Chapter 3

Derivatives

Section 3.1 Derivative of a Function
    (pp. 101–110)

Exploration 1 Reading the Graphs

1. The graph in Figure 3.3b represents the rate of change of the depth of the water in the ditch with respect to time. Since y is measured in inches and x is measured in days,

    the derivative $\dfrac{dy}{dx}$ would be measured in

    inches per day. Those are the units that should be used along the y-axis in Figure 3.3b.

2. The water in the ditch is 1 inch deep at the start of the first day and rising rapidly. It continues to rise, at a gradually decreasing rate, until the end of the second day, when it achieves a maximum depth of about

    $4\dfrac{3}{4}$ inches. During days 3, 4, 5, and 6, the

    water level goes down, until it reaches a depth of 1 inch at the end of day 6. During the seventh day it rises again, almost to a depth of 2 inches.

3. The weather appears to have been wettest at the beginning of day 1 (when the water level was rising fastest) and driest at the end of day 4 (when the water level was declining the fastest).

4. The highest point on the graph of the derivative shows where the water is rising the fastest, while the lowest point (most negative) on the graph of the derivative shows where the water is declining the fastest.

5. The y-coordinate of point C gives the maximum depth of the water level in the ditch over the 7-day period, while the x-coordinate of C gives the time during the 7-day period that the maximum depth occurred. The derivative of the function changes sign from positive to negative at C', indicating that this is when the water level stops rising and begins falling.

6. Water continues to run down sides of hills and through underground streams long after the rain has stopped falling. Depending on how much high ground is located near the ditch, water from the first day's rain could still be flowing into the ditch several days later. Engineers responsible for flood control of major rivers must take this into consideration when they predict when floodwaters will "crest," and at what levels.

Quick Review 3.1

1. $\displaystyle\lim_{h\to 0}\frac{(2+h)^2-4}{4} = \lim_{h\to 0}\frac{(4+4h+h^2)-4}{h}$

    $\displaystyle\qquad = \lim_{h\to 0}\frac{4h+h^2}{h}$

    $\displaystyle\qquad = \lim_{h\to 0}\frac{h(4+h)}{h}$

    $\displaystyle\qquad = \lim_{h\to 0}(4+h)$

    $\displaystyle\qquad = 4+0$

    $\displaystyle\qquad = 4$

2. $\displaystyle\lim_{x\to 2^+}\frac{x+3}{2} = \frac{2+3}{2} = \frac{5}{2}$

3. Since $\dfrac{|y|}{y}=-1$ for $y<0,\ \displaystyle\lim_{y\to 0^-}\frac{|y|}{y}=-1.$

4. $\displaystyle\lim_{x\to 4}\frac{2x-8}{\sqrt{x}-2} = \lim_{x\to 4}\frac{2(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2}$

    $\displaystyle\qquad = \lim_{x\to 4} 2(\sqrt{x}+2)$

    $\displaystyle\qquad = 2(\sqrt{4}+2)$

    $\displaystyle\qquad = 8$

5. The vertex of the parabola is at $(0, 1)$. The slope of the line through $(0, 1)$ and another point $(h, h^2 + 1)$ on the parabola is

    $\dfrac{(h^2+1)-1}{h-0}=h.$ Since $\displaystyle\lim_{h\to 0}h=0,$ the slope of

    the line tangent to the parabola at its vertex is 0.

6. Use the graph of f to find that $(0, 2)$ is the coordinate of the high point and $(2, -2)$ is the coordinate of the low point. Therefore, f is increasing on $(-\infty, 0]$ and $[2, \infty)$.

7. $\displaystyle\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}(x-1)^2=(1-1)^2=0$

    $\displaystyle\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}(x+2)=1+2=3$

8. $\lim\limits_{h\to 0^+} f(1+h) = \lim\limits_{x\to 1^+} f(x) = 0$

(see Exercise 7).

9. No, the two one-sided limits are different (see Exercise 7).

10. No, f is discontinuous at $x = 1$ because $\lim\limits_{x\to 1} f(x)$ does not exist.

Section 3.1 Exercises

1. $f'(2) = \lim\limits_{h\to 0} \dfrac{f(2+h) - f(2)}{h}$

$= \lim\limits_{h\to 0} \dfrac{\dfrac{1}{2+h} - \dfrac{1}{2}}{h}$

$= \lim\limits_{h\to 0} \dfrac{1}{h}\left[\dfrac{2}{2(2+h)} - \dfrac{2+h}{2(2+h)}\right]$

$= \lim\limits_{h\to 0} \dfrac{1}{h}\left(\dfrac{h}{2(2+h)}\right)$

$= \lim\limits_{h\to 0} \dfrac{1}{2(2+h)}$

$= \dfrac{1}{4}$

2. $f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h) - f(1)}{h}$

$= \lim\limits_{h\to 0} \dfrac{[(1+h)^2 + 4] - [1^2 + 4]}{h}$

$= \lim\limits_{h\to 0} \dfrac{1 + 2h + h^2 + 4 - 5}{h}$

$= \lim\limits_{h\to 0} \dfrac{2h + h^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{h(2+h)}{h}$

$= \lim\limits_{h\to 0} (2+h)$

$= 2$

3. $f'(-1) = \lim\limits_{h\to 0} \dfrac{f(-1+h) - f(-1)}{h}$

$= \lim\limits_{h\to 0} \dfrac{[3-(-1+h)^2] - [3-(-1)^2]}{h}$

$= \lim\limits_{h\to 0} \dfrac{[3-(1-2h+h^2)] - (3-1)}{h}$

$= \lim\limits_{h\to 0} \dfrac{2h - h^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{h(2-h)}{h}$

$= \lim\limits_{h\to 0} (2-h)$

$= 2$

4. $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f(h) - f(0)}{h}$

$= \lim\limits_{h\to 0} \dfrac{(h^3 + h) - (0^3 + 0)}{h}$

$= \lim\limits_{h\to 0} \dfrac{h(h^2 + 1)}{h}$

$= \lim\limits_{h\to 0} (h^2 + 1) = 1$

5. $f'(2) = \lim\limits_{x\to 2} \dfrac{f(x) - f(2)}{x - 2}$

$= \lim\limits_{x\to 2} \dfrac{\dfrac{1}{x} - \dfrac{1}{2}}{x - 2}$

$= \lim\limits_{x\to 2} \dfrac{1}{x - 2}\left[\dfrac{2}{2x} - \dfrac{x}{2x}\right]$

$= \lim\limits_{x\to 2} \dfrac{1}{x - 2} \cdot \dfrac{2 - x}{2x}$

$= \lim\limits_{x\to 2} \dfrac{1}{x - 2} \cdot \dfrac{-(x - 2)}{2x}$

$= \lim\limits_{x\to 2} \dfrac{-1}{2x}$

$= -\dfrac{1}{4}$

6. $f'(1) = \lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1}$

$= \lim\limits_{x \to 1} \dfrac{(x^2 + 4) - (1^2 + 4)}{x - 1}$

$= \lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

$= \lim\limits_{x \to 1} \dfrac{(x-1)(x+1)}{x - 1}$

$= \lim\limits_{x \to 1} (x + 1)$

$= 2$

7. $f'(3) = \lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$

$= \lim\limits_{x \to 3} \dfrac{\sqrt{x+1} - \sqrt{3+1}}{x - 3}$

$= \lim\limits_{x \to 3} \dfrac{(\sqrt{x+1} - 2)}{(x - 3)} \cdot \dfrac{(\sqrt{x+1} + 2)}{(\sqrt{x+1} + 2)}$

$= \lim\limits_{x \to 3} \dfrac{(x+1) - 4}{(x - 3)(\sqrt{x+1} + 2)}$

$= \lim\limits_{x \to 3} \dfrac{x - 3}{(x - 3)(\sqrt{x+1} + 2)}$

$= \lim\limits_{x \to 3} \dfrac{1}{\sqrt{x+1} + 2}$

$= \dfrac{1}{4}$

8. $f'(-1) = \lim\limits_{x \to -1} \dfrac{f(x) - f(-1)}{x - (-1)}$

$= \lim\limits_{x \to -1} \dfrac{(2x + 3) - (2(-1) + 3)}{x + 1}$

$= \lim\limits_{x \to -1} \dfrac{2x + 3 - 1}{x + 1}$

$= \lim\limits_{x \to -1} \dfrac{2x + 2}{x + 1}$

$= \lim\limits_{x \to -1} \dfrac{2(x + 1)}{x + 1}$

$= \lim\limits_{x \to -1} 2$

$= 2$

9. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[3(x+h) - 12] - (3x - 12)}{h}$

$= \lim\limits_{h \to 0} \dfrac{3h}{h}$

$= \lim\limits_{h \to 0} 3$

$= 3$

10. $\dfrac{dy}{dx} = \lim\limits_{h \to 0} \dfrac{y(x+h) - y(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{7(x+h) - 7x}{h}$

$= \lim\limits_{h \to 0} \dfrac{7h}{h}$

$= \lim\limits_{h \to 0} 7$

$= 7$

11. Let $f(x) = x^2$

$\dfrac{d}{dx}(x^2) = f'(x)$

$= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(x+h)^2 - x^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{x^2 + 2xh + h^2 - x^2}{h}$

$= \lim\limits_{h \to 0} (2x + h)$

$= 2x$

12. $\dfrac{d}{dx} f(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{3(x+h)^2 - 3x^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{3x^2 + 6xh + h^2 - 3x^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{6xh + h^2}{h}$

$= \lim\limits_{h \to 0} (6x + h)$

$= 6x$

13. The graph of $y = f_1(x)$ is decreasing for $x < 0$ and increasing for $x > 0$, so its derivative is negative for $x < 0$ and positive for $x > 0$. (b)

14. The graph of $y = f_2(x)$ is always increasing, so its derivative is always ≥ 0. (a)

15. The graph of $y = f_3(x)$ oscillates between increasing and decreasing, so its derivative oscillates between positive and negative. (d)

16. The graph of $y = f_4(x)$ is decreasing, then increasing, then decreasing, and then increasing, so its derivative is negative, then positive, then negative, and then positive. (c)

17. (a) The tangent line has slope 5 and passes through (2, 3).

$$y = 5(x - 2) + 3$$
$$y = 5x - 7$$

(b) The normal line has slope $-\dfrac{1}{5}$ and passes through (2, 3).

$$y = -\frac{1}{5}(x - 2) + 3$$
$$y = -\frac{1}{5}x + \frac{17}{5}$$

18.
$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{[2(x + h)^2 - 13(x + h) + 5] - (2x^2 - 13x + 5)}{h}$$
$$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 13x - 13h + 5 - 2x^2 + 13x - 5}{h}$$
$$= \lim_{h \to 0} \frac{4xh + 2h^2 - 13h}{h}$$
$$= \lim_{h \to 0} (4x + 2h - 13)$$
$$= 4x - 13$$

At $x = 3$, $\dfrac{dy}{dx} = 4(3) - 13 = -1$, so the tangent line has slope -1 and passes through $(3, y(3)) = (3, -16)$.

$$y = -1(x - 3) - 16$$
$$y = -x - 13$$

19. Let $f(x) = x^3$.

$$f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{(1 + h)^3 - 1^3}{h}$$
$$= \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h}$$
$$= \lim_{h \to 0} (3 + 3h + h^2)$$
$$= 3$$

(a) The tangent line has slope 3 and passes through (1, 1). Its equation is $y = 3(x - 1) + 1$, or $y = 3x - 2$.

(b) The normal line has slope $-\dfrac{1}{3}$ and passes through (1, 1). Its equation is $y = -\dfrac{1}{3}(x - 1) + 1$,

or $y = -\dfrac{1}{3}x + \dfrac{4}{3}$.

20. Let $f(x) = \sqrt{x}$.

$$\begin{aligned}
f'(4) &= \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \\
&= \lim_{h \to 0} \frac{(\sqrt{4+h} - 2)}{h} \cdot \frac{(\sqrt{4+h} + 2)}{(\sqrt{4+h} + 2)} \\
&= \lim_{h \to 0} \frac{(4+h) - 4}{h(\sqrt{4+h} + 2)} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{4+h} + 2)} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2} \\
&= \frac{1}{4}
\end{aligned}$$

(a) The tangent line has slope $\frac{1}{4}$ and passes through $(4, 2)$. Its equation is $y = \frac{1}{4}x + 1$.

(b) The normal line has slope -4 and passes through $(4, 2)$. Its equation is $y = -4x + 18$.

21. (a) The amount of daylight is increasing at the fastest rate when the slope of the graph is largest. This occurs about one-fourth of the way through the year, sometime around April 1. The rate at this time is approximately $\frac{4 \text{ hours}}{24 \text{ days}}$ or $\frac{1}{6}$ hour per day.

(b) Yes, the rate of change is zero when the tangent to the graph is horizontal. This occurs near the beginning of the year and halfway through the year, around January 1 and July 1.

(c) Positive: January 1 through July 1
Negative: July 1 through December 31

22. The slope of the given graph is zero at $x \approx -2$ and at $x \approx 1$, so the derivative graph includes $(-2, 0)$ and $(1, 0)$. The slopes at $x = -3$ and at $x = 2$ are about 5 and the slope at $x = -0.5$ is about -2.5, so the derivative graph includes $(-3, 5)$, $(2, 5)$, and $(-0.5, -2.5)$. Connecting the points smoothly, we obtain the graph shown.
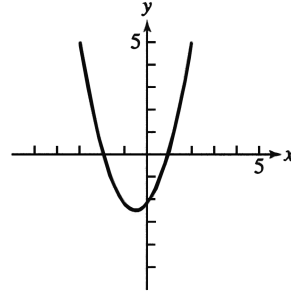
23. (a) Using Figure 3.10a, the number of hares is largest after 2 years and smallest after 6 years. Using Figure 3.10b, the derivative is 0 at these times.

(b) Using Figure 3.10a, there were 120,000 hares when the population was the largest and 60,000 hares when the population was the smallest.

(c) The peak for the hares occurs at 2 years and the peak for the lynx occurs at 4 years. Therefore, 2 years elapse between the peaks.

24. Since the graph of $y = x \ln x - x$ is decreasing for $0 < x < 1$ and increasing for $x > 1$, its derivative is negative for $0 < x < 1$ and positive for $x > 1$. The only one of the given functions with this property is $y = \ln x$. Note also that $y = \ln x$ is undefined for $x < 0$, which further agrees with the given graph. (ii)

25. Each of the functions $y = \sin x$, $y = x$, $y = \sqrt{x}$, has the property that $y(0) = 0$ but the graph has nonzero slope (or undefined slope) at $x = 0$, so none of these functions can be its own derivative. The function $y = x^2$ is not its own derivative because $y(1) = 1$ but

$$\begin{aligned}
y'(1) &= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} \\
&= \lim_{h \to 0} \frac{2h + h^2}{h} \\
&= \lim_{h \to 0} (2 + h) \\
&= 2.
\end{aligned}$$

This leaves only e^x, which can plausibly be its own derivative because both the function value and the slope increase from very small positive values to very large values as we move from left to right along the graph. (iv)

26. (a) The slope from $x = -4$ to $x = 0$ is

$$\frac{2-0}{0-(-4)} = \frac{1}{2}.$$

The slope from $x = 0$ to $x = 1$ is
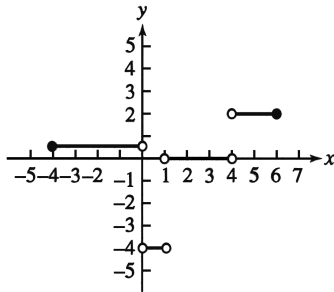
$$\frac{-2-2}{1-0} = -4.$$

The slope from $x = 1$ to $x = 4$ is

$$\frac{-2-(-2)}{4-1} = 0.$$

The slope from $x = 4$ to $x = 6$ is

$$\frac{2-(-2)}{6-4} = 2.$$

Note that the derivative is undefined at $x = 0$, $x = 1$, and $x = 4$. (The function is differentiable at $x = -4$ and at $x = 6$ because these are endpoints of the domain and the one-sided derivatives exist.) The graph of the derivative is shown.
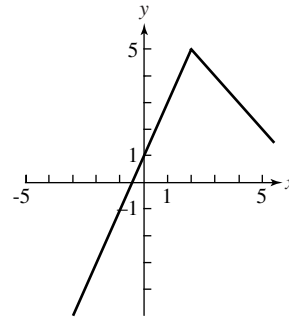

(b) $x = 0, 1, 4$

27. For $x > -1$, the graph of $y = f(x)$ must lie on a line of slope -2 that passes through $(0, -1)$: $y = -2x - 1$. Then $y(-1) = -2(-1) - 1 = 1$, so for $x < -1$, the graph of $y = f(x)$ must lie on a line of slope 1 that passes through $(-1, 1)$: $y = 1(x + 1) + 1$ or $y = x + 2$.

Thus $f(x) = \begin{cases} x+2, & x < -1 \\ -2x-1, & x \geq -1 \end{cases}$
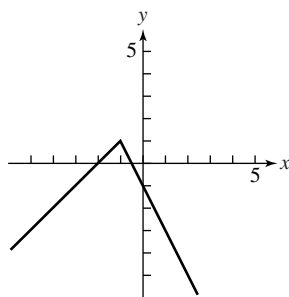
28.
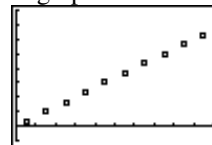

29.

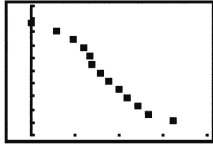| Midpoint of Interval (x)	Slope $\left(\frac{\Delta y}{\Delta x}\right)$
0.5	$\frac{3.3-0}{1-0} = 3.3$
1.5	$\frac{13.3-3.3}{2-1} = 10.0$
2.5	$\frac{29.9-13.3}{3-2} = 16.6$
3.5	$\frac{53.2-29.9}{4-3} = 23.3$
4.5	$\frac{83.2-53.2}{5-4} = 30.0$
5.5	$\frac{119.8-83.2}{6-5} = 36.6$
6.5	$\frac{163.0-119.8}{7-6} = 43.2$
7.5	$\frac{212.9-163.0}{8-7} = 49.9$
8.5	$\frac{269.5-212.9}{9-8} = 56.6$
9.5	$\frac{332.7-269.5}{10-9} = 63.2$

A graph of the derivative data is shown.

[0,10] by [−10,80]

(a) The derivative represents the speed of the skier.

(b) Since the distances are given in feet and the times are given in seconds, the units are feet per second.

(c) The graph appears to be approximately linear and passes through (0, 0) and (9.5, 63.2), so the slope is $\frac{63.2-0}{9.5-0} \approx 6.65$.

The equation of the derivative is approximately $D = 6.65t$.

30. (a)

[−0.5, 4] by [700, 1700]

(b)

Midpoint of Interval (x)	Slope $\left(\frac{\Delta y}{\Delta x}\right)$
$\frac{0.00+0.56}{2} = 0.28$	$\frac{1512-1577}{0.56-0.00} \approx -116.07$
$\frac{0.56+0.92}{2} = 0.74$	$\frac{1448-1512}{0.92-0.56} \approx -177.78$
$\frac{0.92+1.19}{2} = 1.055$	$\frac{1384-1448}{1.19-0.92} \approx -237.04$
$\frac{1.19+1.30}{2} = 1.245$	$\frac{1319-1384}{1.30-1.19} \approx -590.91$
$\frac{1.30+1.39}{2} = 1.345$	$\frac{1255-1319}{1.39-1.30} \approx -711.11$
$\frac{1.39+1.57}{2} = 1.48$	$\frac{1191-1255}{1.57-1.39} \approx -355.56$
$\frac{1.57+1.74}{2} = 1.655$	$\frac{1126-1191}{1.74-1.57} \approx -382.35$
$\frac{1.74+1.98}{2} = 1.86$	$\frac{1062-1126}{1.98-1.74} \approx -266.67$
$\frac{1.98+2.18}{2} = 2.08$	$\frac{998-1062}{2.18-1.98} = -320.00$
$\frac{2.18+2.41}{2} = 2.295$	$\frac{933-998}{2.41-2.18} \approx -282.61$
$\frac{2.41+2.64}{2} = 2.525$	$\frac{869-933}{2.64-2.41} \approx -278.26$
$\frac{2.64+3.24}{2} = 2.94$	$\frac{805-869}{3.24-2.64} \approx -106.67$

A graph of the derivative data is shown.
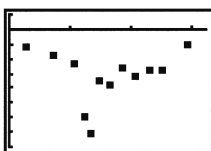
[0, 3.24] by [−800, 100]

(c) Since the elevation y is given in feet and the distance x down river is given in miles, the units of the gradient are feet per mile.

(d) Since the elevation y is given in feet and the distance x downriver is given in miles, the units of the derivative $\frac{dy}{dx}$ are feet per mile.

(e) Look for the steepest part of the curve. This is where the elevation is dropping most rapidly, and therefore the most likely location for significant "rapids."

(f) Look for the lowest point on the graph. This is where the elevation is dropping most rapidly, and therefore the most likely location for significant "rapids."

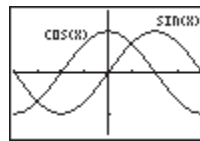**31.** We show that the right-hand derivative at 1 does not exist.

$$\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^+} \frac{3(1+h)-2-2}{h}$$
$$= \lim_{h \to 0^+} \frac{3h-1}{h}$$
$$= -\infty$$

Does not exist.

32. We show that the right-hand derivative at 1 does not exist.

$$\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^+} \frac{3(1+h)-(1)^3}{h}$$
$$= \lim_{h \to 0^+} \frac{2+3h}{h}$$
$$= \lim_{h \to 0^+} \left(\frac{2}{h}+3\right)$$
$$= \infty$$

33.

[−π, π] by [−1.5, 1.5]

The cosine function could be the derivative of the sine function. The values of cosine are positive where sine is increasing, zero where sine has horizontal tangents, and negative where sine is decreasing.

34. $\displaystyle\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^+} \frac{\sqrt{h}-\sqrt{0}}{h}$

$\displaystyle\qquad\qquad = \lim_{h \to 0^+} \frac{\sqrt{h}}{h}$

$\displaystyle\qquad\qquad = \lim_{h \to 0^+} \frac{1}{\sqrt{h}}$

$\displaystyle\qquad\qquad = \infty$

Thus, the right-hand derivative at 0 does not exist.

