y = c + De^{-(kA/V)t}$$

Initial condition $y = y_0$ *when* $t = 0$

$$y_0 = c + D$$
$$y_0 - c = D$$

Solution: $y = c + (y_0 - c)e^{-(kA/V)t}$

(b) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} [c + (y_0 - c)e^{-(kA/V)t}] = c$

59. (a) $P(t) = \frac{150}{1+e^{4.3-t}} = \frac{150}{1+e^{4.3}e^{-t}}$

This is $P = \frac{M}{1+Ae^{-Mkt}}$ where M = 150, $A = e^{4.3}$, and $k = \frac{1}{150}$. Therefore, it is a solution of the logistic differential equation.

$$\frac{dP}{dt} = kP(M-P), \text{ or}$$
$$\frac{dP}{dt} = \frac{1}{150}P(150-P).$$

The carrying capacity is 150.

(b) $P(0) = \frac{150}{1+e^{4.3}} \approx 2$

Initially there were 2 infected students.

(c)

$$\frac{150}{1+e^{4.3-t}} = 125$$
$$\frac{6}{5} = 1 + e^{4.3-t}$$
$$\frac{1}{5} = e^{4.3-t}$$
$$-\ln 5 = 4.3 - t$$
$$t = 4.3 + \ln 5 \approx 5.9 \text{ days.}$$

It took about 6 days.

60. Use the Fundamental Theorem of Calculus.

$$y' = \frac{d}{dx}\left(\int_0^x \sin t^2 dt\right) + \frac{d}{dx}\left(x^3 + x + 2\right)$$
$$= (\sin x^2) + (3x^2 + 1)$$
$$y'' = \frac{d}{dx}(\sin x^2 + 3x^2 + 1)$$
$$= (\cos x^2)(2x) + 6x$$
$$= 2x\ \cos(x^2) + 6x$$

Thus, the differential equation is satisfied. Verify the initial conditions:

$$y'(0) = (\sin 0^2) + 3(0)^2 + 1 = 1$$
$$y(0) = \int_0^0 \sin(t^2)\ dt + 0^3 + 0 + 2 = 2$$

61.

$$\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{800}\right)$$
$$\frac{dP}{dt} = 0.002P\left(\frac{800 - P}{800}\right)$$
$$\frac{800}{P(800 - P)}dP = 0.002dt$$
$$\frac{A}{P} + \frac{B}{800 - P} = \frac{800}{P(800 - P)}$$
$$A(800 - P) + BP = 800$$
$$P = 0, A = 1$$
$$P = 800, B = 1$$
$$\int\left(\frac{1}{P} + \frac{1}{800 - P}\right)dP = \int 0.002\ dt$$
$$\ln|P| - \ln|800 - P| = 0.002t + C$$
$$\ln\left|\frac{P}{800 - P}\right| = 0.002t + C$$
$$\ln\left|\frac{800 - P}{P}\right| = -0.002t - C$$
$$\left|\frac{800 - P}{P}\right| = e^{-0.002t - C}$$
$$\frac{800 - P}{P} = \pm e^{-C}e^{-0.002t}$$
$$\frac{800}{P} - 1 = Ae^{-0.002t}$$
$$P = \frac{800}{1 + Ae^{-0.002t}}$$

Initial condition: $P(0) = 50$

$$50 = \frac{800}{1 + Ae^0}$$
$$1 + A = 16$$
$$A = 15$$

Solution: $P = \dfrac{800}{1 + 15e^{-0.002t}}$

62. Method 1–Compare graph of $y_1 = x^2 \ln x$ with $y_2 = \text{NDER}\left(\dfrac{x^3 \ln x}{3} - \dfrac{x^3}{9}\right)$. The graphs should be the same. Method 2–Compare graph of $y_1 = \text{NINT}(x^2 \ln x)$ with $y_2 = \dfrac{x^3 \ln x}{3} - \dfrac{x^3}{9}$. The graphs should be the same or differ only by a vertical translation.

63. **(a)**

$$20{,}000 = 10{,}000(1.063)^t$$
$$2 = 1.063^t$$
$$\ln 2 = t \ln 1.063$$
$$t = \frac{\ln 2}{\ln 1.063} \approx 11.345$$

It will take about 11.3 years.

(b)

$$20{,}000 = 10{,}000e^{0.063t}$$
$$2 = e^{0.063t}$$
$$\ln 2 = 0.063t$$
$$t = \frac{\ln 2}{0.063} \approx 11.002$$

It will take about 11.0 years.

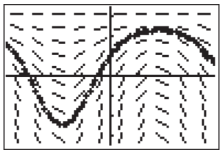
64. **(a)**

$$f'(x) = \frac{d}{dx}\int_0^x u(t)\ dt = u(x)$$
$$g'(x) = \frac{d}{dx}\int_3^x u(t)\ dt = u(x)$$

(b)

$$C = f(x) - g(x)$$
$$= \int_0^x u(t)\ dt - \int_3^x u(t)\ dt$$
$$= \int_0^x u(t)\ dt + \int_x^3 u(t)\ dt$$
$$= \int_0^3 u(t)\ dt$$

65. **(a)**



[–3.5, 3.5] by [–3.5, 3.5]

(b) At (0, 1), $\dfrac{dy}{dx} = (3 - 1)\cos 0 = 2.$

The tangent line has equation $y - 1 = 2(x - 0)$ or $y = 2x + 1$. Thus $f(0.2) \approx 2(0.2) + 1 = 1.4$.