35. Two parabolas are parallel if they have the same derivative at every value of x. This means that their tangent lines are parallel at each value of x. Two such parabolas are given by $y = x^2$ and $y = x^2 + 4$.
They are graphed below.
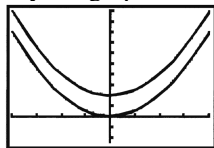
[−4, 4] by [−5, 20]

The parabolas are "everywhere equidistant," as long as the distance between them is always measured along a vertical line.

36. True. $f'(x) = 2x + 1$

37. False. Let $f(x) = |x|$. The left-hand derivative at $x = 0$ is −1 and the right-hand derivative at $x = 0$ is 1. $f'(0)$ does not exist.

38. C; $\displaystyle f'(-1) = \lim_{h \to 0} \frac{f(-1+h)-f(-1)}{h}$

$\displaystyle\qquad = \lim_{h \to 0} \frac{[4-3(-1+h)]-[4-3(-1)]}{h}$

$\displaystyle\qquad = \lim_{h \to 0} \frac{4+3-3h-7}{h}$

$\displaystyle\qquad = \lim_{h \to 0} \frac{-3h}{h}$

$\displaystyle\qquad = \lim_{h \to 0} -3$

$\displaystyle\qquad = -3$

39. A; $\displaystyle f'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$

$\displaystyle\qquad = \lim_{h \to 0} \frac{[1-3(1+h)^2]-[1-3(1)^2]}{h}$

$\displaystyle\qquad = \lim_{h \to 0} \frac{-6h-3h^2}{h}$

$\displaystyle\qquad = \lim_{h \to 0} \frac{h(-6-3h)}{h}$

$\displaystyle\qquad = \lim_{h \to 0} (-6-3h)$

$\displaystyle\qquad = -6$

40. B;

$\displaystyle\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^-} \frac{f(h)-f(0)}{h}$

$\displaystyle\qquad\qquad = \lim_{h \to 0^-} \frac{(h^2-1)-(-1)}{h}$

$\displaystyle\qquad\qquad = \lim_{h \to 0^-} \frac{h^2}{h}$

$\displaystyle\qquad\qquad = \lim_{h \to 0^-} h$

$\displaystyle\qquad\qquad = 0$

41. C;

$\displaystyle\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^+} \frac{f(h)-f(0)}{h}$

$\displaystyle\qquad\qquad = \lim_{h \to 0^+} \frac{(2h-1)-(-1)}{h}$

$\displaystyle\qquad\qquad = \lim_{h \to 0^+} \frac{2h}{h}$

$\displaystyle\qquad\qquad = \lim_{h \to 0^+} 2$

$\displaystyle\qquad\qquad = 2$

42. (a) $\displaystyle f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$\displaystyle\qquad = \lim_{h \to 0} \frac{(x+h)^2-x^2}{h}$

$\displaystyle\qquad = \lim_{h \to 0} \frac{2xh+h^2}{h}$

$\displaystyle\qquad = \lim_{h \to 0} (2x+h)$

$\displaystyle\qquad = 2x$

(b) $\displaystyle f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$\displaystyle\qquad = \lim_{h \to 0} \frac{2(x+h)-2x}{h}$

$\displaystyle\qquad = \lim_{h \to 0} 2$

$\displaystyle\qquad = 2$

(c) $\lim\limits_{x\to 1^-} f'(x) = \lim\limits_{x\to 1^-} 2x = 2(1) = 2$

(d) $\lim\limits_{x\to 1^+} f'(x) = \lim\limits_{x\to 1^+} 2 = 2$

(e) Yes, the one-sided limits exist and are the same, so $\lim\limits_{x\to 1} f'(x) = 2$.

(f) $\lim\limits_{h\to 0^-} \dfrac{f(1+h) - f(1)}{h}$

$= \lim\limits_{h\to 0^-} \dfrac{(1+h)^2 - 1^2}{h}$

$= \lim\limits_{h\to 0^-} \dfrac{1 + 2h + h^2 - 1}{h}$

$= \lim\limits_{h\to 0^-} \dfrac{2h + h^2}{h}$

$= \lim\limits_{h\to 0^-} \dfrac{h(2+h)}{h}$

$= \lim\limits_{h\to 0^-} (2 + h)$

$= 2$

(g) $\lim\limits_{h\to 0^+} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to 0^+} \dfrac{2(1+h) - 1^2}{h}$

$= \lim\limits_{h\to 0^+} \dfrac{1 + 2h}{h}$

$= \lim\limits_{h\to 0^+} \left(\dfrac{1}{h} + 2\right)$

$= \infty$

The right-hand derivative does not exist.

(h) It does not exist because the right-hand derivative does not exist.

43. (e) The y-intercept of the derivative is $b - a$.

44. Since the function must be continuous at $x = 1$, we have $\lim\limits_{x\to 1^+} (3x + k) = f(1) = 1$, so $3 + k = 1$,

or $k = -2$. This gives $f(x) = \begin{cases} x^3, & x \le 1 \\ 3x - 2, & x > 1. \end{cases}$

Now we confirm that $f(x)$ is differentiable at $x = 1$.

$\lim\limits_{h\to 0^-} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to 0^-} \dfrac{(1+h)^3 - (1)^3}{h}$

$= \lim\limits_{h\to 0^-} \dfrac{3h + 3h^2 + h^3}{h}$

$= \lim\limits_{h\to 0^-} (3 + 3h + h^2)$

$= 3$

$\lim\limits_{h\to 0^+} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to 0^+} \dfrac{[3(1+h) - 2] - (1)^3]}{h}$

$= \lim\limits_{h\to 0^+} \dfrac{(1 + 3h) - 1}{h}$

$= \lim\limits_{h\to 0^+} 3$

$= 3$

Since the right-hand derivative equals the left-hand derivative at $x = 1$, the derivative exists (and is equal to 3) when $k = -2$.

45. (a) $1 \cdot \dfrac{364}{365} \cdot \dfrac{363}{365} \approx 0.992$

Alternate method: $\dfrac{_{365}P_3}{365^3} \approx 0.992$

(b) Using the answer to part (a), the probability is about $1 - 0.992 = 0.008$.

(c) Let P represent the answer to part (b), $P \approx 0.008$. Then the probability that three people all have different birthdays is $1 - P$. Adding a fourth person, the probability that all have different birthdays is $(1 - P)\left(\dfrac{362}{365}\right)$, so the probability of a shared birthday is $1 - (1 - P)\left(\dfrac{362}{365}\right) \approx 0.016$.

(d) No; clearly February 29 is a much less likely birth date. Furthermore, census data do not support the assumption that the other 365 birth dates are equally likely. However, this simplifying assumption may still give us some insight into this problem even if the calculated probabilities aren't completely accurate.

46. The derivative looks like a quadratic with roots at $x = 2$ and $x = 6$. All such quadratics have the form

$a(x - 2)(x - 6) = a(x^2 - 8x + 12)$.

Since $f'(0) = 4$, then $a = \dfrac{1}{3}$, and

$f'(x) = \dfrac{1}{3}(x^2 - 8x + 12)$.

Using the method of Example 1, a function that has f' as its derivative is

$\dfrac{1}{3}\left(\dfrac{x^3}{3} - 4x^2 + 12x\right)$. Adding a constant to this

trial version of f does not change f', so since $f(0) = 1$, the function is

$$f(x) = \frac{1}{3}\left(\frac{x^3}{3} - 4x^2 + 12x\right) + 1.$$

## Section 3.2   Differentiability
(pp. 111–117)

Exploration 1   Zooming in to "See" Differentiability

1. Zooming in on the graph of f at the point $(0, 1)$ always produces a graph exactly like the one shown below, provided that a square window is used. The corner shows no sign of straightening out.
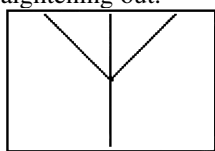

[–0.25, 0.25] by [0.836, 1.164]

2. Zooming in on the graph of g at the point $(0, 1)$ begins to reveal a smooth turning point. This graph shows the result of three zooms, each by a factor of 4 horizontally and vertically, starting with the window. [–4, 4] by [–1.624, 3.624].
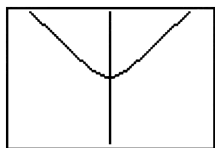
[–0.0625, 0.0625] by [0.959, 1.041]

3. On our grapher, the graph became horizontal after 8 zooms. Results can vary on different machines.

4. As we zoom in on the graphs of f and g together, the differentiable function gradually straightens out to resemble its tangent line, while the nondifferentiable function stubbornly retains its same shape.
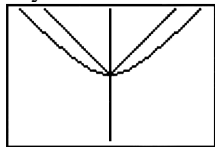
[–0.03125, 0.03125] by [0.9795, 1.0205]

Exploration 2   Looking at the Symmetric
Difference Quotient Analytically

1. $\dfrac{f(10+h) - f(10)}{h} = \dfrac{(10.01)^2 - 10^2}{0.01} = 20.01$

$f'(10) = 2 \cdot 10 = 20$

The difference quotient is 0.01 away from $f'(10)$.

2. $\dfrac{f(10+h) - f(10-h)}{2h} = \dfrac{(10.01)^2 - (9.99)^2}{0.02}$
$$= 20$$

The symmetric difference quotient exactly equals $f'(10)$.

3. $\dfrac{f(10+h) - f(10)}{h} = \dfrac{(10.01)^3 - 10^3}{0.01}$
$$= 300.3001$$

$f'(10) = 3 \cdot 10^2 = 300$

The difference quotient is 0.3001 away from $f'(10)$.

$\dfrac{f(10+h) - f(10-h)}{2h} = \dfrac{(10.01)^3 - (9.99)^3}{0.02}$
$$= 300.0001.$$

The symmetric difference quotient is 0.0001 away from $f'(10)$.

Quick Review 3.2

1. Yes

2. No (The $f(h)$ term in the numerator is incorrect.)

3. Yes

4. Yes

5. No (The denominator for this expression should be $2h$).

6. All reals

7. $[0, \infty)$

8. $[3, \infty)$

9. The equation is equivalent to $y = 3.2x + (3.2\pi + 5)$, so the slope is 3.2.

10. $\dfrac{f(3+0.001)-f(3-0.001)}{0.002}$

$=\dfrac{5(3+0.001)-5(3-0.001)}{0.002}$

$=\dfrac{5(0.002)}{0.002}$

$=5$

Section 3.2 Exercises

1. Left-hand derivative:

$$\lim_{h\to0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0^-}\frac{h^2-0}{h}$$
$$=\lim_{h\to0^-}h$$
$$=0$$

Right-hand derivative:

$$\lim_{h\to0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0^+}\frac{h-0}{h}$$
$$=\lim_{h\to0^+}1$$
$$=1$$

Since $0\ne1$, the function is not differentiable at the point P.

2. Left-hand derivative:

$$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^-}\frac{2-2}{h}$$
$$=\lim_{h\to0^-}0$$
$$=0$$

Right-hand derivative:

$$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^+}\frac{2(1+h)-2}{h}$$
$$=\lim_{h\to0^+}2$$
$$=2$$

Since $0\ne2$, the function is not differentiable at the point P.

3. Left-hand derivative:

$$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to0^-}\frac{\sqrt{1+h}-1}{h}$$
$$=\lim_{h\to0^-}\frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)}$$
$$=\lim_{h\to0^-}\frac{(1+h)-1}{h(\sqrt{1+h}+1)}$$
$$=\lim_{h\to0^-}\frac{1}{\sqrt{1+h}+1}=\frac{1}{2}$$

Right-hand derivative:

$$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^+}\frac{[2(1+h)-1]-1}{h}$$
$$=\lim_{h\to0^+}\frac{2h}{h}$$
$$=\lim_{h\to0^+}2=2$$

Since $\dfrac{1}{2}\ne2$, the function is not differentiable at the point P.

4. Left-hand derivative:

$$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^-}\frac{(1+h)-1}{h}$$
$$=\lim_{h\to0^-}1$$
$$=1$$

Right-hand derivative:

$$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^+}\frac{\frac{1}{1+h}-1}{h}$$
$$=\lim_{h\to0^+}\frac{1-(1+h)}{h(1+h)}$$
$$=\lim_{h\to0^+}\frac{-h}{h(1+h)}$$
$$=\lim_{h\to0^+}-\frac{1}{1+h}$$
$$=-1$$

Since $1\ne-1$, the function is not differentiable at the point P.

5. **(a)** All points in $[-3,2]$

   (b) None

   (c) None

6. **(a)** All points in $[-2,3]$

   (b) None

   (c) None

7. **(a)** All points in $[-3,3]$ except $x=0$

   (b) None

   (c) $x=0$

8. **(a)** All points in $[-2,3]$ except $x=-1,0,2$

   (b) $x=-1$

   (c) $x=0,\,x=2$

9. (a) All points in $[-1, 2]$ except $x = 0$

(b) $x = 0$

(c) None

10. (a) All points in $[-3, 3]$ except $x = -2, 2$

(b) $x = -2, x = 2$

(c) None

11. Since $\lim\limits_{x \to 0} \tan^{-1} x = \tan^{-1} 0 = 0 \neq y(0),$ the
problem is a discontinuity.

12.
$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^{4/5}}{h}$$
$$= \lim_{h \to 0^-} \frac{1}{h^{1/5}}$$
$$= -\infty$$
$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^{4/5}}{h}$$
$$= \lim_{h \to 0^+} \frac{1}{h^{1/5}}$$
$$= \infty$$
The problem is a cusp.

13. Note that $y = x + \sqrt{x^2} + 2$
$$= x + |x| + 2$$
$$= \begin{cases} 2, & x \leq 0 \\ 2x + 2, & x > 0. \end{cases}$$
$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{2 - 2}{h}$$
$$= \lim_{h \to 0^-} 0$$
$$= 0$$
$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(2h + 2) - 2}{h}$$
$$= \lim_{h \to 0^+} 2$$
$$= 2$$
The problem is a corner.

14.
$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{(3 - \sqrt[3]{h}) - 3}{h}$$
$$= \lim_{h \to 0} -\frac{\sqrt[3]{h}}{h}$$
$$= \lim_{h \to 0} \left(-\frac{1}{h^{2/3}}\right)$$
$$= -\infty$$
The problem is a vertical tangent.

15. Note that $y = 3x - 2|x| - 1 = \begin{cases} 5x - 1, & x \leq 0 \\ x - 1, & x > 0 \end{cases}$
$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(5h - 1) - (-1)}{h}$$
$$= \lim_{h \to 0^-} 5$$
$$= 5$$
$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(h - 1) - (-1)}{h}$$
$$= \lim_{h \to 0^+} 1$$
$$= 1$$
The problem is a corner.

16.
$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{\sqrt[3]{|h|} - 0}{h}$$
$$= \lim_{h \to 0^-} \frac{-\sqrt[3]{h}}{h}$$
$$= \lim_{h \to 0^-} \left(-\frac{1}{h^{2/3}}\right)$$
$$= -\infty$$
$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{\sqrt[3]{|h|} - 0}{h}$$
$$= \lim_{h \to 0^+} \frac{\sqrt[3]{h}}{h}$$
$$= \lim_{h \to 0^+} \frac{1}{h^{2/3}}$$
$$= \infty$$
The problem is a cusp.

17. $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h} = \dfrac{4(0.001)-(0.001)^2-(4(-0.001)-(-0.001)^2)}{0.002}$

$= 4,$ yes it is differentiable.

18. $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h} = \dfrac{4(3.001)-(3.001)^2-(4(2.999)-(2.999)^2)}{0.002}$

$= -2,$ yes it is differentiable.

19. $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h} = \dfrac{4(1.001)+(1.001)^2-4(0.999)-(0.99)^2}{0.002}$

$= 2,$ yes it is differentiable.

20. $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h} = \dfrac{(0.001)^3-4(0.001)-((-0.001)^3-4(-0.001))}{0.002}$

$= -3.999999,$ yes it is differentiable.

21. $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h} = \dfrac{(-1.999)^3-4(-1.999)-((-2.001)^3-4(-2.001))}{0.002}$

$= 8.000001,$ yes it is differentiable.

22. $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h} = \dfrac{(2.001)^3-4(2.001)-((1.999)^3-4(1.999))}{0.002}$

$= 8.000001,$ yes it is differentiable.

23. $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h} = \dfrac{(0.001)^{2/3}-(-0.001)^{2/3}}{0.002}$

$= 0,$ no it is not differentiable. (CUSP)

24. $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h} = \dfrac{\left|3.001-3\right|-\left|2.999-3\right|}{0.002}$

$= 0,$ no it is not differentiable. (CORNER)

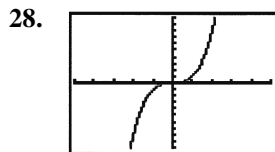**25.** $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h} = \dfrac{(0.001)^{2/5}-(-0.001)^{2/5}}{0.002}$

$= 0,$ no it is not differentiable. (CUSP)

26. $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h} = \dfrac{(0.001)^{4/5}-(-0.001)^{4/5}}{0.002}$
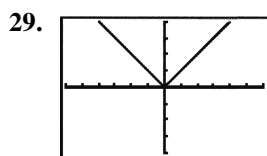
$= 0,$ no it is not differentiable. (CUSP)

27.

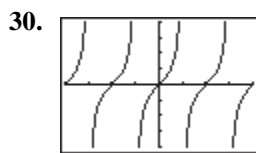$[-2\pi, 2\pi]$ by $[-1.5, 1.5]$

$\dfrac{dy}{dx} = \sin x$

28.

[−5, 5] by [−10, 10]

$$\frac{dy}{dx} = x^3$$

29.
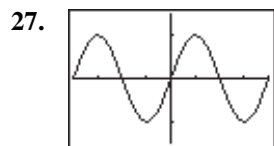
[−6, 6] by [−4, 4]

$$\frac{dy}{dx} = \text{abs }(x) \text{ or } |x|$$

30.

[−2π, 2π] by [−4, 4]

$$\frac{dy}{dx} = \tan x$$

Note: Due to the way NDER is defined, the graph of $y = $ NDER (x) actually has two asymptotes for each asymptote of $y = \tan x$. The asymptotes of $y = $ NDER (x) occur at

$x = \dfrac{\pi}{2} + k\pi \pm 0.001$, where k is an integer. A

good window for viewing this behavior is [1.566, 1.576] by [−1000, 1000].

31. Find the zeros of the denominator.
$$x^2 - 4x - 5 = 0$$
$$(x+1)(x-5) = 0$$
$$x = -1 \text{ or } x = 5$$
The function is a rational function, so it is differentiable for all x in its domain: all reals except $x = -1, 5$.

32. The function is differentiable except possibly where $3x - 6 = 0$, that is, at $x = 2$. We check for differentiability at $x = 2$, using k instead of the usual h, in order to avoid confusion with the function $h(x)$.

$$\lim_{k \to 0} \frac{h(2+k) - h(2)}{k}$$
$$= \lim_{k \to 0} \frac{[\sqrt[3]{3(2+k) - 6} + 5] - 5}{k}$$
$$= \lim_{k \to 0} \frac{\sqrt[3]{3k}}{k}$$
$$= \sqrt[3]{3} \lim_{k \to 0} \frac{1}{k^{2/3}}$$
$$= \infty$$
The function has a vertical tangent at $x = 2$. It is differentiable for all reals except $x = 2$.

33. Note that the sine function is odd, so
$$P(x) = \sin(|x|) - 1 = \begin{cases} -\sin x - 1, x < 0 \\ \sin x - 1, \quad x \geq 0. \end{cases}$$
The graph of $P(x)$ has a corner at $x = 0$. The function is differentiable for all reals except $x = 0$.

34. Since the cosine function is even,
$$Q(x) = 3\cos(|x|) = 3\cos x. \text{ The function is}$$
differentiable for all reals.

35. The function is piecewise-defined in terms of polynomials, so it is differentiable everywhere except possibly at $x = 0$ and at $x = 3$. Check $x = 0$:

$$\lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^-} \frac{(h+1)^2 - 1}{h}$$
$$= \lim_{h \to 0^-} \frac{h^2 + 2h}{h}$$
$$= \lim_{h \to 0^-} (h+2)$$
$$= 2$$

$$\lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^+} \frac{(2h+1) - 1}{h}$$
$$= \lim_{h \to 0^+} 2$$
$$= 2$$

The function is differentiable at $x = 0$.
Check $x = 3$:
Since $g(3) = (4 - 3)^2 = 1$ and
$\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-} (2x+1) = 2(3) + 1 = 7$, the
function is not continuous (and hence not differentiable) at $x = 3$. The function is differentiable for all reals except $x = 3$.

36. Note that $C(x) = x|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$, so it is

Check $x = 0$:

$$\lim_{h \to 0} \frac{C(0+h) - C(0)}{h} = \lim_{h \to 0} \frac{h|h| - 0}{h}$$
$$= \lim_{h \to 0} |h|$$
$$= 0$$

The function is differentiable for all reals.

37. The function $f(x)$ does not have the intermediate value property. Choose some a in $(-1, 0)$ and b in $(0, 1)$. Then $f(a) = 0$ and $f(b) = 1$, but f does not take on any value between 0 and 1. Therefore, by the Intermediate Value Theorem for Derivatives, f cannot be the derivative of any function on $[-1, 1]$.

38. (a) $x = 0$ is not in their domains, or, they are both discontinuous at $x = 0$.

(b) For $\dfrac{1}{x}$: $\text{NDER}\left(\dfrac{1}{x}, 0\right) = 1{,}000{,}000$

For $\dfrac{1}{x^2}$: $\text{NDER}\left(\dfrac{1}{x^2}, 0\right) = 0$

(c) It returns an incorrect response because even though these functions are not defined at $x = 0$, they are defined at $x = \pm 0.001$. The responses differ from each other because $\dfrac{1}{x^2}$ is even (which automatically makes NDER $\left(\dfrac{1}{x^2}, 0\right) = 0$) and $\dfrac{1}{x}$ is odd.

39. (a) $\lim_{x \to 1^-} f(x) = f(1)$

$$\lim_{x \to 1^-} (3 - x) = a(1)^2 + b(1)$$
$$2 = a + b$$

The relationship is $a + b = 2$.

(b) Since the function needs to be continuous, we may assume that $a + b = 2$ and $f(1) = 2$.

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0^-} \frac{(3 - (1+h)) - 2}{h}$$
$$= \lim_{h \to 0^-} (-1)$$
$$= -1$$

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0^+} \frac{a(1+h)^2 + b(1+h) - 2}{h}$$
$$= \lim_{h \to 0^+} \frac{a + 2ah + ah^2 + b + bh - 2}{h}$$
$$= \lim_{h \to 0^+} \frac{2ah + ah^2 + bh + (a + b - 2)}{h}$$
$$= \lim_{h \to 0^+} (2a + ah + b)$$
$$= 2a + b$$

Therefore, $2a + b = -1$. Substituting $2 - a$ for b gives $2a + (2 - a) = -1$, so $a = -3$. Then $b = 2 - a = 2 - (-3) = 5$. The values are $a = -3$ and $b = 5$.

40. True. See Theorem 1.

41. False. The function $f(x) = |x|$ is continuous at $x = 0$ but is not differentiable at $x = 0$.
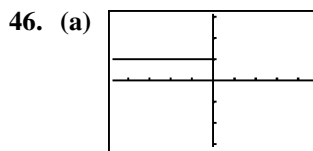
42. B

43. A; $\text{NDER}(f, a) = \dfrac{f(a+h) - f(a-h)}{2h}$

$$= \frac{\sqrt[3]{1.001 - 1} - \sqrt[3]{0.999 - 1}}{0.002}$$
$$= 100$$

The symmetric difference quotient gets larger as h gets smaller, so $f'(1)$ is undefined.

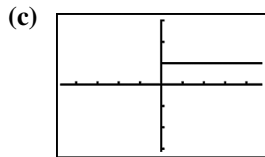**44.** B; $\lim_{h \to 0^-} \dfrac{2(0+h) + 1 - (2(0) + 1)}{h} = \lim_{h \to 0^-} \dfrac{2h}{h} = 2$

45. C; $\lim_{h \to 0^+} \dfrac{(0+h)^2 + 1 - (0^2 + 1)}{h} = \lim_{h \to 0^+} \dfrac{h^2}{h} = 0$

46. (a)

[−4.7, 4.7] by [−3.1, 3.1]

(b) You can use Trace to help see that the value of Y1 is 1 for every $x < 0$ and is 0 for every $x \geq 0$. It appears to be the graph

of $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$

(c)

[−4.7, 4.7] by [−3.1, 3.1]

(d) You can use Trace to help see that the value of Y1 is 0 for every $x < 0$ and is 1 for every $x \geq 0$. It appears to be the graph
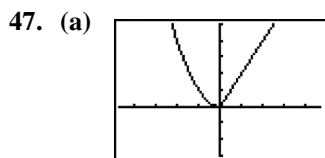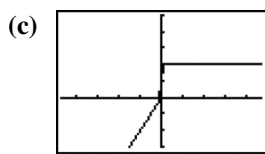
of $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$

47. (a)

[−4.7, 4.7] by [−3, 5]

(b) See exercise 46.

(c)

[−4.7, 4.7] by [−3, 5]

(d) NDER(Y1, –0.1) = –0.1,
NDER(Y1, 0) = 0.9995,
NDER(Y1, 0.1) = 2.
The numerical derivative gets it right at −0.1 and 0.1, but gets it wrong at 0 where the function is discontinuous and nondifferentiable.

48. (a) Note that $-|x| \leq x \sin(1/x) \leq |x|$ for all x

except 0, so $\displaystyle\lim_{x \to 0}\left(x \sin\frac{1}{x}\right) = 0$ by the

Sandwich Theorem. Therefore, f is continuous at $x = 0$.

(b) $\dfrac{f(0+h) - f(0)}{h} = \dfrac{h\sin\frac{1}{h} - 0}{h} = \sin\frac{1}{h}$

(c) The limit does not exist because $\sin\dfrac{1}{h}$

oscillates between −1 and 1 an infinite number of times arbitrarily close to $h = 0$

(that is, for h in any open interval containing 0).

(d) No, because the one-sided limits (as in part (c)) do not exist.

(e) $\dfrac{g(0+h) - g(0)}{h} = \dfrac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = h\sin\frac{1}{h}$

As noted in part (a), the limit of this as h approaches zero is 0, so $g'(0) = 0$.

Section 3.3 Rules for Differentiation
 (pp. 118–128)

Quick Review 3.3

1. $(x^2 - 2)(x^{-1} + 1) = x^2 x^{-1} + x^2 \cdot 1 - 2x^{-1} - 2 \cdot 1$
 $= x + x^2 - 2x^{-1} - 2$

2. $\left(\dfrac{x}{x^2 + 1}\right)^{-1} = \dfrac{x^2 + 1}{x} = \dfrac{x^2}{x} + \dfrac{1}{x} = x + x^{-1}$

3. $3x^2 - \dfrac{2}{x} + \dfrac{5}{x^2} = 3x^2 - 2x^{-1} + 5x^{-2}$

4. $\dfrac{3x^4 - 2x^3 + 4}{2x^2} = \dfrac{3x^4}{2x^2} - \dfrac{2x^3}{2x^2} + \dfrac{4}{2x^2}$
 $= \dfrac{3}{2}x^2 - x + 2x^{-2}$

5. $(x^{-1} + 2)(x^{-2} + 1)$
 $= x^{-1}x^{-2} + x^{-1} \cdot 1 + 2x^{-2} + 2 \cdot 1$
 $= x^{-3} + x^{-1} + 2x^{-2} + 2$

6. $\dfrac{x^{-1} + x^{-2}}{x^{-3}} = x^3(x^{-1} + x^{-2}) = x^2 + x$

7.
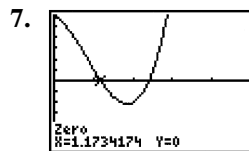
[0, 5] by [−6, 6]

At $x \approx 1.173$, $500x^6 \approx 1305$.
At $x \approx 2.394$, $500x^6 \approx 94{,}212$.
After rounding, we have:
At $x \approx 1$, $500x^6 \approx 1305$.
At $x \approx 2$, $500x^6 \approx 94{,}212$.

8. (a) $f(10) = 7$

(b) $f(0) = 7$

(c) $f(x+h) = 7$

(d) $\displaystyle \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{7 - 7}{x - a} = \lim_{x \to a} 0 = 0$

9. These are all constant functions, so the graph of each function is a horizontal line and the derivative of each function is 0.

10. (a) $\displaystyle f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\displaystyle = \lim_{h \to 0} \frac{\frac{x+h}{\pi} - \frac{x}{\pi}}{h}$

$\displaystyle = \lim_{h \to 0} \frac{x + h - x}{\pi h}$

$\displaystyle = \lim_{h \to 0} \frac{1}{\pi}$

$\displaystyle = \frac{1}{\pi}$

(b) $\displaystyle f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\displaystyle = \lim_{h \to 0} \frac{\frac{\pi}{x+h} - \frac{\pi}{x}}{h}$

$\displaystyle = \lim_{h \to 0} \frac{\pi x - \pi(x+h)}{hx(x+h)}$

$\displaystyle = \lim_{h \to 0} \frac{-\pi h}{hx(x+h)}$

$\displaystyle = \lim_{h \to 0} -\frac{\pi}{x(x+h)}$

$\displaystyle = -\frac{\pi}{x^2}$

$\displaystyle = -\pi x^{-2}$

Section 3.3 Exercises

1. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}(-x^2) + \frac{d}{dx}(3) = -2x + 0 = -2x$

2. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{3}x^3\right) - \frac{d}{dx}(x) = x^2 - 1$

3. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}(2x) + \frac{d}{dx}(1) = 2 + 0 = 2$

4. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(1)$

$= 2x + 1 + 0$

$= 2x + 1$

5. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{3}x^3\right) + \frac{d}{dx}\left(\frac{1}{2}x^2\right) + \frac{d}{dx}(x)$

$= x^2 + x + 1$

6. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}(1) - \frac{d}{dx}(x) + \frac{d}{dx}(x^2) - \frac{d}{dx}(x^3)$

$= 0 - 1 + 2x - 3x^2$

$= -1 + 2x - 3x^2$

7. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}(x^3 - 2x^2 + x + 1)$

$= 3x^2 - 4x + 1$

$= 0$

Horizontal tangents at $x = \dfrac{1}{3},\ 1.$

8. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}(x^3 - 4x^2 + x + 2)$

$= 3x^2 - 8x + 1$

$= 0$

Horizontal tangents at $x = \dfrac{4 - \sqrt{13}}{3},\ \dfrac{4 + \sqrt{13}}{3}.$

9. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}(x^4 - 4x^2 + 1) = 4x^3 - 8x = 0$

Horizontal tangents at $x = 0,\ \pm\sqrt{2}.$

10. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}(4x^3 - 6x^2 - 1) = 12x^2 - 12x = 0$

Horizontal tangents at $x = 0,\ 1.$

11. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}(5x^3 - 3x^5) = 15x^2 - 15x^4 = 0$

Horizontal tangents at $x = -1,\ 0,\ 1.$

12. $\displaystyle \frac{dy}{dx} = \frac{d}{dx}(x^4 - 7x^3 + 2x^2 + 15)$

$= 4x^3 - 21x^2 + 4x$

$= 0$

Horizontal tangents at $x = 0,\ \dfrac{21 - \sqrt{377}}{8},$

$\dfrac{21 + \sqrt{377}}{8}.$

13. (a) $\dfrac{dy}{dx} = \dfrac{d}{dx}[(x+1)(x^2+1)]$

$= (x+1)\dfrac{d}{dx}(x^2+1) + (x^2+1)\dfrac{d}{dx}(x+1)$

$= (x+1)(2x) + (x^2+1)(1)$

$= 2x^2 + 2x + x^2 + 1$

$= 3x^2 + 2x + 1$

(b) $\dfrac{dy}{dx} = \dfrac{d}{dx}[(x+1)(x^2+1)]$

$= \dfrac{d}{dx}(x^3 + x^2 + x + 1)$

$= 3x^2 + 2x + 1$

14. (a) $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2+3}{x}\right)$

$= \dfrac{x\dfrac{d}{dx}(x^2+3) - (x^2+3)\dfrac{d}{dx}(x)}{x^2}$

$= \dfrac{x(2x) - (x^2+3)}{x^2}$

$= \dfrac{x^2-3}{x^2}$

(b) $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2+3}{x}\right)$

$= \dfrac{d}{dx}(x + 3x^{-1})$

$= 1 - 3x^{-2}$

$= 1 - \dfrac{3}{x^2}$

This is equivalent to the answer in part (a).

15. $(x^3+x+1)(x^4+x^2+1)$

$\dfrac{d}{dx}(x^7 + 2x^5 + x^4 + 2x^3 + x^2 + x + 1)$

$= 7x^6 + 10x^4 + 4x^3 + 6x^2 + 2x + 1$

16. $(x^2+1)(x^3+1)$

$\dfrac{d}{dx}(x^5 + x^3 + x^2 + 1) = 5x^4 + 3x^2 + 2x$

17. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{2x+5}{3x-2}\right)$

$= \dfrac{(3x-2)(2) - (2x+5)(3)}{(3x-2)^2}$

$= -\dfrac{19}{(3x-2)^2}$

18. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2+5x-1}{x^2}\right)$

$= \dfrac{d}{dx}(1 + 5x^{-1} - x^{-2})$

$= 0 - 5x^{-2} + 2x^{-3}$

$= -\dfrac{5}{x^2} + \dfrac{2}{x^3}$

19. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{(x-1)(x^2+x+1)}{x^3}\right)$

$= \dfrac{d}{dx}\left(\dfrac{x^3-1}{x^3}\right)$

$= \dfrac{d}{dx}(1 - x^{-3})$

$= 0 + 3x^{-4}$

$= \dfrac{3}{x^4}$

20. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1-x}{1+x^2}\right)$

$= \dfrac{(1+x^2)(-1) - (1-x)(2x)}{(1+x^2)^2}$

$= \dfrac{x^2-2x-1}{(1+x^2)^2}$

21. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2}{1-x^3}\right)$

$= \dfrac{(1-x^3)(2x) - x^2(-3x^2)}{(1-x^3)^2}$

$= \dfrac{x^4+2x}{(1-x^3)^2}$

22. $\dfrac{dy}{dx}$

$= \dfrac{d}{dx}\left(\dfrac{(x+1)(x+2)}{(x-1)(x-2)}\right)$

$= \dfrac{d}{dx}\left(\dfrac{x^2+3x+2}{x^2-3x+2}\right)$

$= \dfrac{(x^2-3x+2)(2x+3)-(x^2+3x+2)(2x-3)}{(x^2-3x+2)^2}$

$= \dfrac{(2x^3-3x^2-5x+6)-(2x^3+3x^2-5x-6)}{(x^2-3x+2)^2}$

$= \dfrac{12-6x^2}{(x^2-3x+2)^2}$

23. (a) At $x=0$, $\dfrac{d}{dx}(uv)=u(0)v'(0)+v(0)u'(0)$

$\qquad = (5)(2)+(-1)(-3)$

$\qquad = 13$

(b) At $x=0$, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v(0)u'(0)-u(0)v'(0)}{[v(0)]^2}$

$\qquad = \dfrac{(-1)(-3)-(5)(2)}{(-1)^2}$

$\qquad = -7$

(c) At $x=0$, $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)=\dfrac{u(0)v'(0)-v(0)u'(0)}{[u(0)]^2}$

$\qquad = \dfrac{(5)(2)-(-1)(-3)}{(5)^2}$

$\qquad = \dfrac{7}{25}$

(d) At $x=0$, $\dfrac{d}{dx}(7v-2u)=7v'(0)-2u'(0)$

$\qquad = 7(2)-2(-3)$

$\qquad = 20$

24. (a) At $x=2$, $\dfrac{d}{dx}(uv)=u(2)v'(2)+v(2)u'(2)$

$\qquad = (3)(2)+(1)(-4)$

$\qquad = 2$

(b) At $x=2$, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v(2)u'(2)-u(2)v'(2)}{[v(2)]^2}$

$\qquad = \dfrac{(1)(-4)-(3)(2)}{(1)^2}$

$\qquad = -10$

(c) At $x=2$, $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)=\dfrac{u(2)v'(2)-v(2)u'(2)}{[u(2)]^2}$

$\qquad = \dfrac{(3)(2)-(1)(-4)}{(3)^2}$

$\qquad = \dfrac{10}{9}$

(d) Use the result from part (a) for $\dfrac{d}{dx}(uv)$.

$\quad$ At $x=2$, $\dfrac{d}{dx}(3u-2v+2uv)$

$\qquad = 3u'(2)-2v'(2)+2\dfrac{d}{dx}(uv)\Big|_{x=2}$

$\qquad = 3(-4)-2(2)+2(2)$

$\qquad = -12$

25. $y'(x)=2x+5$

$\quad y'(3)=2(3)+5=11$

$\quad$ The slope is 11. (iii)

26. The given equation is equivalent to

$\quad y=\dfrac{3}{2}x+6$, so the slope is $\dfrac{3}{2}$. (iii)

27. $\dfrac{dy}{dx}=\dfrac{d}{dx}\left(\dfrac{x^3+1}{2x}\right)$

$\qquad = \dfrac{(3x^2)2x-2(x^3+1)}{4x^2}$

$\qquad = \dfrac{4x^3-2}{4x^2}$

$\quad y'(1)=\dfrac{4(1)^3-2}{4(1)^2}=\dfrac{1}{2}$

$\quad y(1)=\dfrac{(1)^3+1}{2(1)}=1$

$\quad y=\dfrac{1}{2}(x-1)+1=\dfrac{1}{2}x+\dfrac{1}{2}$

28. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^4+2}{x^2}\right)$

$= \dfrac{(4x^3)x^2 - 2x(x^4+2)}{x^4}$

$= \dfrac{2x^4-4}{x^3}$

$y'(-1) = \dfrac{2(-1)^4-4}{(-1)^3} = 2$

$y(-1) = \dfrac{(-1)^4+2}{(-1)^2} = 3$

$y = 2(x+1)+3$

$y = 2x+5$

29. $\dfrac{dy}{dx} = \dfrac{d}{dx}(4x^{-2}-8x+1) = -8x^{-3}-8$

30. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^{-4}}{4} - \dfrac{x^{-3}}{3} + \dfrac{x^{-2}}{2} - x^{-1} + 3\right)$

$= -x^{-5}+x^{-4}-x^{-3}+x^{-2}$

31. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right)$

$= \dfrac{d}{dx}\left(\dfrac{x^{1/2}-1}{x^{1/2}+1}\right)$

$= \dfrac{(x^{1/2}+1)\frac{1}{2}x^{-1/2} - (x^{1/2}-1)\frac{1}{2}x^{-1/2}}{(x^{1/2}+1)^2}$

$= \dfrac{\frac{1}{2}x^{-1/2}[(x^{1/2}+1)-(x^{1/2}-1)]}{(x^{1/2}+1)^2}$

$= \dfrac{\frac{1}{2}x^{-1/2}\cdot 2}{(x^{1/2}+1)^2}$

$= \dfrac{1}{\sqrt{x}(\sqrt{x}+1)^2}$

32. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(2\sqrt{x}-\dfrac{1}{\sqrt{x}}\right) = \dfrac{1}{\sqrt{x}} + \dfrac{1}{2x^{3/2}}$

33. $y = x^4+x^3-2x^2+x-5$

$y' = 4x^3+3x^2-4x+1$

$y'' = 12x^2+6x-4$

$y''' = 24x+6$

$y'''' = 24$

34. $y = x^2+x+3$

$y' = 2x+1$

$y'' = 2$

$y''' = 0$

$y'''' = 0$

35. $y = x^{-1}+x^2$

$y' = -x^{-2}+2x$

$y'' = 2x^{-3}+2$

$y''' = -6x^{-4}$

$y'''' = -24x^{-5}$

36. $y = \dfrac{x+1}{x} = \dfrac{x}{x}+\dfrac{1}{x} = 1+x^{-1}$

$y' = -x^{-2} = -\dfrac{1}{x^2}$

$y'' = 2x^{-3} = \dfrac{2}{x^3}$

$y''' = -6x^{-4} = -\dfrac{6}{x^4}$

$y^{\mathrm{iv}} = 24x^{-5} = \dfrac{24}{x^5}$

37. $y'(x) = 3x^2-3$

$y'(2) = 3(2)^2-3 = 9$

The tangent line has slope 9, so the

perpendicular line has slope $-\dfrac{1}{9}$ and passes

through (2, 3).

$y = -\dfrac{1}{9}(x-2)+3$

$y = -\dfrac{1}{9}x + \dfrac{29}{9}$

38. $y'(x) = 3x^2+1$

The slope is 4 when $3x^2+1 = 4$, at $x = \pm 1$. The
tangent at $x = -1$ has slope 4 and passes
through (−1, −2), so its equation is
$y = 4(x+1)-2$, or $y = 4x+2$. The tangent at
$x = 1$ has slope 4 and passes through (1, 2), so
its equation is $y = 4(x-1)+2$, or $y = 4x-2$.

The smallest slope occurs when $3x^2+1$ is
minimized, so the smallest slope is 1 and
occurs at $x = 0$.

39. $y'(x) = 6x^2 - 6x - 12$

$\qquad = 6(x^2 - x - 2)$

$\qquad = 6(x+1)(x-2)$

The tangent is parallel to the x-axis when $y' = 0$, at $x = -1$ and at $x = 2$. Since $y(-1) = 27$ and $y(2) = 0$, the two points where this occurs are $(-1, 27)$ and $(2, 0)$.

40. $y'(x) = 3x^2$

$y'(-2) = 12$

The tangent line has slope 12 and passes through $(-2, -8)$, so its equation is $y = 12(x + 2) - 8$, or

$y = 12x + 16$. The x-intercept is $-\dfrac{4}{3}$ and the y-intercept is 16.

41. $y'(x) = \dfrac{(x^2 + 1)(4) - 4x(2x)}{(x^2 + 1)^2} = \dfrac{-4x^2 + 4}{(x^2 + 1)^2}$

At the origin: $y'(0) = 4$

The tangent is $y = 4x$.

At $(1, 2)$: $y'(1) = 0$.

The tangent is $y = 2$.

42. $y'(x) = \dfrac{(4 + x^2)(0) - 8(2x)}{(4 + x^2)^2} = -\dfrac{16x}{(4 + x^2)^2}$

$y'(2) = -\dfrac{1}{2}$

The tangent has slope $-\dfrac{1}{2}$ and passes through $(2, 1)$. Its equation is $y = -\dfrac{1}{2}(x - 2) + 1$, or $y = -\dfrac{1}{2}x + 2$.

43. (a) Let $f(x) = x$.

$\dfrac{d}{dx}(x) = f'(x)$

$\qquad = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\qquad = \lim_{h \to 0} \dfrac{(x+h) - x}{h}$

$\qquad = \lim_{h \to 0} \dfrac{h}{h}$

$\qquad = \lim_{h \to 0} (1)$

$\qquad = 1$

(b) Note that $u = u(x)$ is a function of x.

$\dfrac{d}{dx}(-u) = \lim_{h \to 0} \dfrac{-u(x+h) - [-u(x)]}{h}$

$\qquad = \lim_{h \to 0} \left(-\dfrac{u(x+h) - u(x)}{h} \right)$

$\qquad = -\lim_{h \to 0} \dfrac{u(x+h) - u(x)}{h}$

$\qquad = -\dfrac{du}{dx}$

44. $\dfrac{d}{dx}(c \cdot f(x)) = c \cdot \dfrac{d}{dx} f(x) + f(x) \cdot \dfrac{d}{dx}(c)$

$\qquad\qquad = c \cdot \dfrac{d}{dx} f(x) + 0$

$\qquad\qquad = c \cdot \dfrac{d}{dx} f(x)$

45. $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right) = \dfrac{f(x) \cdot 0 - 1 \cdot \frac{d}{dx} f(x)}{[f(x)]^2} = -\dfrac{f'(x)}{[f(x)]^2}$

46. $\dfrac{dP}{dV} = \dfrac{d}{dV}\left(\dfrac{nRT}{V - nb} - \dfrac{an^2}{V^2}\right)$

$\qquad = \dfrac{(V - nb)\frac{d}{dV}(nRT) - (nRT)\frac{d}{dV}(V - nb)}{(V - nb)^2} - \dfrac{d}{dV}(an^2 V^{-2})$

$\qquad = \dfrac{0 - nRT}{(V - nb)^2} + 2an^2 V^{-3}$

$\qquad = -\dfrac{nRT}{(V - nb)^2} + \dfrac{2an^2}{V^3}$

47. $\dfrac{ds}{dt} = \dfrac{d}{dt}(4.9t^2) = 9.8t$

$\dfrac{d^2 s}{dt^2} = \dfrac{d}{dt}(9.8t) = 9.8$

48. $\dfrac{dR}{dM} = \dfrac{d}{dM}\left[M^2\left(\dfrac{C}{2} - \dfrac{M}{3}\right)\right]$

$\qquad = \dfrac{d}{dM}\left(\dfrac{C}{2}M^2 - \dfrac{1}{3}M^3\right)$

$\qquad = CM - M^2$

49. If the radius of a circle is changed by a very small amount Δr, the change in the area can be thought of as a very thin strip with length given by the circumference, $2\pi r$, and width Δr. Therefore, the change in the area can be approximated by $(2\pi r)(\Delta r)$, which means that the change in the area divided by the change in the radius is approximately $2\pi r$.