(c) $$\frac{dy}{dx} = (3-y)\cos x$$

$$\frac{1}{3-y}\,dy = \cos x\,dx$$

$$\int \frac{1}{3-y}\,dy = \int \cos x\,dx$$

$$\ln\left|\frac{1}{3-y}\right| = \sin x + C$$

$$\ln|3-y| = -\sin x - C$$

$$|3-y| = e^{-\sin x - C}$$

$$3-y = \pm e^{-C}e^{-\sin x}$$

$$3-y = Ae^{-\sin x}$$

$$y = 3 - Ae^{-\sin x}$$

Initial condition: $y(0) = 1$

$$1 = 3 - Ae^{-\sin 0}$$

$$1 = 3 - A$$

$$A = 2$$

The solution is $y = 3 - 2e^{-\sin x}$.

66. **(a)** At (1, 0), $\frac{dy}{dx} = e^0(3(1)^2 - 6(1)) = -3$.

The tangent line has equation

$y - 0 = -3(x - 1)$ or $y = -3x + 3$.

Thus, $f(1.2) \approx -3(1.2) + 3 = -0.6$.

(b) $$\frac{dy}{dx} = e^y(3x^2 - 6x)$$

$$e^{-y}dy = (3x^2 - 6x)dx$$

$$\int e^{-y}dy = \int (ex^2 - 6x)dx$$

$$-e^{-y} = x^3 - 3x^2 + C$$

$$e^{-y} = -x^3 + 3x^2 - C$$

$$-y = \ln(-x^3 + 3x^2 - C)$$

$$y = -\ln(-x^3 + 3x^2 - C)$$

Initial condition: $y(1) = 0$

$$0 = -\ln(-(1)^3 + 3(1)^2 - C)$$

$$0 = -\ln(2 - C)$$

$$1 = 2 - C$$

$$C = 1$$

The solution is $y = -\ln(-x^3 + 3x^2 - 1)$.

67. **(a)** $\frac{1}{2}$ of the town has heard the rumor when it is spreading the fastest.

(b) $$\int \frac{dy}{y(1-y)} = \int 1.2\,dt$$

$$\frac{A}{y} + \frac{B}{1-y} = \frac{1}{y(1-y)}$$

$$A(1-y) + By = 1$$

$y = 0, A = 1$

$y = 1, B = 1$

$$\int\left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \int 1.2\,dt$$

$$\ln\left|\frac{y}{1-y}\right| = 1.2t + C$$

$$\frac{y}{1-y} = e^c e^{1.2t}$$

$$\frac{1-y}{y} = e^{-c}e^{-1.2t}$$

$$\frac{1-y}{y} = Ae^{-1.2t}$$

$$y = \frac{1}{1 + Ae^{-1.2t}}$$

$$y(0) = \frac{1}{10} = \frac{1}{1 + Ae^0}$$

$$A = 9$$

$$y = \frac{1}{1 + 9e^{-1.2t}}$$

(c) $$\frac{1}{2} = \frac{1}{1 + 9e^{-1.2t}}$$

Solve for t to obtain

$$t = \frac{5\ln 3}{3} \approx 1.83 \text{ days.}$$

68. **(a)** $\frac{dP}{dt} = k(600 - P)$. Separate the variables to obtain

$$\frac{dP}{600 - P} = kdt$$

$$\frac{dP}{P - 600} = -kdt$$

$$\ln|P - 600| = -kt + C_1$$

$$P - 600 = Ce^{-kt}$$

$$200 - 600 = Ce^0 \Rightarrow C = -400$$

$$P - 600 = -400e^{-kt}$$

$$P(t) = 600 - 400e^{-kt}$$

(b) $$500 = 600 - 400e^{-k \cdot 2}$$

$$\frac{1}{4} = e^{-2k}$$

$$k = \ln 2 \approx 0.693$$

(c) $\lim_{t\to\infty}(600-400e^{-0.693t})=600$

69. **(a)** Separate the variables to obtain

$$\frac{dv}{v+17}=-2dt$$

$$\ln|v+17|=-2t+C_1$$

$$v+17=Ce^{-2t}$$

$$-47+17=Ce^{0}\Rightarrow C=-30$$

$$v+17=-30e^{-2t}$$

$$v=-30e^{-2t}-17$$

(b) $\lim_{t\to\infty}(-30e^{-2t}-17)=-17$ feet per second

(c) $-20=-30e^{-2t}-17$

$$t=\frac{\ln 10}{2}\approx 1.151 \text{ seconds}$$