50. If the radius of a sphere is changed by a very small amount Δr, the change in the volume can be thought of as a very thin layer with an area given by the surface area, $4\pi r^2$, and a thickness given by Δr. Therefore, the change in the volume can be approximated by $(4\pi r^2)(\Delta r)$, which means that the change in the volume divided by the change in the radius is approximately $4\pi r^2$.

51. Let $t(x)$ be the number of trees and $y(x)$ be the yield per tree x years from now. Then $t(0) = 156$, $y(0) = 12$, $t'(0) = 13$, and $y'(0) = 1.5$. The rate of increase of production is

$\dfrac{d}{dx}(ty)\bigg|_{x=0} = t(0)\,y'(0) + y(0)t'(0)$

$\qquad\qquad = (156)(1.5) + (12)(13)$

$\qquad\qquad = 390$ bushels or annual production per year.

52. Let $m(x)$ be the number of members and $c(x)$ be the pavilion cost x years from now. Then $m(0) = 65$, $c(0) = 250$, $m'(0) = 6$, and $c'(0) = 10$. The rate of change of each member's share is

$$\frac{d}{dx}\left(\frac{c}{m}\right)\bigg|_{x=0} = \frac{m(0)c'(0) - c(0)m'(0)}{[m(0)]^2}$$

$$= \frac{(65)(10) - (250)(6)}{(65)^2}$$

$$\approx -0.201 \text{ dollars}$$

per year. Each member's share of the cost is decreasing by approximately 20 cents per year.

53. False; π is a constant, so π^3 is also a constant and hence $\frac{d}{dx}(\pi^3) = 0$.

54. True; $f'(x) = -\dfrac{1}{x^2}$ is never zero, so there are no horizontal tangents.

55. B; $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$$= (2)(1) + (-1)(3)$$

$$= -1$$

56. D; $f(x) = x - \dfrac{1}{x}$

$$f'(x) = 1 + \dfrac{1}{x^2}$$

$$f''(x) = -\dfrac{2}{x^3}$$

57. E; $\dfrac{d}{dx}\left(\dfrac{x+1}{x-1}\right) = \dfrac{(x-1) - (x+1)}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$

58. B; $f'(x) = (x^2 - 1) \cdot 2x + (x^2 + 1) \cdot 2x$

$$= 2x[(x^2 - 1) + (x^2 + 1)]$$

$$= 2x \cdot 2x^2$$

$$= 4x^3$$

$f'(x) = 0$ only when $x = 0$

There is one horizontal tangent at $x = 0$.

59. (a) It is insignificant in the limiting case and can be treated as zero (and removed from the expression).

(b) It was "rejected" because it is incomparably smaller than the other terms: $v\,du$ and $u\,dv$.

(c) $\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$. This is equivalent to the product rule given in the text.

(d) Because dx is "infinitely small," and this could be thought of as dividing by zero.

(e) $d\left(\dfrac{u}{v}\right) = \dfrac{u + du}{v + dv} - \dfrac{u}{v}$

$$= \dfrac{(u + du)(v) - (u)(v + dv)}{(v + dv)(v)}$$

$$= \dfrac{uv + vdu - uv - udv}{v^2 + vdv}$$

$$= \dfrac{vdu - udv}{v^2}$$

Quick Quiz Sections 3.1–3.3

1. D

2. A; Slope of normal: $m_1 = \dfrac{1-2}{-1-1} = \dfrac{1}{2}$

Slope of tangent: $m_2 = -\dfrac{1}{m_1} = -2$

Therefore $f'(1) = -2$.

3. C; $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{4x-3}{2x+1}\right)$

$$= \dfrac{4(2x+1) - 2(4x-3)}{(2x+1)^2}$$

$$= \dfrac{10}{(2x+1)^2}$$

4. (a) $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^4 - 4x^2)$

$$= 4x^3 - 8x$$

$$= 4x(x^2 - 2)$$

$$= 0$$

$$x = 0, \ \pm\sqrt{2}$$

(b) $f'(1) = 4(1)(1^2 - 2) = -4$

$x = 1 \qquad y = (1)^4 - 4(1)^2$

$\qquad\qquad y = -3$

$\qquad y = m(x - x_1) + y_1$

$\qquad y = -4(x - 1) - 3$

$\qquad y = -4x + 1$

(c) $m_2 = -\dfrac{1}{m_1} = \dfrac{1}{4}$

$\qquad y = \dfrac{1}{4}(x - 1) - 3$

$\qquad\quad = \dfrac{1}{4}x - \dfrac{1}{4} - 3$

$\qquad\quad = \dfrac{1}{4}x - \dfrac{13}{4}$

Section 3.4   Velocity and Other Rates of Change
(pp. 129−142)

Exploration 1   Growth Rings on a Tree

1. Figure 3.22 is a better model, as it shows rings of equal *area* as opposed to rings of equal *width*. It is not likely that a tree could sustain increased growth year after year, although climate conditions do produce some years of greater growth than others.

2. Rings of equal area suggest that the tree adds approximately the same amount of wood to its girth each year. With access to approximately the same raw materials from which to make the wood each year, this is how most trees actually grow.

3. Since change in area is constant, so also is $\dfrac{\text{change in area}}{2\pi}$.

   If we denote this latter constant by k, we have $\dfrac{k}{\text{change in radius}} = r$, which means that r varies inversely as the change in the radius. In other words, the change in radius must get smaller when r gets bigger, and vice-versa.

**Exploration 2**   Modeling Horizontal Motion

1. The particle reverses direction at about $t = 0.61$ and $t = 2.06$.
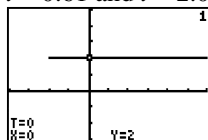
2. When the trace cursor is moving to the right the particle is moving to the right, and when the cursor is moving to the left the particle is moving to the left. Again we find the particle reverses direction at about $t = 0.61$ and $t = 2.06$.
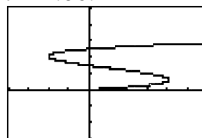
3. When the trace cursor is moving upward the particle is moving to the right, and when the cursor is moving downward the particle is moving to the left. Again we find the same values of t for when the particle reverses direction.
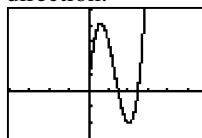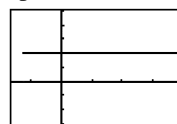

4. We can represent the velocity by graphing the parametric equations

   $x_4(t) = x_1'(t) = 12t^2 - 32t + 15, \; y_4(t) = 2$
   (part 1),

   $x_5(t) = x_1'(t) = 12t^2 - 32t + 15, \; y_5(t) = t$
   (part 2),

   $x_6(t) = t, \; y_6(t) = x_1'(t) = 12t^2 - 32t + 15$
   (part 3)

[−8, 20] by [−3, 5]
(x_4, y_4)
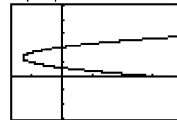
[−8, 20] by [−3, 5]
(x_5, y_5)

[−2, 5] by [−10, 20]
(x_6, y_6)

For (x_4, y_4) and (x_5, y_5), the particle is moving to the right when the x-coordinate of the graph (velocity) is positive, moving to the left when the x-coordinate of the graph (velocity) is negative, and is stopped when the x-coordinate of the graph (velocity) is 0. For

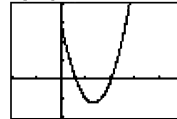(x_6, y_6), the particle is moving to the right when the y-coordinate of the graph (velocity) is positive, moving to the left when the y-coordinate of the graph (velocity) is negative, and is stopped when the y-coordinate of the graph (velocity) is 0.

Exploration 3   Seeing Motion on a Graphing Calculator

1. Let tMin = 0 and tMax = 10.

2. Since the rock achieves a maximum height of 400 feet, set yMax to be slightly greater than 400, for example yMax = 420.

4. The grapher proceeds with constant increments of t (time), so pixels appear on the screen at regular time intervals. When the rock is moving more slowly, the pixels appear closer together. When the rock is moving faster, the pixels appear farther apart. We observe faster motion when the pixels are farther apart.

Quick Review 3.4

1. The coefficient of x^2 is negative, so the parabola opens downward.

2. The y-intercept is $f(0) = -256$.

3. The x-intercepts occur when $f(x) = 0$.
$$-16x^2 + 160x - 256 = 0$$
$$-16(x^2 - 10x + 16) = 0$$
$$-16(x - 2)(x - 8) = 0$$
$$x = 2 \text{ or } x = 8$$
The x-intercepts are 2 and 8

4. Since $f(x) = -16(x^2 - 10x + 16)$
$$= -16(x^2 - 10x + 25 - 9)$$
$$= -16(x - 5)^2 + 144,$$
the range is $(-\infty, \ 144]$.

5. Since $f(x) = -16(x^2 - 10x + 16)$
$$= -16(x^2 - 10x + 25 - 9)$$
$$= -16(x - 5)^2 + 144,$$
the vertex is at $(5, 144)$.

6.
$$f(x) = 80$$
$$-16x^2 + 160x - 256 = 80$$
$$-16x^2 + 160x - 336 = 0$$
$$-16(x^2 - 10x + 21) = 0$$
$$-16(x - 3)(x - 7) = 0$$
$$x = 3 \text{ or } x = 7$$
$f(x) = 80$ at $x = 3$ and at $x = 7$.

7.
$$\frac{dy}{dx} = 100$$
$$-32x + 160 = 100$$
$$60 = 32x$$
$$x = \frac{15}{8}$$
$\dfrac{dy}{dx} = 100$ at $x = \dfrac{15}{8}$

8.
$$\frac{dy}{dx} > 0$$
$$-32x + 160 > 0$$
$$-32x > -160$$
$$x < 5$$
$\dfrac{dy}{dx} > 0$ when $x < 5$.

9. Note that $f'(x) = -32x + 160$.
$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = f'(3)$$
$$= -32(3) + 160$$
$$= 64$$

10. $f'(x) = -32x + 160 \qquad f''(x) = -32$
At $x = 7$ (and, in fact, at any other value of x),
$$\frac{d^2 y}{dx^2} = -32.$$

Section 3.4 Exercises

1. (a) $V(s) = s^3$

    (b) $\dfrac{dV}{ds} = 3s^2$

    (c) $\left. \dfrac{dV}{ds} \right|_{s=1} = 3(1)^2 = 3$

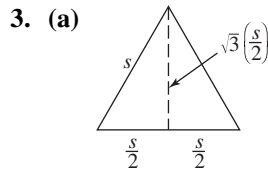 $\left. \dfrac{dV}{ds} \right|_{s=5} = 3(5)^2 = 75$

(d) $\dfrac{in^3}{in.}$

2. (a) $C = 2\pi r \;\Rightarrow\; r = \dfrac{C}{2\pi}$

$A = \pi r^2 = \pi \left(\dfrac{C}{2\pi}\right)^2 = \dfrac{C^2}{4\pi}$

$A(C) = \dfrac{C^2}{4\pi}$

(b) $\dfrac{dA}{dC} = \dfrac{2C}{4\pi} = \dfrac{C}{2\pi}$

(c) $\left.\dfrac{dA}{dC}\right|_{C=\pi} = \dfrac{\pi}{2\pi} = \dfrac{1}{2}$

$\left.\dfrac{dA}{dC}\right|_{C=6\pi} = \dfrac{6\pi}{2\pi} = 3$

(d) $\dfrac{in^2}{in.}$ or square inches per inch
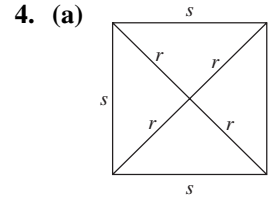
3. (a)

$A = \dfrac{1}{2}bh = \dfrac{1}{2}s\sqrt{3}\left(\dfrac{s}{2}\right)$

$A(s) = \dfrac{\sqrt{3}}{4}s^2$

(b) $\dfrac{dA}{ds} = \dfrac{\sqrt{3}}{2}s$

(c) $\left.\dfrac{dA}{ds}\right|_{s=2} = \dfrac{\sqrt{3}}{2}(2) = \sqrt{3}$

$\left.\dfrac{dA}{ds}\right|_{s=10} = \dfrac{\sqrt{3}}{2}(10) = 5\sqrt{3}$

(d) $\dfrac{in^2}{in.}$ or square inches per inch

4. (a)

Use Pythagorean Theorem on lower right triangle:

$s^2 + s^2 = (2r)^2$

$2s^2 = 4r^2$

$s^2 = 2r^2$

$A = s^2 = 2r^2$

$A(r) = 2r^2$

(b) $\dfrac{dA}{dr} = 4r$

(c) $\left.\dfrac{dA}{dr}\right|_{r=1} = 4(1) = 4$

$\left.\dfrac{dA}{dr}\right|_{r=8} = 4(8) = 32$

(d) $\dfrac{in^2}{in.}$ or square inches per inch

5. (a)

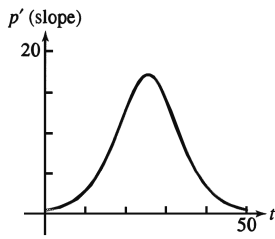**(b)** $s'(1) = 18$, $s'(2.5) = 0$, $s'(3.5) = -12$

6. (a)
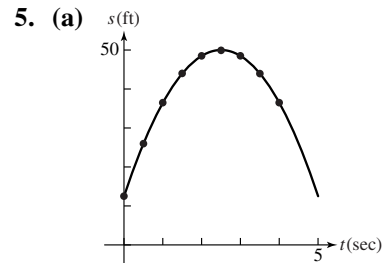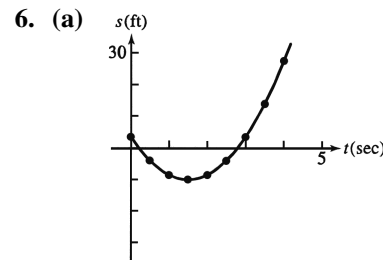
(b) $s'(1) = -6$, $s'(2.5) = 12$, $s'(3.5) = 24$

7. (a) We estimate the slopes at several points as follows, then connect the points to create a smooth curve.

t (days)	0	10	20	30	40	50
Slope (flies/ day)	0.5	3.0	13.0	14.0	3.5	0.5

Horizontal axis: Days
Vertical axis: Flies per day

(b) Fastest: Around the 25th day
Slowest: Day 50 or day 0

8. $Q(t) = 200(30 - t)^2$
$\quad = 200(900 - 60t + t^2)$
$\quad = 180,000 - 12,000t + 200t^2$
$Q'(t) = -12,000 + 400t$

The rate of change of the amount of water in the tank after 10 minutes is $Q'(10) = -8000$ gallons per minute.

Note that $Q'(10) < 0,$ so the rate at which the water is running *out* is positive. The water is running out at the rate of 8000 gallons per minute.

The average rate for the first 10 minutes is
$\dfrac{Q(10) - Q(0)}{10 - 0} = \dfrac{80,000 - 180,000}{10}$
$\quad = -10,000$ gal/min.

The water is flowing out at an average rate of 10,000 gallons per minute over the first 10 min.

9. **(a)** The particle moves forward when $v > 0$, for $0 \le t < 1$ and for $5 < t < 7$.
The particle moves backward when $v < 0$, for $1 < t < 5$.
The particle speeds up when v is negative and decreasing, for $1 < t < 2$, and when v is positive and increasing, for $5 < t < 6$.
The particle slows down when v is positive and decreasing, for $0 \le t < 1$ and for $6 < t < 7$, and when v is negative and increasing, for $3 < t < 5$.

(b) Note that the acceleration $a = \dfrac{dv}{dt}$ is undefined at $t = 2$, $t = 3$, and $t = 6$.
The acceleration is positive when v is

increasing, for $3 < t < 6$.
The acceleration is negative when v is decreasing, for $0 \le t < 2$ and for $6 < t < 7$.
The acceleration is zero when v is constant, for $2 < t < 3$ and for $7 < t \le 9$.

(c) The particle moves at its greatest speed when $|v|$ is maximized, at $t = 0$ and for $2 < t < 3$.

(d) The particle stands still for more than an instant when v stays at zero, for $7 < t \le 9$.

10. **(a)** The particle is moving left when the graph of s has negative slope, for $2 < t < 3$ and for $5 < t \le 6$.
The particle is moving right when the graph of s has positive slope, for $0 \le t < 1$.
The particle is standing still when the graph of s is horizontal, for $1 < t < 2$ and for $3 < t < 5$.

(b) For $0 \le t < 1$: $v = \dfrac{2 - 0}{1 - 0} = 2$ cm/sec
$\quad$ Speed $= |v| = 2$ cm/sec
For $1 < t < 2$: $v = \dfrac{2 - 2}{2 - 1} = 0$ cm/sec
$\quad$ Speed $= |v| = 0$ cm/sec
For $2 < t < 3$: $v = \dfrac{-2 - 2}{3 - 2} = -4$ cm/sec
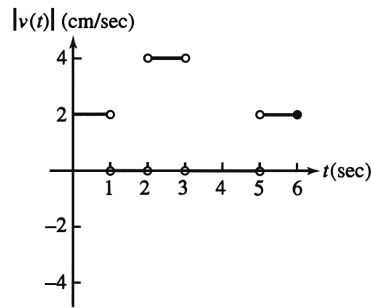$\quad$ Speed $= |v| = 4$ cm/sec
For $3 < t < 5$: $v = \dfrac{-2 - (-2)}{5 - 3} = 0$ cm/sec
$\quad$ Speed $= |v| = 0$ cm/sec
For $5 < t \le 6$: $v = \dfrac{-4 - (-2)}{6 - 5} = -2$ cm/sec
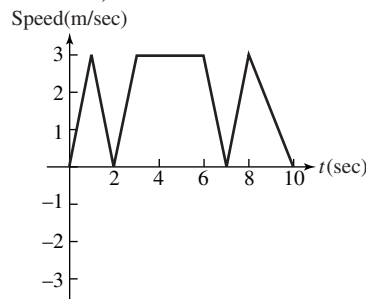$\quad$ Speed $= |v| = 2$ cm/sec
Velocity graph:
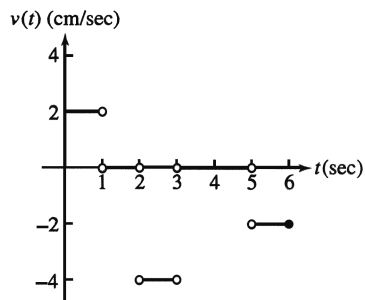
Speed graph:

11. (a) The body reverses direction when v changes sign, at $t = 2$ and at $t = 7$.

(b) The body is moving at a constant speed, $|v| = 3$ m/sec, between $t = 3$ and $t = 6$.

(c) The speed graph is obtained by reflecting the negative portion of the velocity graph, $2 < t < 7$, about the x-axis.

Speed(m/sec)

(d) For $0 \leq t < 1$: $a = \dfrac{3-0}{1-0} = 3$ m/sec^2

For $1 < t < 3$: $a = \dfrac{-3-3}{3-1} = -3$ m/sec^2

For $3 < t < 6$: $a = \dfrac{-3-(-3)}{6-3} = 0$ m/sec^2

For $6 < t < 8$: $a = \dfrac{3-(-3)}{8-6} = 3$ m/sec^2

For $8 < t \leq 10$: $a = \dfrac{0-3}{10-8} = -1.5$ m/sec^2

Acceleration (m/sec^2)
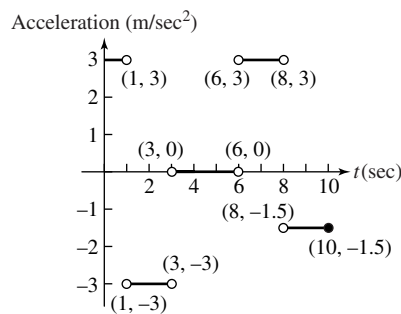
12. (a) It takes 135 seconds.

(b) Average speed $= \dfrac{\Delta F}{\Delta t}$

$= \dfrac{5-0}{73-0}$

$= \dfrac{5}{73}$

≈ 0.068 furlongs/sec.

(c) Using a symmetric difference quotient, the horse's speed is approximately

$\dfrac{\Delta F}{\Delta t} = \dfrac{4-2}{59-33}$

$= \dfrac{2}{26}$

$= \dfrac{1}{13}$

≈ 0.077 furlongs/sec.

(d) The horse is running the fastest during the last furlong (between 9th and 10th furlong markers). This furlong takes only 11 seconds to run, which is the least amount of time for a furlong.

(e) The horse accelerates the fastest during the first furlong (between markers 0 and 1).

13. (a) Velocity: $v(t) = \dfrac{ds}{dt}$

$= \dfrac{d}{dt}(24t - 0.8t^2)$

$= 24 - 1.6t$ m/sec

Acceleration: $a(t) = \dfrac{dv}{dt}$

$= \dfrac{d}{dt}(24 - 1.6t)$

$= -1.6$ m/sec^2

(b) The rock reaches its highest point when $v(t) = 24 - 1.6t = 0$, at $t = 15$. It took 15 seconds.

(c) The maximum height was $s(15) = 180$ meters.

(d) $s(t) = \dfrac{1}{2}(180)$

$24t - 0.8t^2 = 90$

$0 = 0.8t^2 - 24t + 90$

$$t = \frac{24 \pm \sqrt{(-24)^2 - 4(0.8)(90)}}{2(0.8)}$$

$$\approx 4.393, \ 25.607$$

It took about 4.393 seconds to reach half its maximum height.

(e) $\qquad s(t) = 0$

$$24t - 0.8t^2 = 0$$

$$0.8t(30 - t) = 0$$

$$t = 0 \text{ or } t = 30$$

The rock was aloft from $t = 0$ to $t = 30$, so it was aloft for 30 seconds.

14. On Mars:

$$\text{Velocity} = \frac{ds}{dt} = \frac{d}{dt}(1.86t^2) = 3.72t$$

Solving $3.72t = 16.6$, the downward velocity reaches 16.6 m/sec after about 4.462 sec.

On Jupiter:

$$\text{Velocity} = \frac{ds}{dt} = \frac{d}{dt}(11.44t^2) = 22.88t$$

Solving $22.88t = 16.6$, the downward velocity reaches 16.6 m/sec after about 0.726 sec.

15. The rock reaches its maximum height when the velocity $s'(t) = 24 - 9.8t = 0$, at $t \approx 2.449$. Its maximum height is about $s(2.449) \approx 29.388$ meters.

16. Moon:

$$s(t) = 0$$

$$832t - 2.6t^2 = 0$$

$$2.6t(320 - t) = 0$$

$$t = 0 \text{ or } t = 320$$

It takes 320 seconds to return.

Earth:

$$s(t) = 0$$

$$832t - 16t^2 = 0$$

$$16t(52 - t) = 0$$

$$t = 0 \text{ or } t = 52$$

It takes 52 seconds to return.

17. The following is one way to simulate the problem situation.

For the moon:

$$x_1(t) = 3(t < 160) + 3.1(t \geq 160)$$

$$y_1(t) = 832t - 2.6t^2$$

t-values: 0 to 320

window: [0, 6] by [−10,000, 70,000]

For the earth:

$$x_1(t) = 3(t < 26) + 3.1(t \geq 26)$$

$$y_1(t) = 832t - 16t^2$$

t-values : 0 to 52

window: [0, 6] by [−1000, 11,000]

18. (a) 190 ft/sec

(b) 2 seconds

(c) After 8 seconds, and its velocity was 0 ft/sec then

(d) After about 11 seconds, and it was falling 90 ft/sec then

(e) About 3 seconds (from the rocket's highest point)

(f) The acceleration was greatest just before the engine stopped. The acceleration was constant from $t = 2$ to $t = 11$, while the rocket was in free fall.

19. (a) Displacement: $s(5) - s(0) = 12 - 2 = 10$ m

(b) Average velocity $= \dfrac{10 \text{ m}}{5 \text{ sec}} = 2$ m/sec

(c) Velocity $= s'(t) = 2t - 3$

At $t = 4$,

velocity $= s'(4) = 2(4) - 3 = 5$ m/sec

(d) Acceleration $= s''(t) = 2$ m/sec^2

(e) The particle changes direction when

$s'(t) = 2t - 3 = 0$, so $t = \dfrac{3}{2}$ sec.

(f) Since the acceleration is always positive, the position s is at a minimum when the particle changes direction, at $t = \dfrac{3}{2}$ sec. Its position at this time is $s\left(\dfrac{3}{2}\right) = -\dfrac{1}{4}$ m.
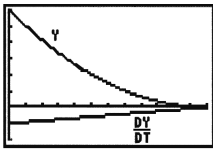
20. (a) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(-t^3 + 7t^2 - 14t + 8)$

$v(t) = -3t^2 + 14t - 14$

**(b)** $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(-3t^2 + 14t - 14)$

$a(t) = -6t + 14$

(c) $v(t) = -3t^2 + 14t - 14 = 0$

$t \approx 1.451, \ 3.215$

(d) The particle starts at the point $s = 8$ when $t = 0$ and moves left until it stops at $s = -0.631$ when $t = 1.451$, then it moves right to the point $s = 2.113$ when $t = 3.215$ where it stops again, and finally continues left from there on.

21. (a) $v(t) = \dfrac{ds}{dt}$

$= \dfrac{d}{dt}[(t-2)^2(t-4)]$

$= (t-2)^2(1) + (t-4) \cdot 2(t-2)$

$= (t-2)[(t-2) + 2(t-4)]$

$= (t-2)(3t-10)$

(b) $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}[(t-2)(3t-10)]$

$a(t) = 6t - 16$

(c) $v(t) = (t-2)(3t-10) = 0$

$t = 2, \dfrac{10}{3}$

(d) The particle starts at the point $s = -16$ when $t = 0$ and move right until it stops at $s = 0$ when $t = 2$, then it moves left to the point $s = -1.185$ when $t = \dfrac{10}{3}$ where it stops again, and finally continues right from there on.

22. (a) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(t^3 - 6t^2 + 8t + 2)$

$v(t) = 3t^2 - 12t + 8$

(b) $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(3t^2 - 12t + 8)$

$a(t) = 6t - 12$

(c) $v(t) = 3t^2 - 12t + 8 = 0$

$t \approx 0.845, \ 3.155$

(d) The particle starts at the point $s = 2$ when $t = 0$ and moves right until it stops at $s = 5.079$ when $t = 0.845$, then it moves left to the point $s = -1.079$ when $t = 3.155$ where it stops again, and finally continues right from there on.

23. $v(t) = s'(t) = 3t^2 - 12t + 9$

$a(t) = v'(t) = 6t - 12$

Find when velocity is zero.

$3t^2 - 12t + 9 = 0$

$3(t^2 - 4t + 3) = 0$

$3(t-1)(t-3) = 0$

$t = 1$ or $t = 3$

At $t = 1$, the acceleration is $a(1) = -6$ m/sec^2

At $t = 3$, the acceleration is $a(3) = 6$ m/sec^2

24. $a(t) = v'(t) = 6t^2 - 18t + 12$

Find when acceleration is zero.

$6t^2 - 18t + 12 = 0$

$6(t^2 - 3t + 2) = 0$

$6(t-1)(t-2) = 0$

$t = 1$ or $t = 2$

At $t = 1$, the speed is $|v(1)| = |0| = 0$ m/sec.

At $t = 2$, the speed is $|v(2)| = |-1| = 1$ m/sec.

25. (a) $\dfrac{dy}{dt} = \dfrac{d}{dt}\left[6\left(1 - \dfrac{t}{12}\right)^2\right]$

$= \dfrac{d}{dt}\left[6\left(1 - \dfrac{t}{6} + \dfrac{t^2}{144}\right)\right]$

$= \dfrac{d}{dt}\left(6 - t + \dfrac{1}{24}t^2\right)$

$= 0 - 1 + \dfrac{t}{12}$

$= \dfrac{t}{12} - 1$

(b) The fluid level is falling fastest when $\dfrac{dy}{dt}$ is the most negative, at $t = 0$, when $\dfrac{dy}{dt} = -1$. The fluid level is falling slowest at $t = 12$, when $\dfrac{dy}{dt} = 0$.

(c)

[0, 12] by [−2, 6]

y is decreasing and $\dfrac{dy}{dt}$ is negative over the entire interval; y decreases more rapidly early in the interval,

and the magnitude of $\dfrac{dy}{dt}$ is larger then. $\dfrac{dy}{dt}$ is 0 at $t = 12$, where the graph of y seems to have a

horizontal tangent.

26. (a) To graph the velocity, we estimate the slopes at several points as follows, then connect the points to create a smooth curve.

t (hours)	0	2.5	5	7.5	10	12.5	15
v (km/hour)	0	56	75	56	0	-4	-225
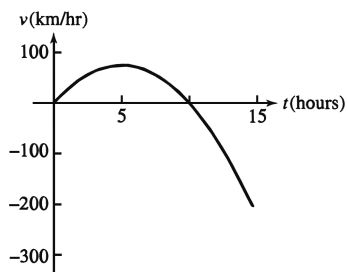
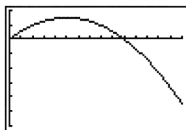To graph the acceleration, we estimate the slope of the velocity graph at several points as follows, and then connect the points to create a smooth curve.

t (hours)	0	2.5	5	7.5	10	12.5	15
a (km/hour^2)	30	15	0	-5	-0	-5	-0
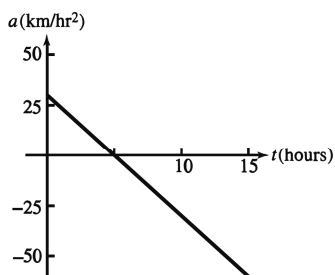
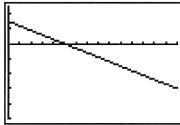**(b)** $\dfrac{ds}{dt} = 30t - 3t^2$

[0, 15] by [−300, 100]

$\dfrac{d^2 s}{dt^2} = 30 - 6t$

[0, 15] by [−100, 50]

The graphs are very similar.

27. **(a)** Average cost $= \dfrac{c(100)}{100}$

$= \dfrac{11,000}{100}$

$= \$110$ per machine

(b) $c'(x) = 100 - 0.2x$

Marginal cost $= c'(100) = \$80$ per machine

(c) Actual cost of 101st machine is
$c(101) - c(100) = \$79.90$, which is very close to the marginal cost calculated in part (b).
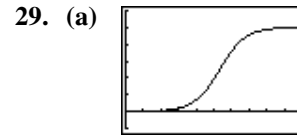
28. **(a)**
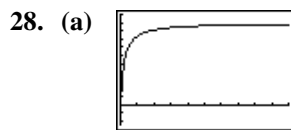
[0, 50] by [−500, 2200]

The values of x which make sense are the whole numbers, $x \geq 0$.

(b) Marginal revenue

$r'(x) = \dfrac{d}{dx}\left[2000\left(1 - \dfrac{1}{x+1}\right)\right]$

$= \dfrac{d}{dx}\left(2000 - \dfrac{2000}{x+1}\right)$

$= 0 - \dfrac{(x+1)(0) - (2000)(1)}{(x+1)^2}$

$= \dfrac{2000}{(x+1)^2}$

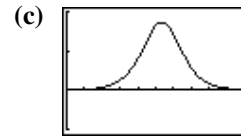**(c)** $r'(5) = \dfrac{2000}{(5+1)^2} = \dfrac{2000}{36} \approx 55.56$

The increase in revenue is approximately $\$55.56$.

(d) The limit is 0. This means that as x gets large, one reaches a point where very little extra revenue can be expected from selling more desks.

29. **(a)**

[0, 200] by [−2, 12]

(b) The values of x which make sense are the whole numbers, $x \geq 0$.

(c)

[0, 200] by [−0.1, 0.2]

P is most sensitive to changes in x when $|P'(x)|$ is largest. It is relatively sensitive to changes in x between approximately $x = 60$ and $x = 160$.

(d) The marginal profit, $P'(x)$, is greatest at $x = 106.44$. Since x must be an integer, $P(106) \approx 4.924$ thousand dollars or $\$4924$.

(e) $P'(50) \approx 0.013$, or $\$13$ per package sold
$P'(100) \approx 0.165$, or $\$165$ per package sold
$P'(125) \approx 0.118$, or $\$118$ per package sold
$P'(150) \approx 0.031$, or $\$31$ per package sold
$P'(175) \approx 0.006$, or $\$6$ per package sold
$P'(300) \approx 10^{-6}$, or $\$0.001$ per package sold

(f) The limit is 10. The maximum possible profit is $\$10,000$ monthly.

(g) Yes; in order to sell more and more packages, the company might need to lower the price to a point where they won't make any additional profit.

30. Since the particle moves along the line $y = 2$, it will be at the point (5, 2) when
$x(t) = 4t^3 - 16t^2 + 15t = 5.$ Use a grapher to see that this occurs when $t = 2.83$.

31. Graph C is position, graph A is velocity, and graph B is acceleration.
A is the derivative of C because it is positive, negative, and zero where C is increasing, decreasing, and has horizontal tangents, respectively. The relationship between B and A is similar.

32. Graph C is position, graph B is velocity, and graph A is acceleration.
B is the derivative of C because it is negative and zero where C is decreasing and has horizontal tangents, respectively.
A is the derivative of B because it is positive, negative, and zero where B is increasing, decreasing, and has horizontal tangents, respectively.

33. Note that "downward velocity" is positive when McCarthy is falling downward. His downward velocity increases steadily until the parachute opens, and then decreases to a constant downward velocity. One possible sketch:
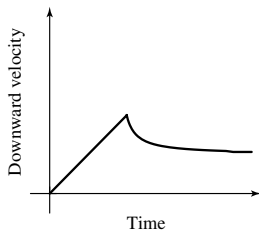
34. **(a)** $\dfrac{dV}{dr} = \dfrac{d}{dr}\left(\dfrac{4}{3}\pi r^3\right) = 4\pi r^2$

When $r = 2$, $\dfrac{dv}{dr} = 4\pi(2)^2 = 16\pi$ cubic feet of volume per foot of radius.

(b) The increase in the volume is

$\dfrac{4}{3}\pi(2.2)^3 - \dfrac{4}{3}\pi(2)^3 \approx 11.092$ cubic feet.

35. Let v_0 be the exit velocity of a particle of lava.

Then $s(t) = v_0 t - 16t^2$ feet, so the velocity is $\dfrac{ds}{dt} = v_0 - 32t$. Solving $\dfrac{ds}{dt} = 0$ gives $t = \dfrac{v_0}{32}$.

Then the maximum height, in feet, is

$s\left(\dfrac{v_0}{32}\right) = v_0\left(\dfrac{v_0}{32}\right) - 16\left(\dfrac{v_0}{32}\right)^2 = \dfrac{v_0^2}{64}$. Solving

$\dfrac{v_0^2}{64} = 1900$ gives $v_0 \approx \pm 348.712$. The exit velocity was about 348.712 ft/sec. Multiplying by $\dfrac{3600 \text{ sec}}{1\text{h}} \cdot \dfrac{1\text{mi}}{5280 \text{ ft}}$, we find that this is equivalent to about 237.758 mi/h.

36. By estimating the slope of the velocity graph at that point.

37. The motion can be simulated in parametric mode using $x_1(t) = 2t^3 - 13t^2 + 22t - 5$ and $y_1(t) = 2$ in [−6, 8] by [−3, 5].

(a) It begins at the point (−5, 2) moving in the positive direction. After a little more than one second, it has moved a bit past (6, 2) and it turns back in the negative direction for approximately 2 seconds. At the end of that time, it is near (−2, 2) and it turns back again in the positive direction. After that, it continues moving in the positive direction indefinitely, speeding up as it goes.

(b) The particle speeds up when its *speed* is increasing, which occurs during the approximate intervals $1.153 \le t \le 2.167$ and $t \ge 3.180$. It slows down during the approximate intervals $0 \le t \le 1.153$ and $2.167 \le t \le 3.180$. One way to determine the endpoints of these intervals is to use a grapher to find the minimums and maximums for the speed,

$\left|\dfrac{dx}{dt}\right| = \left|6t^2 - 26t + 22\right|$ using function

mode in the window [0, 5] by [0, 10].

(c) The particle changes direction at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

(d) The particle is at rest "instantaneously" at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

(e) The velocity starts out positive but decreasing, it becomes negative, then starts to increase, and becomes positive again and continues to increase. The speed is decreasing, reaches 0 at $t \approx 1.15$ sec, then increases until $t \approx 2.17$ sec, decreases until $t \approx 3.18$ sec when it is 0 again, and then increases after that.

(f) The particle is at (5, 2) when $2t^3 - 13t^2 + 22t - 5 = 5$ at $t \approx 0.745$ sec, $t \approx 1.626$ sec, and at $t \approx 4.129$ sec.

38. **(a)** Solving $160 = 490t^2$ gives $t = \pm\dfrac{4}{7}$. It took $\dfrac{4}{7}$ of a second. The average velocity was $\dfrac{160\,\text{cm}}{\left(\frac{4}{7}\right)}$ sec $= 280$ cm/sec.

(b) $V = \dfrac{ds}{dt} = 980t$

$a = \dfrac{dV}{dt} = 980$

At $s = 160$ cm, $t = \dfrac{4}{7}$ sec (from part (a)) and

$V = 980\left(\dfrac{4}{7}\right) = 560$ cm/sec

$a = 980$ cm/sec^2

(c) Once the balls begin falling, each flash will produce a different image. There are 16 images of the balls falling, so

$\dfrac{16 \text{ flashes}}{\frac{4}{7} \text{ seconds}} = 28$ flashes per second.

39. Since profit = revenue – cost, the Sum and Difference Rule gives

$\dfrac{d}{dx}(\text{profit}) = \dfrac{d}{dx}(\text{revenue}) - \dfrac{d}{dx}(\text{cost}),$ where x

is the number of units produced. This means that marginal profit
= marginal revenue – marginal cost.

40. False; it is the absolute value of the velocity.

41. True. The acceleration is the first derivative of the velocity which, in turn, is the second derivative of the position function.

42. C; $f'(x) = 2x + \dfrac{2}{x^2}$

$f'(-1) = 2(-1) + \dfrac{2}{(-1)^2} = 0$

43. D; $V(x) = x^3$

$\dfrac{dv}{dx} = 3x^2$

44. E; $\dfrac{ds}{dt} = \dfrac{d}{dt}(2 + 7t - t^2)$

$v(t) = 7 - 2t < 0$

$7 < 2t$

$\dfrac{7}{2} < t$

$4 > \dfrac{7}{2}$

45. C; $v(t) = 7 - 2t = 0$

$7 = 2t$

$t = \dfrac{7}{2}$

46. The growth rate is given by
$b'(t) = 10^4 - 2 \cdot 10^3 (t) = 10,000 - 2000t.$
At $t = 0$: $b'(0) = 10,000$ bacteria/hour
At $t = 5$: $b'(5) = 0$ bacteria/hour
At $t = 10$: $b'(10) = -10,000$ bacteria/hour

47. (a) $g'(x) = \dfrac{d}{dx}(x^3) = 3x^2$

$h'(x) = \dfrac{d}{dx}(x^3 - 2) = 3x^2$

$t'(x) = \dfrac{d}{dx}(x^3 + 3) = 3x^2$

(b) The graphs of NDER $g(x)$, NDER $h(x)$, and NDER $t(x)$ are all the same, as shown.
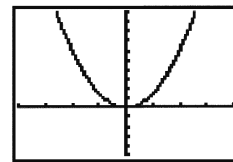
$[-4, 4]$ by $[-10, 20]$

(c) $f(x)$ must be of the form
$f(x) = x^3 + c,$ where c is a constant.

(d) Yes; $f(x) = x^3$

(e) Yes. $f(x) = x^3 + 3$

48. For $t > 0$, the runners' speed in feet per second is $v(t) = 2(1.5)t = 3t.$
At a speed of 22 feet per second,
$22 = 3t$
$7.33 \approx t,$
her approach was 7.333 seconds in time.
distance $= 1.5t^2 = 1.5(7.33\overline{3})^2 \approx 80.667$
Her approach was 80.667 feet in distance.

49. (a) Assume that f is even. Then,

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x-h) - f(x)}{h},$$

and substituting $k = -h$,

$$= \lim_{k \to 0} \frac{f(x+k) - f(x)}{-k}$$

$$= \lim_{k \to 0} \frac{f(x+k) - f(x)}{k} = -f'(x)$$

So, f' is an odd function.

(b) Assume that f is odd. Then,

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h}$$

$$= \lim_{h \to 0} \frac{-f(x-h) + f(x)}{h},$$

and substituting $k = -h$,

$$= \lim_{k \to 0} \frac{-f(x+k) + f(x)}{-k}$$

$$= \lim_{k \to 0} \frac{f(x+k) - f(x)}{k}$$

$$= f'(x)$$

So, f' is an even function.

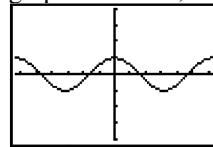**50.** $\dfrac{d}{dx}(fgh) = \dfrac{d}{dx}[f(gh)]$

$$= f \cdot \frac{d}{dx}(gh) + gh \cdot \frac{d}{dx}(f)$$

$$= f\left(g \cdot \frac{dh}{dx} + h \cdot \frac{dg}{dx}\right) + gh \cdot \frac{df}{dx}$$

$$= \left(\frac{df}{dx}\right)gh + f\left(\frac{dg}{dx}\right)h + fg\left(\frac{dh}{dx}\right)$$

Section 3.5 Derivatives of Trigonometric Functions (pp. 143–149)

Exploration 1 Making a Conjecture by Graphing the Derivative

1. When the graph of sin x is increasing, the graph of $\dfrac{d}{dx}$ (sin x) is positive (above the x-axis).

2. When the graph of sin x is decreasing, the graph of $\dfrac{d}{dx}$ (sin x) is negative (below the x-axis).

3. When the graph of sin x stops increasing and starts decreasing, the graph of $\dfrac{d}{dx}$ (sin x) crosses the x-axis from above to below.

4. The slope of the graph of sin x matches the value of $\dfrac{d}{dx}$ (sin x) at these points.

5. We conjecture that $\dfrac{d}{dx}$ (sin x) = cos x. The graphs coincide, supporting our conjecture.

$[-2\pi, 2\pi]$ by $[-4, 4]$

6. When the graph of cos x is increasing, the graph of $\dfrac{d}{dx}$ (cos x) is positive (above the x-axis).
When the graph of cos x is decreasing, the graph of $\dfrac{d}{dx}$ (cos x) is negative (below the x-axis).
When the graph of cos x stops increasing and starts decreasing, the graph of $\dfrac{d}{dx}$ (cos x) crosses the x-axis from above to below.
The slope of the graph of cos x matches the value of $\dfrac{d}{dx}$ (cos x) at these points.

We conjecture that $\dfrac{d}{dx}$ (cos x) = −sin x. The graphs coincide, supporting our conjecture.
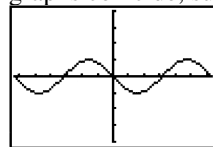
$[-2\pi, 2\pi]$ by $[-4, 4]$

Quick Review 3.5

1. $135° \cdot \dfrac{\pi}{180°} = \dfrac{3\pi}{4} \approx 2.356$

2. $1.7 \cdot \dfrac{180°}{\pi} = \left(\dfrac{306}{\pi}\right)° \approx 97.403°$

3. $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

4. Domain: All reals
Range: $[-1, 1]$

5. Domain: $x \neq \dfrac{k\pi}{2}$ for odd integers k
Range: All reals

6. $\cos a = \pm\sqrt{1 - \sin^2 a} = \pm\sqrt{1 - (-1)^2} = \pm\sqrt{0} = 0$

7. If $\tan a = -1$, then $a = \dfrac{3\pi}{4} + k\pi$ for some
integer k, so $\sin a = \pm\dfrac{1}{\sqrt{2}}$.

8. $\dfrac{1 - \cos h}{h} = \dfrac{(1 - \cos h)(1 + \cos h)}{h(1 + \cos h)}$

$\qquad = \dfrac{1 - \cos^2 h}{h(1 + \cos h)}$

$\qquad = \dfrac{\sin^2 h}{h(1 + \cos h)}$

9. $y'(x) = 6x^2 - 14x$
$y'(3) = 12$
The tangent line has slope 12 and passes
through $(3, 1)$, so its equation
is $y = 12(x - 3) + 1$, or $y = 12x - 35$.

10. $a(t) = v'(t) = 6t^2 - 14t$
$a(3) = 12$

Section 3.5 Exercises

1. $\dfrac{d}{dx}(1 + x - \cos x) = 0 + 1 - (-\sin x) = 1 + \sin x$

2. $\dfrac{d}{dx}(2\sin x - \tan x) = 2\cos x - \sec^2 x$

3. $\dfrac{d}{dx}\left(\dfrac{1}{x} + 5\sin x\right) = -\dfrac{1}{x^2} + 5\cos x$

4. $\dfrac{d}{dx}(x\sec x) = x\dfrac{d}{dx}(\sec x) + \sec x\dfrac{d}{dx}(x)$
$\qquad = x\sec x\tan x + \sec x$

5. $\dfrac{d}{dx}(4 - x^2\sin x)$

$= \dfrac{d}{dx}(4) - \left[x^2\dfrac{d}{dx}(\sin x) + (\sin x)\dfrac{d}{dx}(x^2)\right]$

$= 0 - [x^2\cos x + (\sin x)(2x)]$

$= -x^2\cos x - 2x\sin x$

6. $\dfrac{d}{dx}(3x + x\tan x)$

$= \dfrac{d}{dx}(3x) + \left[x\dfrac{d}{dx}(\tan x) + (\tan x)\dfrac{d}{dx}(x)\right]$

$= 3 + x\sec^2 x + \tan x$

7. $\dfrac{d}{dx}\left(\dfrac{4}{\cos x}\right) = \dfrac{d}{dx}(4\sec x) = 4\sec x\tan x$

8. $\dfrac{d}{dx}\left(\dfrac{x}{1 + \cos x}\right)$

$= \dfrac{(1 + \cos x)\frac{d}{dx}(x) - x\frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2}$

$= \dfrac{1 + \cos x + x\sin x}{(1 + \cos x)^2}$

9. $\dfrac{d}{dx}\left(\dfrac{\cot x}{1 + \cot x}\right)$

$= \dfrac{(1 + \cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1 + \cot x)}{(1 + \cot x)^2}$

$= \dfrac{(1 + \cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1 + \cot x)^2}$

$= -\dfrac{\csc^2 x}{(1 + \cot x)^2}$

$= -\dfrac{\csc^2 x\sin^2 x}{(1 + \cot x)^2\sin^2 x}$

$= -\dfrac{1}{(\sin x + \cos x)^2}$

10. $\dfrac{d}{dx}\left(\dfrac{\cos x}{1+\sin x}\right)$

$= \dfrac{(1+\sin x)\frac{d}{dx}(\cos x)-(\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$

$= \dfrac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$

$= \dfrac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$

$= \dfrac{-(1+\sin x)}{(1+\sin x)^2}$

$= -\dfrac{1}{1+\sin x}$

11. $v(t)=\dfrac{ds}{dt}=\dfrac{d}{dx}(5\,\sin t)$

$v(t)=5\cos t$

$a(t)=\dfrac{dv}{dt}=\dfrac{d}{dx}(5\,\cos t)$

$a(t)=-5\sin t$

The weight starts at 0, goes to 5, and the oscillates between 5 and –5. The period of the motion is 2π. The speed is greatest when $\cos t=\pm1$ $(t=k\pi)$, zero when $\cos t=0\left(t=\dfrac{k\pi}{2},\ k\text{ odd}\right)$. The acceleration is greatest when $\sin t=\pm1\left(t=\dfrac{k\pi}{2},\ k\text{ odd}\right)$, zero when $\sin t=0$ $(t=k\pi)$.

12. $v(t)=\dfrac{ds}{dt}=\dfrac{d}{dx}(7\,\cos t)$

$v(t)=-7\sin t$

$a(t)=\dfrac{dv}{dt}=\dfrac{d}{dx}(-7\,\sin t)$

$a(t)=-7\cos t$

The weight starts at 7, goes to –7, and then oscillates between –7 and 7. The period of the motion is 2π. The speed is greatest when $\sin t=\pm1\left(t=\dfrac{k\pi}{2},\ k\text{ odd}\right)$, zero when $\sin t=0$ $(t=k\pi)$. The acceleration is greatest when $\cos t=\pm1$ $(t=k\pi)$, zero when $\cos t=0\left(t=\dfrac{k\pi}{2},\ k\text{ odd}\right)$.

13. (a) $v(t)=\dfrac{ds}{dt}=\dfrac{d}{dt}(2+3\sin t)$

$v(t)=3\cos t,\text{speed}=|3\cos t|$

$a(t)=\dfrac{dv}{dt}=\dfrac{d}{dt}(3\cos t)=-3\sin t$

(b) $v\left(\dfrac{\pi}{4}\right)=3\cos\left(\dfrac{\pi}{4}\right)=\dfrac{3\sqrt{2}}{2}$, speed $=\dfrac{3\sqrt{2}}{2}$

$a\left(\dfrac{\pi}{4}\right)=-3\sin\left(\dfrac{\pi}{4}\right)=-\dfrac{3\sqrt{2}}{2}$

(c) The body starts at 2, goes up to 5, goes down to –1, and then oscillates between –1 and 5. The period of motion is 2π.

14. (a) $v(t)=\dfrac{ds}{dt}=\dfrac{d}{dt}(1-4\cos t)$

$v(t)=4\,\sin t,\ \text{speed}=|4\,\sin t|$

$a(t)=\dfrac{dv}{dt}=\dfrac{d}{dt}(4\,\sin t)$

$a(t)=4\,\cos t$

(b) $v\left(\dfrac{\pi}{4}\right)=4\sin\left(\dfrac{\pi}{4}\right)=2\sqrt{2}$, speed $=2\sqrt{2}$

$a\left(\dfrac{\pi}{4}\right)=4\cos\left(\dfrac{\pi}{4}\right)=2\sqrt{2}$

(c) The body starts at –3, goes up to 5, and then oscillates between 5 and –3. The period of the motion is 2π.

15. (a) $v(t)=\dfrac{ds}{dt}=\dfrac{d}{dt}(2\sin t+3\cos t)$

$v(t)=2\cos t-3\sin t,$

speed $=|2\cos t-3\sin t|$

$a(t)=\dfrac{dv}{dt}=\dfrac{d}{dt}(2\cos t-3\sin t)$

$a(t)=-2\sin t-3\cos t$

(b) $v\left(\dfrac{\pi}{4}\right)=2\cos\dfrac{\pi}{4}-3\sin\dfrac{\pi}{4}$

$v\left(\dfrac{\pi}{4}\right)=-\dfrac{\sqrt{2}}{2}$

speed $=\dfrac{\sqrt{2}}{2}$

$a\left(\dfrac{\pi}{4}\right)=-2\sin\dfrac{\pi}{4}-3\cos\dfrac{\pi}{4}$

$a\left(\dfrac{\pi}{4}\right)=\dfrac{-5\sqrt{2}}{2}$

(c) The body starts at 3, goes to 3.606, and then oscillates between -3.606 and 3.606. The period of the motion is 2π.

16. (a) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(\cos t - 3\sin t)$

$v(t) = -\sin t - 3\cos t,$

speed $= |{-\sin t - 3\cos t}|$

$a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(-\sin t - 3\cos t)$

$a(t) = -\cos t + 3\sin t$

(b) $v\left(\dfrac{\pi}{4}\right) = -\sin\dfrac{\pi}{4} - 3\cos\dfrac{\pi}{4} = -2\sqrt{2}$

speed $= 2\sqrt{2}$

$a\left(\dfrac{\pi}{4}\right) = -\cos\dfrac{\pi}{4} + 3\sin\dfrac{\pi}{4} = \sqrt{2}$

(c) The body starts at 1, goes to -3.162, and then oscillates between 3.162 and -3.162. The period of the motion is 2π.

17. $j(t) = \dfrac{da}{dt} = \dfrac{d^3 s}{dt^3}$

$f(t) = 2\cos t$

$f'(t) = -2\sin t$

$f''(t) = -2\cos t$

$f'''(t) = 2\sin t$

18. $j(t) = \dfrac{da}{dt} = \dfrac{d^3 s}{dt^3}$

$f(t) = 1 + 2\cos t$

$f'(t) = -2\sin t$

$f''(t) = -2\cos t$

$f'''(t) = 2\sin t$

19. $j(t) = \dfrac{da}{dt} = \dfrac{d^3 s}{dt^3}$

$f(t) = \sin t - \cos t$

$f'(t) = \cos t + \sin t$

$f''(t) = -\sin t + \cos t$

$f'''(t) = -\cos t - \sin t$

20. $j(t) = \dfrac{da}{dt} = \dfrac{d^3 s}{dt}$

$f(t) = 2 + 2\sin t$

$f'(t) = 2\cos t$

$f''(t) = -2\sin t$

$f'''(t) = -2\cos t$

21. $y = \sin x + 3$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin x + 3) = \cos x$

$y(\pi) = \sin \pi + 3 = 3$

$y'(\pi) = \cos \pi = -1$

tangent: $y = -1(x - \pi) + 3 = -x + \pi + 3$

normal: $m_2 = -\dfrac{1}{m_1} = 1$

$y = (x - \pi) + 3$

22. $y = \sec x$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(\sec x) = \sec x \tan x$

$y\left(\dfrac{\pi}{4}\right) = \sec\dfrac{\pi}{4} = 1.414$

$y'\left(\dfrac{\pi}{4}\right) = \sec\dfrac{\pi}{4} \tan\dfrac{\pi}{4} = 1.414$

tangent: $y = 1.414\left(x - \dfrac{\pi}{4}\right) + 1.414$

$y = 1.414x + 0.303$

normal: $m_2 = -\dfrac{1}{m_1} = -0.707$

$y = -0.707\left(x - \dfrac{\pi}{4}\right) + 1.414 = -0.707x + 1.970$

23. $y = x^2 \sin x$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2 \sin x) = 2x\sin x + x^2 \cos x$

$y(3) = (3)^2 \sin 3 = 1.270$

$y'(3) = 2(3)\sin 3 + (3)^2 \cos 3 = -8.063$

tangent:

$y = -8.063(x - 3) + 1.270 = -8.063x + 25.460$

normal: $m_2 = -\dfrac{1}{m_1} = 0.124$

$y = 0.124(x - 3) + 1.270$

$y = 0.124x + 0.898$

24. $\displaystyle\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$

$\displaystyle=\lim_{h\to 0}\frac{(\cos x\cos h-\sin x\sin h)-\cos x}{h}$

$\displaystyle=\lim_{h\to 0}\frac{\cos x(\cos h-1)-\sin x\sin h}{h}$

$\displaystyle=\lim_{h\to 0}\left((\cos x)\frac{\cos h-1}{h}-(\sin x)\frac{\sin h}{h}\right)$

$\displaystyle=(\cos x)\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right)-(\sin x)\left(\lim_{h\to 0}\frac{\sin h}{h}\right)$

$=(\cos x)(0)-(\sin x)(1)=-\sin x$

25. **(a)** $\displaystyle\frac{d}{dx}\tan x$

$\displaystyle=\frac{d}{dx}\frac{\sin x}{\cos x}$

$\displaystyle=\frac{(\cos x)\frac{d}{dx}(\sin x)-(\sin x)\frac{d}{dx}(\cos x)}{(\cos x)^2}$

$\displaystyle=\frac{(\cos x)(\cos x)-(\sin x)(-\sin x)}{\cos^2 x}$

$\displaystyle=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$

$\displaystyle=\frac{1}{\cos^2 x}$

$=\sec^2 x$

(b) $\displaystyle\frac{d}{dx}\sec x=\frac{d}{dx}\frac{1}{\cos x}$

$\displaystyle=\frac{(\cos x)\frac{d}{dx}(1)-(1)\frac{d}{dx}(\cos x)}{(\cos x)^2}$

$\displaystyle=\frac{(\cos x)(0)-(1)(-\sin x)}{\cos^2 x}$

$\displaystyle=\frac{\sin x}{\cos^2 x}$

$=\sec x\tan x$

26. **(a)** $\displaystyle\frac{d}{dx}\cot x$

$\displaystyle=\frac{d}{dx}\frac{\cos x}{\sin x}$

$\displaystyle=\frac{(\sin x)\frac{d}{dx}(\cos x)-(\cos x)\frac{d}{dx}(\sin x)}{(\sin x)^2}$

$\displaystyle=\frac{(\sin x)(-\sin x)-(\cos x)(\cos x)}{\sin^2 x}$

$\displaystyle=\frac{-(\sin^2 x+\cos^2 x)}{\sin^2 x}$

$\displaystyle=-\frac{1}{\sin^2 x}$

$=-\csc^2 x$

(b) $\displaystyle\frac{d}{dx}\csc x=\frac{d}{dx}\frac{1}{\sin x}$

$\displaystyle=\frac{(\sin x)\frac{d}{dx}(1)-(1)\frac{d}{dx}(\sin x)}{(\sin x)^2}$

$\displaystyle=\frac{(\sin x)(0)-(1)(\cos x)}{\sin^2 x}$

$\displaystyle=-\frac{\cos x}{\sin^2 x}$

$=-\csc x\cot x$

27. $\displaystyle\frac{d}{dx}\sec x=\sec x\tan x$ which is 0 at $x=0$, so the slope of the tangent line is 0.

$\displaystyle\frac{d}{dx}\cos x=-\sin x$ which is 0 at $x=0$, so the slope of the tangent line is 0.

28. $\displaystyle\frac{d}{dx}\tan x=\sec^2 x=\frac{1}{\cos^2 x}$, which is never 0.

$\displaystyle\frac{d}{dx}\cot x=-\csc^2 x=-\frac{1}{\sin^2 x}$, which is never 0.

29. $\displaystyle y'(x)=\frac{d}{dx}\left(\sqrt{2}\cos x\right)=-\sqrt{2}\sin x$

$\displaystyle y'\left(\frac{\pi}{4}\right)=-\sqrt{2}\sin\frac{\pi}{4}=-\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)=-1$

The tangent line has slope -1 and passes through $\displaystyle\left(\frac{\pi}{4},\ \sqrt{2}\cos\frac{\pi}{4}\right)=\left(\frac{\pi}{4},\ 1\right)$, so its equation is $\displaystyle y=-1\left(x-\frac{\pi}{4}\right)+1,$ or

$y = -x + \dfrac{\pi}{4} + 1.$

The normal line has slope 1 and passes

through $\left(\dfrac{\pi}{4},\ 1\right)$, so its equation

is $y = 1\left(x - \dfrac{\pi}{4}\right) + 1$, or $y = x + 1 - \dfrac{\pi}{4}.$

30. $y'(x) = \dfrac{d}{dx}\tan x = \sec^2 x$

$y'(x) = \dfrac{d}{dx}(2x) = 2$

$\sec^2 x = 2$

$\sec x = \pm\sqrt{2}$

$\cos x = \pm\dfrac{1}{\sqrt{2}}$

On $\left(-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right)$, the solutions are $x = \pm\dfrac{\pi}{4}$. The

points on the curve are $\left(-\dfrac{\pi}{4},\ -1\right)$ and

$\left(\dfrac{\pi}{4},\ 1\right).$

31. $y'(x) = \dfrac{d}{dx}(4 + \cot x - 2\csc x)$

$\qquad = 0 - \csc^2 x + 2\csc x \cot x$

$\qquad = -\csc^2 x + 2\csc x \cot x$

(a) $y'\left(\dfrac{\pi}{2}\right) = -\csc^2\dfrac{\pi}{2} + 2\csc\dfrac{\pi}{2}\cot\dfrac{\pi}{2}$

$\qquad = -1^2 + 2(1)(0)$

$\qquad = -1$

The tangent line has slope -1 and passes

through $P\left(\dfrac{\pi}{2},\ 2\right)$. Its equation is

$y = -1\left(x - \dfrac{\pi}{2}\right) + 2$, or $y = -x + \dfrac{\pi}{2} + 2.$

(b) $\qquad\qquad f'(x) = 0$

$-\csc^2 x + 2\csc x \cot x = 0$

$-\dfrac{1}{\sin^2 x} + \dfrac{2\cos x}{\sin^2 x} = 0$

$\dfrac{1}{\sin^2 x}(2\cos x - 1) = 0$

$\cos x = \dfrac{1}{2}$

$x = \dfrac{\pi}{3}$

at point Q

$y\left(\dfrac{\pi}{3}\right) = 4 + \cot\dfrac{\pi}{3} - 2\csc\dfrac{\pi}{3}$

$\qquad = 4 + \dfrac{1}{\sqrt{3}} - 2\left(\dfrac{2}{\sqrt{3}}\right)$

$\qquad = 4 - \dfrac{3}{\sqrt{3}}$

$\qquad = 4 - \sqrt{3}$

The coordinates of Q are $\left(\dfrac{\pi}{3},\ 4 - \sqrt{3}\right).$

The equation of the horizontal line

is $y = 4 - \sqrt{3}.$

32. $y'(x) = \dfrac{d}{dx}(1 + \sqrt{2}\csc x + \cot x)$

$\qquad = 0 + \sqrt{2}(-\csc x \cot x) + (-\csc^2 x)$

$\qquad = -\sqrt{2}\csc x \cot x - \csc^2 x$

(a) $y'\left(\dfrac{\pi}{4}\right) = -\sqrt{2}\csc\dfrac{\pi}{4}\cot\dfrac{\pi}{4} - \csc^2\dfrac{\pi}{4}$

$\qquad = -\sqrt{2}(\sqrt{2})(1) - (\sqrt{2})^2$

$\qquad = -2 - 2$

$\qquad = -4$

The tangent line has slope -4 and passes

through $P\left(\dfrac{\pi}{4},\ 4\right)$. Its equation is

$y = -4\left(x - \dfrac{\pi}{4}\right) + 4$, or $y = -4x + \pi + 4.$

(b) $y'(x) = 0$

$$-\sqrt{2}\csc x \cot x - \csc^2 x = 0$$

$$-\frac{\sqrt{2}\cos x}{\sin^2 x} - \frac{1}{\sin^2 x} = 0$$

$$-\frac{1}{\sin^2 x}(\sqrt{2}\cos x + 1) = 0$$

$$\cos x = -\frac{1}{\sqrt{2}}$$

$$x = \frac{3\pi}{4} \text{ at point } Q$$

$$y\left(\frac{3\pi}{4}\right) = 1 + \sqrt{2}\csc\frac{3\pi}{4} + \cot\frac{3\pi}{4}$$

$$= 1 + \sqrt{2}(\sqrt{2}) + (-1)$$

$$= 2$$

The coordinates of Q are $\left(\frac{3\pi}{4}, \ 2\right)$.

The equation of the horizontal line is
$y = 2$.

33. (a) Velocity: $s'(t) = -2\cos t$ m/sec
Speed: $|s'(t)| = |2\cos t|$ m/sec
Acceleration: $s''(t) = 2\sin t$ m/sec^2

Jerk: $s'''(t) = 2\cos t$ m/sec^3

(b) Velocity: $-2\cos\frac{\pi}{4} = -\sqrt{2}$ m/sec

Speed: $\left|-\sqrt{2}\right| = \sqrt{2}$ m/sec

Acceleration: $2\sin\frac{\pi}{4} = \sqrt{2}$ m/sec^2

Jerk: $2\cos\frac{\pi}{4} = \sqrt{2}$ m/sec^3

(c) The body starts at 2, goes to 0 and then
oscillates between 0 and 4.
Speed: *Greatest* when
$\cos t = \pm 1$ (or $t = k\pi$), at the center of
the interval of motion.

Zero when $\cos t = 0$ (or $t = \frac{k\pi}{2}$, k odd),

at the endpoints of the interval of motion.
Acceleration: *Greatest* (in magnitude)

when $\sin t = \pm 1$ (or $t = \frac{k\pi}{2}$, k odd)

Zero when $\sin t = 0$ (or $t = k\pi$)
Jerk: *Greatest* (in magnitude) when
$\cos t = \pm 1$ (or $t = k\pi$).

Zero when $\cos t = 0$ (or $t = \frac{k\pi}{2}$, k odd)

34. (a) Velocity: $s'(t) = \cos t - \sin t$ m/sec
Speed: $s'(t) = |\cos t - \sin t|$ m/sec
Acceleration: $s''(t) = -\sin t - \cos t$ m/sec^2
Jerk: $s'''(t) = -\cos t + \sin t$ m/sec^3

(b) Velocity: $\cos\frac{\pi}{4} - \sin\frac{\pi}{4} = 0$ m/sec

Speed: $|0| = 0$ m/sec

Acceleration:

$-\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\sqrt{2}$ m/sec^2

Jerk: $-\cos\frac{\pi}{4} + \sin\frac{\pi}{4} = 0$ m/sec^3

(c) The body starts at 1, goes to $\sqrt{2}$ and then
oscillates between $\pm\sqrt{2}$.
Speed:

Greatest when $t = \frac{3\pi}{4} + k\pi$

Zero when $t = \frac{\pi}{4} + k\pi$

Acceleration:

Greatest (in magnitude) when $t = \frac{\pi}{4} + k\pi$

Zero when $t = \frac{3\pi}{4} + k\pi$

Jerk:
Greatest (in magnitude) when

$t = \frac{3\pi}{4} + k\pi$

Zero when $t = \frac{\pi}{4} + k\pi$

35. $y' = \dfrac{d}{dx}\csc x = -\csc x \cot x$

$y'' = \dfrac{d}{dx}(-\csc x \cot x)$

$= -(\csc x)\dfrac{d}{dx}(\cot x) - (\cot x)\dfrac{d}{dx}(\csc x)$

$= -(\csc x)(-\csc^2 x) - (\cot x)(-\csc x \cot x)$

$= \csc^3 x + \csc x \cot^2 x$

36. $y' = \dfrac{d}{d\theta}(\theta \tan \theta)$

$\qquad = \theta \dfrac{d}{d\theta}(\tan \theta) + (\tan \theta)\dfrac{d}{d\theta}(\theta)$

$\qquad = \theta \sec^2 \theta + \tan \theta$

$\quad y'' = \dfrac{d}{d\theta}(\theta \sec^2 \theta + \tan \theta)$

$\qquad = \theta \dfrac{d}{d\theta}[(\sec \theta)(\sec \theta)] + (\sec^2 \theta)\dfrac{d}{d\theta}(\theta) + \dfrac{d}{d\theta}(\tan \theta)$

$\qquad = \theta \left[(\sec \theta)\dfrac{d}{d\theta}(\sec \theta) + (\sec \theta)\dfrac{d}{d\theta}(\sec \theta) \right] + \sec^2 \theta + \sec^2 \theta$

$\qquad = 2\theta \sec^2 \theta \tan \theta + 2\sec^2 \theta$

$\qquad = (2\theta \tan \theta + 2)(\sec^2 \theta)$

or, writing in terms of sines and cosines,

$\qquad = \dfrac{2 + 2\theta \tan \theta}{\cos^2 \theta}$

$\qquad = \dfrac{2\cos \theta + 2\theta \sin \theta}{\cos^3 \theta}$

37. Continuous: Note that $g(0) = \lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} \cos x = \cos(0) = 1$, and $\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-}(x + b) = b$. We

require $\lim\limits_{x \to 0^-} g(x) = g(0)$, so $b = 1$. The function is continuous if $b = 1$.

Differentiable: For $b = 1$, the left-hand derivative is 1 and the right-hand derivative is $-\sin(0) = 0$, so the function is not differentiable. For other values of b, the function is discontinuous at $x = 0$ and there is no left-hand derivative. So, there is no value of b that will make the function differentiable at $x = 0$.

38. Observe the pattern:

$\dfrac{d}{dx}\cos x = -\sin x \qquad\qquad \dfrac{d^5}{dx^5}\cos x = -\sin x$

$\dfrac{d^2}{dx^2}\cos x = -\cos x \qquad\qquad \dfrac{d^6}{dx^6}\cos x = -\cos x$

$\dfrac{d^3}{dx^3}\cos x = \sin x \qquad\qquad \dfrac{d^7}{dx^7}\cos x = \sin x$

$\dfrac{d^4}{dx^4}\cos x = \cos x \qquad\qquad \dfrac{d^8}{dx^8}\cos x = \cos x$

Continuing the pattern, we see that

$\dfrac{d^n}{dx^n}\cos x = \sin x$ when $n = 4k + 3$ for any whole number k.

Since $999 = 4(249) + 3$, $\dfrac{d^{999}}{dx^{999}}\cos x = \sin x.$

39. Observe the pattern:

$$\frac{d}{dx}\sin x = \cos x \qquad\qquad \frac{d^5}{dx^5}\sin x = \cos x$$

$$\frac{d^2}{dx^2}\sin x = -\sin x \qquad\qquad \frac{d^6}{dx^6}\sin x = -\sin x$$

$$\frac{d^3}{dx^3}\sin x = -\cos x \qquad\qquad \frac{d^7}{dx^7}\sin x = -\cos x$$

$$\frac{d^4}{dx^4}\sin x = \sin x \qquad\qquad \frac{d^8}{dx^8}\sin x = \sin x$$

Continuing the pattern, we see that

$$\frac{d^n}{dx^n}\sin x = \cos x \text{ when } n = 4k + 1 \text{ for any}$$

whole number k. Since $725 = 4(181) + 1$,

$$\frac{d^{725}}{dx^{725}}\sin x = \cos x.$$

40. The line is tangent to the graph of $y = \sin x$ at $(0, 0)$. Since $y'(0) = \cos(0) = 1$, the line has slope 1 and its equation is $y = x$.

41. **(a)** Using $y = x$, $\sin(0.12) \approx 0.12$.

(b) $\sin(0.12) = 0.1197122073$ to 10 digits. So 0.12 is only off by 0.00028779 of the computed actual value.

42. $\dfrac{d}{dx}\sin 2x$

$$= \frac{d}{dx}(2\sin x \cos x)$$

$$= 2\frac{d}{dx}(\sin x \cos x)$$

$$= 2\left[(\sin x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(\sin x)\right]$$

$$= 2[(\sin x)(-\sin x) + (\cos x)(\cos x)]$$

$$= 2(\cos^2 x - \sin^2 x)$$

$$= 2\cos 2x$$

43. $\dfrac{d}{dx}\cos 2x$

$$= \frac{d}{dx}[(\cos x)(\cos x) - (\sin x)(\sin x)]$$

$$= \left[(\cos x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(\cos x)\right] -$$

$$\qquad \left[(\sin x)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(\sin x)\right]$$

$$= 2(\cos x)(-\sin x) - 2(\sin x)(\cos x)$$

$$= -4\sin x \cos x$$

$$= -2(2\sin x \cos x)$$

$$= -2\sin 2x$$

44. True. $s'(t) = -3\cos t$,

$$s'\left(\frac{3\pi}{4}\right) = -3\cos\left(\frac{3\pi}{4}\right) = \frac{3\sqrt{2}}{2} > 0. \text{ The}$$

derivative is positive at $t = \dfrac{3\pi}{4}$.

45. False. The velocity is negative and the speed is positive at $t = \dfrac{\pi}{4}$.

46. A; $y = \sin x + \cos x$
$y'(x) = \cos x - \sin x$
$y(\pi) = \sin \pi + \cos \pi = -1$
$y'(\pi) = \cos \pi - \sin \pi = -1$
$y = -1(x - \pi) - 1$
$y = -x + \pi - 1$

47. B; See 46.

$$m_2 = -\frac{1}{m_1} = -\frac{1}{-1} = 1$$

$$y = (x - \pi) - 1$$

48. C; $y = x\sin x$
$y' = \sin x + x\cos x$
$y'' = \cos x + \cos x - x\sin x k = -x\sin x + 2\cos x$

49. C; $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(3 + \sin t)$
$v(t) = \cos t = 0$
$t = \dfrac{\pi}{2}$

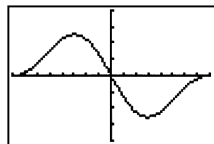**50. (a)**
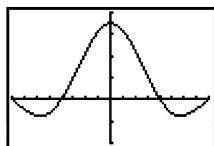
[−360, 360] by [−0.01, 0.02]

The limit is $\dfrac{\pi}{180}$ because this is the conversion factor for changing from degrees to radians.

(b)

[−360, 360] by [−0.02, 0.02]
This limit is still 0.

(c)
$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$
$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$
$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h}$$
$$= \left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) + \left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right)$$
$$= (\sin x)(0) + (\cos x)\left(\frac{\pi}{180}\right)$$
$$= \frac{\pi}{180}\cos x$$

(d)
$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$
$$= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$
$$= \lim_{h\to 0}\frac{(\cos x)(\cos h - 1) - \sin x\sin h}{h}$$
$$= \left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) - \left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right)$$
$$= (\cos x)(0) - (\sin x)\left(\frac{\pi}{180}\right)$$
$$= -\frac{\pi}{180}\sin x$$

(e)
$$\frac{d^2}{dx^2}\sin x = \frac{d}{dx}\frac{\pi}{180}\cos x$$
$$= \frac{\pi}{180}\left(-\frac{\pi}{180}\sin x\right)$$
$$= -\frac{\pi^2}{180^2}\sin x$$

$$\frac{d^3}{dx^3}\sin x = \frac{d}{dx}\left(-\frac{\pi^2}{180^2}\sin x\right)$$
$$= -\frac{\pi^2}{180^2}\left(\frac{\pi}{180}\cos x\right)$$
$$= -\frac{\pi^3}{180^3}\cos x$$

$$\frac{d^2}{dx^2}\cos x = \frac{d}{dx}\left(-\frac{\pi}{180}\sin x\right)$$
$$= -\frac{\pi}{180}\left(\frac{\pi}{180}\cos x\right)$$
$$= -\frac{\pi^2}{180^2}\cos x$$

$$\frac{d^3}{dx^3}\cos x = \frac{d}{dx}\left(-\frac{\pi^2}{180^2}\cos x\right)$$
$$= -\frac{\pi^2}{180^2}\left(-\frac{\pi}{180}\sin x\right)$$
$$= \frac{\pi^3}{180^3}\sin x$$

51. $\displaystyle\lim_{h\to 0}\frac{(\cos h - 1)}{h} = \lim_{h\to 0}\frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}$

$$= \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h + 1)}$$
$$= \lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h + 1)}$$
$$= -\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{\sin h}{\cos h + 1}\right)$$
$$= -(1)\left(\frac{0}{2}\right)$$
$$= 0$$

52. $y' = \dfrac{d}{dx}(A\sin x + B\cos x) = A\cos x - B\sin x$

$$y'' = \frac{d}{dx}(A\cos x - B\sin x)$$
$$= -A\sin x - B\cos x$$

Solve:
$$y'' - y = \sin x$$
$$(-A\sin x - B\cos x) - (A\sin x + B\cos x) = \sin x$$
$$-2A\sin x - 2B\cos x = \sin x$$

At $x = \dfrac{\pi}{2}$, this gives $-2A = 1$, so $A = -\dfrac{1}{2}$.

At $x = 0$, we have $-2B = 0$, so $B = 0$.

Thus, $A = -\dfrac{1}{2}$ and $B = 0$.

Quick Quiz Sections 3.4–3.5

1. $f'(1) = \dfrac{6 - (-4)}{1 - (-1)} = \dfrac{10}{2} = 5;\text{C}$

2. $y' = \dfrac{d}{dx}(\cos x + \tan x)$
$$= -\sin x + \sec^2 x$$
$$= -\sin x + \sec x \cdot \sec x$$
$$y'' = \frac{d}{dx}(-\sin x + \sec x \cdot \sec x)$$
$$= -\cos x + \sec x(\sec x\tan x) + \sec x(\sec x\tan x)$$
$$= -\cos x + 2\sec^2 x\tan x;\ \text{A}$$

3. $\dfrac{dy}{dx} = \dfrac{3(2x+3) - 2(3x+2)}{(2x+3)^2} = \dfrac{5}{(2x+3)^2};\ \text{D}$

4. (a) $s(0) = -0^2 + 0 + 2 = 2$ m

(b) $v(t) = s'(t) = \dfrac{d}{dt}(-t^2 + t + 2)$
$$= -2t + 1 \text{ m/s}$$

(c) The particle moves to the right when
$v(t) > 0$.
$$-2t + 1 > 0$$
$$t < 0.5$$
$$0 \le t < 0.5 \qquad \text{(time must be} \ge 0)$$

(d) $a(t) = v'(t) = \dfrac{d}{dt}(-2t+1) = -2 \text{ m/s}^2$

(e)
$$s(t) = 0$$
$$-t^2 + t + 2 = 0$$
$$-(t^2 - t - 2) = 0$$
$$-(t-2)(t+1) = 0$$
$$t = 2 \qquad \text{or} \qquad t = -1 \text{ (not in domain)}$$
$$\text{speed} = |v(2)| = |-2(2)+1| = |-3| = 3 \text{ m/s}$$

Chapter 3 Review Exercises
(pp. 150–153)

1. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(x^5 - \dfrac{1}{8}x^2 + \dfrac{1}{4}x\right) = 5x^4 - \dfrac{1}{4}x + \dfrac{1}{4}$

2. $\dfrac{dy}{dx} = \dfrac{d}{dx}(3 - 7x^3 + 3x^7) = -21x^2 + 21x^6$

3. $\dfrac{dy}{dx} = \dfrac{d}{dx}(2\sin x \cos x)$

$= 2(\sin x)\dfrac{d}{dx}(\cos x) + 2(\cos x)\dfrac{d}{dx}(\sin x)$

$= -2\sin^2 x + 2\cos^2 x$

$= 2\cos 2x$

4. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{2x+1}{2x-1}\right)$

$= \dfrac{(2x-1)(2)-(2x+1)(2)}{(2x-1)^2}$

$= -\dfrac{4}{(2x-1)^2}$

5. $\dfrac{ds}{dt} = \dfrac{d}{dt}[(t^2-1)(t^2+1)]$

$= \dfrac{d}{dt}[t^4-1]$

$= 4t^3$

6. $\dfrac{ds}{dt} = \dfrac{d}{dt}\left(\dfrac{t^2+1}{1-t^2}\right)$

$= \dfrac{(1-t^2)(2t)-(t^2+1)(-2t)}{(1-t^2)^2}$

$= \dfrac{4t}{(1-t^2)^2}$

7. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\sqrt{x}+1+\dfrac{1}{\sqrt{x}}\right)$

$= \dfrac{d}{dx}(x^{1/2}+1+x^{-1/2})$

$= \dfrac{1}{2}x^{-1/2}-\dfrac{1}{2}x^{-3/2}$

$= \dfrac{1}{2\sqrt{x}}-\dfrac{1}{2x^{3/2}}$

8. $\dfrac{dy}{dx} = \dfrac{d}{dx}[(x^5+1)(3x^2-x)]$

$= 5x^4(3x^2-x)+(6x-1)(x^5+1)$

9. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}(5\theta^2 \sec \theta)$

$= 10\theta \sec \theta + 5\theta^2 \sec \theta \tan \theta$

10. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\left(\dfrac{\tan \theta}{\theta^3+\theta+1}\right)$

$= \dfrac{\sec^2 \theta(\theta^3+\theta+1)-\tan \theta(3\theta^2+1)}{(\theta^3+\theta+1)^2}$

11. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2 \sin x + x\cos x)$

$= x^2 \cos x + 2x\sin x + x(-\sin x)+\cos x$

$= (x^2+1)\cos x + x\sin x$

12. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2 \sin x - x\cos x)$

$= x^2 \cos x + 2x\sin x - [x(-\sin x)+\cos x]$

$= (x^2-1)\cos x + 3x\sin x$

13. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\tan x}{2x^3}\right)$

$= \dfrac{2x^3 \sec^2 x - 6x^2 \tan x}{4x^6}$

$= \dfrac{x\sec^2 x - 3\tan x}{2x^4}$

14. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\tan x - \cot x)$

$= \sec^2 x - (-\csc^2 x)$

$= \sec^2 x + \csc^2 x$

15. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{\sin x + \cos x}\right)$

$= \dfrac{(\sin x + \cos x)\cdot 0 - 1(\cos x - \sin x)}{(\sin x + \cos x)^2}$

$= \dfrac{\sin x - \cos x}{(\sin x + \cos x)^2}$

16. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}+\dfrac{1}{\cos x}\right)$

$= \dfrac{d}{dx}(\csc x + \sec x)$

$= -\csc x \cot x + \sec x \tan x$

$= \sec x \tan x - \csc x \cot x$

17. $\dfrac{dV}{dr} = \dfrac{d}{dr}\left(\dfrac{4}{3}\pi r^3 + 8\pi r^2\right) = 4\pi r^2 + 16\pi r$

18. $\dfrac{dA}{ds} = \dfrac{d}{ds}\left(\dfrac{\sqrt{3}}{4}s^2 + \dfrac{3\pi}{4}s^2\right)$

$= \dfrac{\sqrt{3}}{2}s + \dfrac{3\pi}{4}s$

$= \left(\dfrac{\sqrt{3}}{2}+\dfrac{3\pi}{4}\right)s$

19. $\dfrac{ds}{dt} = \dfrac{d}{dt}\left(\dfrac{1+\sin t}{1+\tan t}\right)$

$= \dfrac{\cos t(1+\tan t) - \sec^2 t(1+\sin t)}{(1+\tan t)^2}$

20. $\dfrac{ds}{dt} = \dfrac{d}{dt}\left(\dfrac{1+\sin t}{1+\cos t}\right)$

$= \dfrac{\cos t(1+\cos t) - (-\sin t)(1+\sin t)}{(1+\cos t)^2}$

$= \dfrac{\cos t + \cos^2 t + \sin t + \sin^2 t}{(1+\cos t)^2}$

$= \dfrac{\cos t + \sin t + 1}{(1+\cos t)^2}$

21. $\dfrac{ds}{dt} = \dfrac{d}{dt}\left(\dfrac{t^{-1}+t^{-2}}{t^{-3}}\right) = \dfrac{d}{dt}(t^2 + t) = 2t + 1$

22. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^{-2}\cos x - 4x^{-3})$

$= -2x^{-3}\cos x - x^{-2}\sin x + 12x^{-4}$

$= \dfrac{-2\cos x}{x^3} - \dfrac{\sin x}{x^2} + \dfrac{12}{x^4}$

$= \dfrac{12 - 2x\cos x - x^2\sin x}{x^4}$

23. $\dfrac{dy}{du} = \dfrac{d}{du}\left(\dfrac{\sin u}{\csc u} + \dfrac{\cos u}{\sec u}\right)$

$= \dfrac{d}{du}(\sin^2 u + \cos^2 u)$

$= \dfrac{d}{du}(1)$

$= 0$

24. $\dfrac{dy}{du} = \dfrac{d}{du}\left(\dfrac{\cot u}{\tan u} - \dfrac{\csc u}{\sin u}\right)$

$= \dfrac{d}{du}(\cot^2 u - \csc^2 u)$

$= \dfrac{d}{du}(-1)$

$= 0$

25. $\dfrac{dy}{dx} = \dfrac{d}{dx}[2x^{-2}(x^5 - x^3)]$

$= \dfrac{d}{dx}(2x^3 - 2x)$

$= 6x^2 - 2$

26. $\dfrac{dy}{dx} = \dfrac{d}{dx}[4x^2(x^{-1} + 3x^{-4})]$

$= \dfrac{d}{dx}(4x + 12x^{-2})$

$= 4 - 24x^{-3}$

27. $\dfrac{dy}{dt} = \dfrac{d}{dt}\left(\dfrac{t^2}{\pi^3} - \dfrac{\pi^2}{t^3}\right)$

$= \dfrac{d}{dt}\left(\dfrac{1}{\pi^3}t^2 - \pi^2 t^{-3}\right)$

$= \dfrac{2}{\pi^3}t + 3\pi^2 t^{-4}$

$= \dfrac{2t}{\pi^3} + \dfrac{3\pi^2}{t^4}$

28. $\dfrac{dy}{dt} = \dfrac{d}{dt}\left(\dfrac{t^3}{\pi^2} - \dfrac{\pi^3}{t^2}\right)$

$= \dfrac{d}{dt}\left(\dfrac{1}{\pi^2}t^3 - \pi^3 t^{-2}\right)$

$= \dfrac{3}{\pi^2}t^2 + 2\pi^3 t^{-3}$

$= \dfrac{3t^2}{\pi^2} + \dfrac{2\pi^3}{t^3}$

29. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sec x \tan x \cos x) = \dfrac{d}{dx}\tan x = \sec^2 x$

30. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\sin x \cot x}{\cos x}\right) = \dfrac{d}{dx}(1) = 0$

31. Since $y = \dfrac{\sin x}{x}$ is defined for all $x \neq 0$ and

$\dfrac{dy}{dx} = \dfrac{x\cos x - \sin x}{x^2}$, the function is

differentiable for all $x \neq 0$.

32. Since $y = \sin x - x\cos x$ is defined for all real x and

$\dfrac{dy}{dx} = \cos x - (x)(-\sin x) - (\cos x)(1) = x\sin x$,

the function is differentiable for all real x.

33. Since $y = \dfrac{3\cos x}{x-2}$ is defined for all $x \neq 2$ and

$\dfrac{dy}{dx} = \dfrac{-3\sin x(x-2) - 3\cos x}{(x-2)^2}$, which is defined

for all $x \neq 2$, the function is differentiable for

all $x \neq 2$.

34. Since $y = (2x-7)^{-1}(x+5) = \dfrac{x+5}{2x-7}$ is defined

for all $x \neq \dfrac{7}{2}$ and

$\dfrac{dy}{dx} = \dfrac{(2x-7)(1)-(x+5)(2)}{(2x-7)^2} = -\dfrac{17}{(2x-7)^2}$, the

function is differentiable for all $x \neq \dfrac{7}{2}$.

35. Slope $= \dfrac{dy}{dx}\Big|_{x=\pi} = \sec\pi\tan\pi = 0$

36. Slope $= \dfrac{dy}{dx}\Big|_{x=\pi} = \cos\pi\cos\pi - \sin\pi\sin\pi = 1$

37. Slope $= \dfrac{dy}{dx}\Big|_{x=\pi} = \dfrac{\pi(-\sin\pi)-1\cdot\cos\pi}{\pi^2} = \dfrac{1}{\pi^2}$

38. Slope $= \dfrac{dy}{dx}\Big|_{x=\pi}$

$= \dfrac{1(\pi+\sin\pi)-(1+\cos\pi)\pi}{\pi^2}$

$= \dfrac{\pi-0}{\pi^2}$

$= \dfrac{1}{\pi}$

39. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{d}{dx}(\sec x) = \sec x\tan x$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(\sec x\tan x)$

$= (\sec x\tan x)\tan x + (\sec x)\sec^2 x$

$= \dfrac{\sin^2 x + 1}{\cos^3 x}$

40. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\csc x) = -\csc x\cot x$

$\dfrac{d^2y}{dx^2}$

$= \dfrac{d}{dx}(-\csc x\cot x)$

$= -(-\csc x\cot x)\cot x + (-\csc x)(-\csc^2 x)$

$= \dfrac{\cos^2 x + 1}{\sin^3 x}$

41. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x\sin x)$

$= 1\cdot\sin x + x\cos x$

$= \sin x + x\cos x$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(\sin x + x\cos x)$

$= \cos x + 1\cdot\cos x + x(-\sin x)$

$= 2\cos x - x\sin x$

42. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x - x\cos x)$

$= 1 - (1\cdot\cos x + x(-\sin x))$

$= 1 - \cos x + x\sin x$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(1 - \cos x + x\sin x)$

$= 0 - (-\sin x) + (1\cdot\sin x + x\cdot\cos x)$

$= 2\sin x + x\cos x$

43. $y' = 2x^3 - 3x - 1,$

$y'' = 6x^2 - 3,$

$y''' = 12x,$

$y^{(4)} = 12,$ and the rest are all zero.

44. $y' = \dfrac{x^4}{24},$

$y'' = \dfrac{x^3}{6},$

$y''' = \dfrac{x^2}{2},$

$y^{(4)} = x,$

$y^{(5)} = 1,$ and the rest are all zero.

45. $\dfrac{dy}{dx} = \dfrac{d}{dx}(8x^{-2}) = -16x^{-3}$

At $x = 2,\ y = 8(2^{-2}) = 2$ and

$\dfrac{dy}{dx} = -16(2^{-3}) = -2.$

(a) Tangent: $y - 2 = -2(x - 2)$ or $y = -2x + 6$

(b) Normal: $y - 2 = \dfrac{1}{2}(x - 2)$ or $y = \dfrac{1}{2}x + 1$

46. $\dfrac{dy}{dx} = \dfrac{d}{dx}(4 + \cot x - 2\csc x)$

$= -\csc^2 x + 2\csc x\cot x$

At $x = \dfrac{\pi}{2},$

$y = 4 + \cot\dfrac{\pi}{2} - 2\csc\dfrac{\pi}{2} = 4 + 0 - 2 = 2$ and

$\dfrac{dy}{dx} = -\csc^2\dfrac{\pi}{2} + 2\csc\dfrac{\pi}{2}\cot\dfrac{\pi}{2}$

$\phantom{\dfrac{dy}{dx}} = -1 + 2(1)(0)$

$\phantom{\dfrac{dy}{dx}} = -1.$

(a) Tangent:

$y - 2 = -1\left(x - \dfrac{\pi}{2}\right)$ or $y = -x + \dfrac{\pi}{2} + 2$

(b) Normal:

$y - 2 = 1\left(x - \dfrac{\pi}{2}\right)$ or $y = x - \dfrac{\pi}{2} + 2$

47. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$

At $x = \dfrac{\pi}{4}$, $y = \sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4} = \sqrt{2}$ and

$\dfrac{dy}{dx} = \cos\dfrac{\pi}{4} - \sin\dfrac{\pi}{4} = 0.$

(a) Tangent: Line is horizontal, so $y = \sqrt{2}$.

(b) Normal: Line is vertical, so $x = \dfrac{\pi}{4}$.

48. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(2x^2 + \dfrac{1}{x^4}\right) = 4x - 4x^{-5}$

At $x = 1$, $y = 2(1^2) + \dfrac{1}{1^4} = 3$ and

$\dfrac{dy}{dx} = 4(1) - 4(1)^{-5} = 0.$

(a) Tangent: Line is horizontal, so $y = 3$.

(b) Normal: Line is vertical, so $x = 1$.

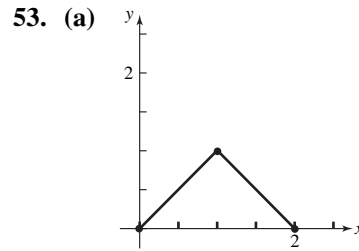**49.** $\dfrac{dy}{dx} = 6x^2 = 6 \Rightarrow x = \pm 1$. The points are $(1, 2)$ and $(-1, -2)$.

50. $\dfrac{dy}{dx} = \dfrac{1}{6}(6x^2 - 6x) = 6 \Rightarrow x^2 - x = 6 \Rightarrow x = 3$ or

$x = -2$. The points are $\left(3, \dfrac{9}{2}\right)$ and

$\left(-2, -\dfrac{14}{3}\right).$

51. $\dfrac{dy}{dx} = \dfrac{6(x+1) - 1\cdot 6x}{(x+1)^2} = \dfrac{6}{(x+1)^2} = 6 \Rightarrow x = 0$ or

$x = -2$. The points are $(0, 0)$ and $(-2, 12)$.

52. $\dfrac{dy}{dx} = 2\cos x = 6 \Rightarrow \cos x = 3$, which is

impossible. There are no points at which the tangent line has slope 6, so "none."

53. (a)

(b) Yes, because both of the one-sided limits as $x \to 1$ are equal to $f(1) = 1$.

(c) No, because the left-hand derivative at $x = 1$ is $+1$ and the right-hand derivative at $x = 1$ is -1.

54. (a) For all m, since $y = \sin 2x$ and $y = mx$ are both continuous on their domains, and they link up at the origin, where

$\lim\limits_{x \to 0^-}\sin 2x = \lim\limits_{x \to 0^+}mx = 0$, regardless of the value of m.

(b) For $m = 2$ only, since the left-hand derivative at 0 (which is $2\cos 0 = 2$) must match the right-hand derivative at 0 (which is m).

55. Note that $\dfrac{dy}{dx} = \dfrac{4}{5}x^{-1/5} = \dfrac{4}{5\sqrt[5]{x}}$ is defined if

and only if $x \neq 0$. The answers are

(a) For all $x \neq 0$

(b) At $x = 0$

(c) Nowhere

56. Note that $\dfrac{dy}{dx} = \dfrac{3}{5}x^{-2/5} = \dfrac{3}{5\sqrt[5]{x^2}}$ is defined if

and only if $x \neq 0$. The answers are

(a) For all $x \neq 0$

(b) At $x = 0$

(c) Nowhere

57. Note that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x-3) = -3$ and

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x-3) = -3.$ Since these

values agree with $f(0),$ the function is

continuous at $x = 0.$ On the other hand,

$f'(x) = \begin{cases} 2, & -1 \le x < 0 \\ 1, & 0 < x \le 4 \end{cases}$, so the derivative is

undefined at $x = 0.$

(a) $[-1, 0) \cup (0, 4]$
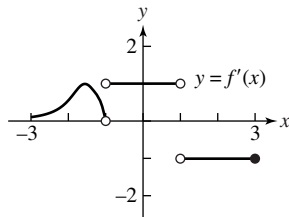
(b) At $x = 0$

(c) Nowhere in its domain

58. Note that the function is undefined at $x = 0.$

(a) $[-2, 0) \cup (0, 2]$

(b) Nowhere

(c) Nowhere in its domain

59.

60.
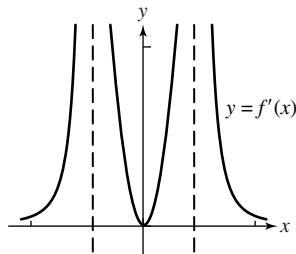
61. (a) iii
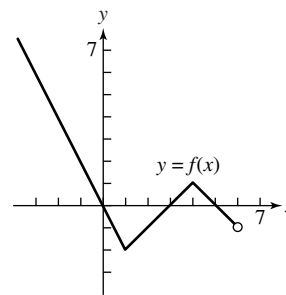
(b) i

(c) ii

62. The graph passes through $(0, 5)$ and has slope -2 for $x < 2$ and slope -0.5 for $x > 2.$
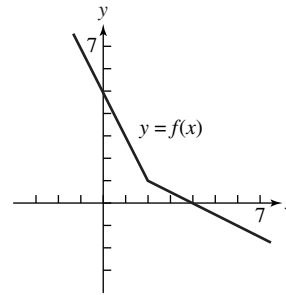
63. The graph passes through $(-1, 2)$ and has slope -2 for $x < 1,$ slope 1 for $1 < x < 4,$ and slope -1 for $4 < x < 6.$

64. i. If $f(x) = \dfrac{9}{28} x^{7/3} + 9,$ then $f'(x) = \dfrac{3}{4} x^{4/3}$

and $f''(x) = x^{1/3}$, which matches the

given equation.

ii. If $f'(x) = \dfrac{9}{28} x^{7/3} - 2,$ then

$f''(x) = \dfrac{3}{4} x^{4/3}$, which contradicts the

given equation.

iii. If $f'(x) = \dfrac{3}{4} x^{4/3} + 6,$ then $f''(x) = x^{1/3}$,
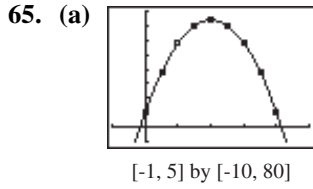
which matches the given equation.

iv. If $f(x) = \dfrac{3}{4} x^{4/3} - 4,$ then $f'(x) = x^{1/3}$ and
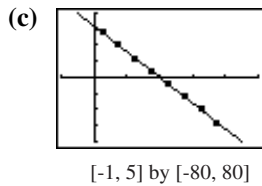
$f''(x) = \dfrac{1}{3} x^{-2/3}$, which contradicts the

given equation.
Answer is D: i and iii only could be true.
Note, however that i and iii could not
simultaneously be true.

65. (a)

[-1, 5] by [-10, 80]

(b)

t interval	avg. vel.
[0, 0.5]	$\dfrac{38-10}{0.5-0}=56$
[0.5, 1]	$\dfrac{58-38}{1-0.5}=40$
[1, 1.5]	$\dfrac{70-58}{1.5-1}=24$
[1.5, 2]	$\dfrac{74-70}{2-1.5}=8$
[2, 2.5]	$\dfrac{70-74}{2.5-2}=-8$
[2.5, 3]	$\dfrac{58-70}{3-2.5}=-24$
[3, 3.5]	$\dfrac{38-58}{3.5-3}=-40$
[3.5, 4]	$\dfrac{10-38}{4-3.5}=-56$

(c)

[-1, 5] by [-80, 80]

(d) Average velocity is a good approximation to velocity.

66. $(x^n)'=nx^{n-1}$; $(x^n)''=n(n-1)x^{n-2}$;

$(x^n)'''=n(n-1)(n-2)x^{n-3}$; ... and

$\dfrac{d^n}{dx^n}(x^n)=n(n-1)(n-2)(n-3)\cdots 2\cdot 1x^0$

$\qquad = n!$.

67. (a) $\dfrac{d}{dx}(3f(x))\Big|_{x=1}=3f'(1)=3\cdot 4=12$

(b) $\dfrac{d}{dx}(xf(x))\Big|_{x=1}=1\cdot f(x)+x\cdot f'(x)\big|_{x=1}$

$\qquad = f(1)+f'(1)$

$\qquad = -3+4$

$\qquad = 1$

(c) $\dfrac{d}{dx}(x^2 f(x))\Big|_{x=1}=2x\cdot f(x)+x^2\cdot f'(x)\big|_{x=1}$

$\qquad = 2f(1)+f'(1)$

$\qquad = -6+4$

$\qquad = -2$

(d) $\dfrac{d}{dx}\left(\dfrac{f(x)}{x}\right)\Big|_{x=1}=\dfrac{f'(x)\cdot x-1\cdot f(x)}{x^2}\Big|_{x=1}$

$\qquad = \dfrac{f'(1)-f(1)}{1}$

$\qquad = 4-(-3)$

$\qquad = 7$

(e) $\dfrac{d}{dx}\left(\dfrac{f(x)}{x^2+2}\right)\Big|_{x=0}$

$\qquad = \dfrac{f'(x)\cdot(x^2+2)-2x\cdot f(x)}{(x^2+2)^2}\Big|_{x=0}$

$\qquad = \dfrac{f'(0)\cdot 2-0\cdot f(0)}{2^2}$

$\qquad = \dfrac{(-2)\cdot 2}{4}$

$\qquad = -1$

(f) $\dfrac{d}{dx}(f(x)\cdot f(x))\Big|_{x=0}$

$\qquad = f'(x)\cdot f(x)+f(x)\cdot f'(x)\big|_{x=0}$

$\qquad = 2f'(0)f(0)$

$\qquad = 2(-2)(9)$

$\qquad = -36$

68. (a) $\dfrac{d}{dx}[3f(x)-g(x)]\Big|_{x=-1}$

$\qquad = 3f'(x)-g'(x)\big|_{x=-1}$

$\qquad = 3(2)-1$

$\qquad = 5$

(b) $\dfrac{d}{dx}(f(x)g(x))\Big|_{x=0}$

$\qquad = f'(x)\cdot g(x)+f(x)\cdot g'(x)\big|_{x=0}$

$\qquad = (-2)(-3)+(-1)(4)$

$\qquad = 2$

(c) $\dfrac{d}{dx}(f(x)g(x))\Big|_{x=-1}$

$\qquad = f'(x)\cdot g(x)+f(x)\cdot g'(x)\big|_{x=-1}$

$\qquad = (2)(-1)+(0)(1)$

$\qquad = -2$

(d) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)\Bigg|_{x=0}$

$= \dfrac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}\Bigg|_{x=0}$

$= \dfrac{(-2)(-3) - (-1)(4)}{(-3)^2}$

$= \dfrac{10}{9}$

(e) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)\Bigg|_{x=-1}$

$= \dfrac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}\Bigg|_{x=-1}$

$= \dfrac{(2)(-1) - (0)(1)}{(-1)^2}$

$= -2$

(f) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)+2}\right)\Bigg|_{x=0}$

$= \dfrac{f'(x)\cdot (g(x)+2) - f(x)\cdot g'(x)}{(g(x)+2)^2}\Bigg|_{x=0}$

$= \dfrac{(-2)(-1) - (-1)(4)}{(-3+2)^2}$

$= 6$

69. Yes; the slope of $f + g$ at $x = 0$ is $(f + g)'(0) = f'(0) + g'(0)$. The sum of two positive numbers must also be positive.

70. No; it depends on the values of $f(0)$ and $g(0)$. For example, let $f(x) = x$ and $g(x) = x - 1$. Both lines have positive slope everywhere, but $(f \cdot g)(x) = x^2 - x$ has a negative slope at $x = 0$.

71. (a) $\dfrac{ds}{dt} = \dfrac{d}{dt}(64t - 16t^2) = 64 - 32t$

$\dfrac{d^2 s}{dt^2} = \dfrac{d}{dt}(64 - 32t) = -32$

(b) The maximum height is reached when $\dfrac{ds}{dt} = 0$, which occurs at $t = 2$ sec.

(c) When $t = 0$, $\dfrac{ds}{dt} = 64$, so the velocity is 64 ft/sec.

(d) Since $\dfrac{ds}{dt} = \dfrac{d}{dt}(64t - 2.6t^2) = 64 - 5.2t$,

the maximum height is reached at

$t = \dfrac{64}{5.2} \approx 12.3 \text{ sec}$. The maximum height

is $s\left(\dfrac{64}{5.2}\right) \approx 393.8$ ft.

72. (a) Solving $160 = 490\,t^2$, it takes $\dfrac{4}{7}$ sec. The

average velocity is $\dfrac{160}{\frac{4}{7}} = 280\,\text{cm/sec}$.

(b) Since $v(t) = \dfrac{ds}{dt} = 980t$, the velocity is

$(980)\left(\dfrac{4}{7}\right) = 560$ cm/sec. Since

$a(t) = \dfrac{dv}{dt} = 980$, the acceleration is

$980 \text{ cm} / \sec^2$.

73. $\dfrac{dV}{dx} = \dfrac{d}{dx}\left[\pi\left(10 - \dfrac{x}{3}\right)x^2\right]$

$= \dfrac{d}{dx}\left[\pi\left(10x^2 - \dfrac{1}{3}x^3\right)\right]$

$= \pi(20x - x^2)$

**74. (a)** $r(x) = \left(3 - \dfrac{x}{40}\right)^2 x = 9x - \dfrac{3}{20}x^2 + \dfrac{1}{1600}x^3$

(b) The marginal revenue is

$r'(x) = 9 - \dfrac{3}{10}x + \dfrac{3}{1600}x^2$

$= \dfrac{3}{1600}(x^2 - 160x + 4800)$

$= \dfrac{3}{1600}(x - 40)(x - 120),$

which is zero when $x = 40$ or $x = 120$. Since the bus holds only 60 people, we require $0 \le x \le 60$. The marginal revenue is 0 when there are 40 people, and the corresponding fare is

$p(40) = \left(3 - \dfrac{40}{40}\right)^2 = \$4.00.$

(c) One possible answer:
If the current ridership is less than 40, then the proposed plan may be good. If the current ridership is greater than or equal to 40, then the plan is not a good idea. Look at the graph of $y = r(x)$.
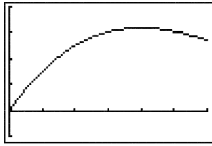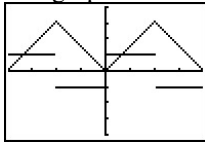

[0, 60] by [−50, 200]

75. (a) Since $x = \tan\theta$, we have

$\dfrac{dx}{dt} = (\sec^2\theta)\dfrac{d\theta}{dt} = -0.6\sec^2\theta$. At point A, we have

$\theta = 0$ and $\dfrac{dx}{dt} = -0.6\sec^2 0 = -0.6\,\text{km/sec}.$

(b) 0.6

$\dfrac{\text{rad}}{\text{sec}}\cdot\dfrac{1\,\text{revolution}}{2\pi\,\text{rad}}\cdot\dfrac{60\,\text{sec}}{1\,\text{min}} = \dfrac{18}{\pi}$ revolutions

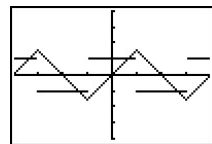per minute or approximately 5.73 revolutions per minute.

76. (a) The graphs:

[−6.3, 6.3] by [−4.1, 4.1]

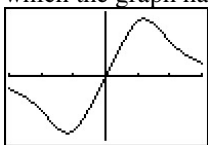It appears that the derivative of y_1 is y_2.

(b) Let $y_2 = \dfrac{|\cos(x)|}{\cos(x)}$. The graphs of y_1 and

y_2 are shown below:

[−6.3, 6.3] by [−4.1, 4.1]

It again appears that the derivative of y_1 is y_2.

77. The graph of the function indicates that the range is confined between the two points at which the graph has horizontal tangents.
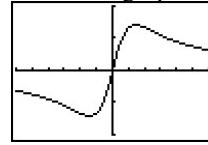

[−3, 3] by [−0.5, 0.5]

Setting

$0 = \dfrac{dy}{dx} = \dfrac{3(x^4+6) - 4x^3(3x)}{(x^4+6)^2} = \dfrac{18 - 9x^4}{(x^4+6)^2},$ we

get $x = \pm\sqrt[4]{2}$. Plugging these values back into the equation of the curve, we get

$\dfrac{3\left(\pm\sqrt[4]{2}\right)}{2+6} = \pm\dfrac{3\sqrt[4]{2}}{8}$. Thus $a = \dfrac{3\sqrt[4]{2}}{8}.$

78. The graph of the function indicates that the range is confined between the two points at which the graph has horizontal tangents.
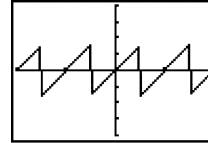

[−6, 6] by [−2, 2]

Setting

$0 = \dfrac{dy}{dx} = \dfrac{4(x^2+2) - 2x(4x)}{(x^2+2)^2} = \dfrac{8 - 4x^2}{(x^2+2)^2},$ we

get $x = \pm\sqrt{2}$. Plugging these values into the

equation of the curve, we get $\dfrac{4\left(\pm\sqrt{2}\right)}{2+2} = \pm\sqrt{2}.$

Thus $a = \sqrt{2}.$

79.

[−π, π] by [−4, 4]

(a) $x \neq k\dfrac{\pi}{4},$ where k is an odd integer

(b) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(c) Where it's not defined, at $x = k\dfrac{\pi}{4},$ k an odd integer

(d) It has period $\dfrac{\pi}{2}$ and continues to repeat the pattern seen in this window.

80. $y'(r) = \dfrac{d}{dr}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right)$

$= \left(\dfrac{1}{2l}\sqrt{\dfrac{T}{\pi d}}\right)\dfrac{d}{dr}\left(\dfrac{1}{r}\right)$

$= -\dfrac{1}{2r^2 l}\sqrt{\dfrac{T}{\pi d}}$

$y'(l) = \dfrac{d}{dl}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right)$

$= \left(\dfrac{1}{2r}\sqrt{\dfrac{T}{\pi d}}\right)\dfrac{d}{dl}\left(\dfrac{1}{l}\right)$

$= -\dfrac{1}{2rl^2}\sqrt{\dfrac{T}{\pi d}}$

$y'(d) = \dfrac{d}{dd}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right)$

$= \left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi}}\right)\dfrac{d}{dd}(d^{-1/2})$

$= \dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi}}\left(-\dfrac{1}{2}d^{-3/2}\right)$

$= -\dfrac{1}{4rl}\sqrt{\dfrac{T}{\pi d^3}}$

$y'(T) = \dfrac{d}{dT}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right)$

$= \left(\dfrac{1}{2rl}\sqrt{\dfrac{1}{\pi d}}\right)\dfrac{d}{dT}(\sqrt{T})$

$= \dfrac{1}{2rl}\sqrt{\dfrac{1}{\pi d}}\left(\dfrac{1}{2\sqrt{T}}\right)$

$= \dfrac{1}{4rl\sqrt{\pi dT}}$

Since $y'(r) < 0$, $y'(l) < 0$, and $y'(d) < 0$, increasing r, l, or d would decrease the frequency. Since $y'(T) > 0$, increasing T would increase the frequency.

81. (a) $v(t) = s'(t) = 3t^2 - 12$

(b) $a(t) = v'(t) = 6t$

(c) Set $v(t) = 0$ and solve for t:

$3t^2 - 12 = 0$

$3(t^2 - 4) = 0$

$3(t - 2)(t + 2) = 0$

$t = 2$ or $\cancel{t = -2}$

The particle is at rest when $t = 2$.

(d) $a(t) = 0$ when $t = 0$

speed $= |v(0)| = |3(0)^2 - 12| = 12$

(e) Towards the origin:

$s(3) = 3^3 - 12(3) + 5 = -4 < 0$

$v(3) = 3(3)^2 - 12 = 15 > 0$

The particle is left of the origin and it is moving to the right.

82. (a) $y - 3 = 5(x - 4)$

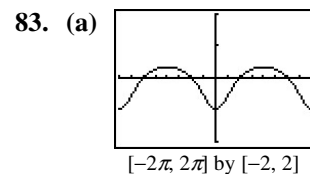**(b)** Yes; since f is differentiable at $x = 3$, it is continuous at $x = 3$.

(c) Yes; since f is continuous on [2, 4], it takes on all values between $f(2) = -1$ and $f(4) = 3$ (Intermediate Value Theorem).

(d) $g'(2) = \dfrac{d}{dx}\left(\dfrac{f(x)}{f(x) - 3}\right)\bigg|_{x=2}$

$= \dfrac{f'(2)(f(2) - 3) - (f'(2) - 0)f(2)}{(f(2) - 3)^2}$

$= -\dfrac{9}{16}$

(e) Since $f(4) - 3 = 0$, the function g is not defined at $x = 4$.

83. (a)

$[-2\pi, 2\pi]$ by $[-2, 2]$

(b) $f'(x) = \dfrac{-\sin x(\cos x - 2) - (-\sin x)\cos x}{(\cos x - 2)^2}$

$= \dfrac{2\sin x}{(\cos x - 2)^2}$

(c) $f'(x) = 0$ where $\sin x = 0$. On this interval, that is at $0, \pm\pi, \pm 2\pi$.

(d) The low turning points are

$f(0) = f(\pm 2\pi) = \dfrac{1}{1 - 2} = -1$, while the

high turning points are

$f(\pm\pi) = \dfrac{-1}{-1 - 2} = \dfrac{1}{3}$. The range is the

interval $\left[-1, \dfrac{1}{3}\right]$.