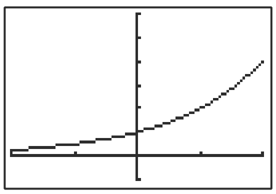
# Chapter 3
# Exponential, Logistic, and Logarithmic Functions

## ■ Section 3.1 Exponential and Logistic Functions

### Exploration 1

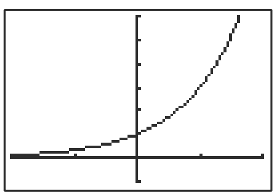
**1.** The point $(0, 1)$ is common to all four graphs, and all four functions can be described as follows:

Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Continuous
Always increasing
Not symmetric
No local extrema
Bounded below by $y = 0$, which is also the only asymptote
$\lim_{x\to\infty} f(x) = \infty,\ \lim_{x\to-\infty} f(x) = 0$



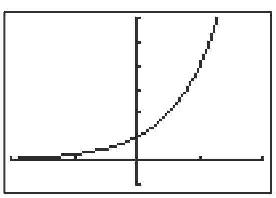
$[-2, 2]$ by $[-1, 6]$

$y_1 = 2^x$



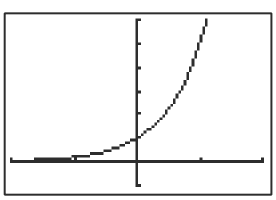
$[-2, 2]$ by $[-1, 6]$

$y_2 = 3^x$



$[-2, 2]$ by $[-1, 6]$

$y_3 = 4^x$

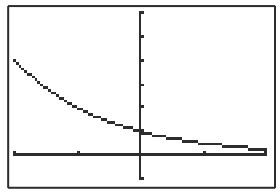


$[-2, 2]$ by $[-1, 6]$

$y_4 = 5^x$

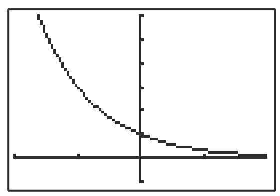
**2.** The point $(0, 1)$ is common to all four graphs, and all four functions can be described as follows:

Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Continuous
Always decreasing
Not symmetric
No local extrema
Bounded below by $y = 0$, which is also the only asymptote
$\lim_{x\to\infty} g(x) = 0,\ \lim_{x\to-\infty} g(x) = \infty$



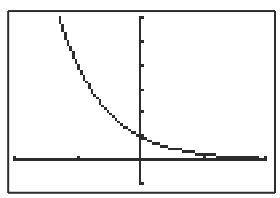
$[-2, 2]$ by $[-1, 6]$

$y_1 = \left(\frac{1}{2}\right)^x$



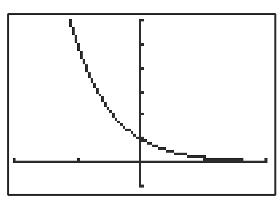
$[-2, 2]$ by $[-1, 6]$

$y_2 = \left(\frac{1}{3}\right)^x$



$[-2, 2]$ by $[-1, 6]$

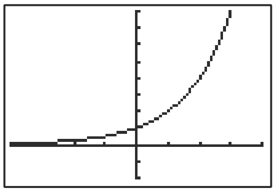
$y_3 = \left(\frac{1}{4}\right)^x$



$[-2, 2]$ by $[-1, 6]$

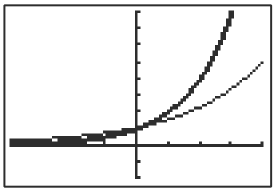
$y_4 = \left(\frac{1}{5}\right)^x$

### Exploration 2

1. 

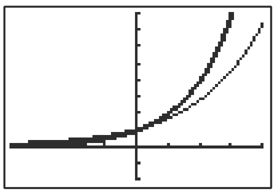
$[-4, 4]$ by $[-2, 8]$

$f(x) = 2^x$

2. 

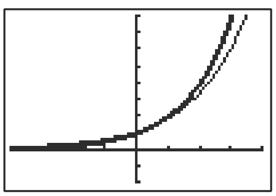
$[-4, 4]$ by $[-2, 8]$

$f(x) = 2^x$
$g(x) = e^{0.4x}$



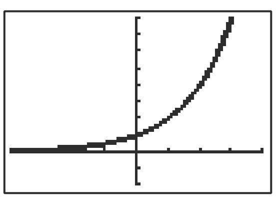
$[-4, 4]$ by $[-2, 8]$

$f(x) = 2^x$
$g(x) = e^{0.5x}$



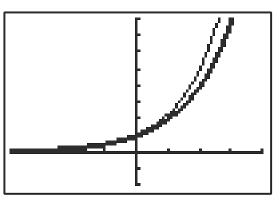
$[-4, 4]$ by $[-2, 8]$

$f(x) = 2^x$
$g(x) = e^{0.6x}$



$[-4, 4]$ by $[-2, 8]$

$f(x) = 2^x$
$g(x) = e^{0.7x}$



$[-4, 4]$ by $[-2, 8]$

$f(x) = 2^x$
$g(x) = e^{0.8x}$

$k = 0.7$ most closely matches the graph of $f(x)$.

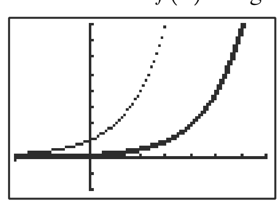
3. $k \approx 0.693$

### Quick Review 3.1

1. $\sqrt[3]{-216} = -6$ since $(-6)^3 = -216$
2. $\sqrt[3]{\frac{125}{8}} = \frac{5}{2}$ since $5^3 = 125$ and $2^3 = 8$
3. $27^{2/3} = (3^3)^{2/3} = 3^2 = 9$
4. $4^{5/2} = (2^2)^{5/2} = 2^5 = 32$
5. $\frac{1}{2^{12}}$
6. $\frac{1}{3^8}$
7. $\frac{1}{a^6}$
8. $b^{15}$
9. $-1.4$, since $(-1.4)^5 = -5.37824$
10. $3.1$, since $(3.1)^4 = 92.3521$

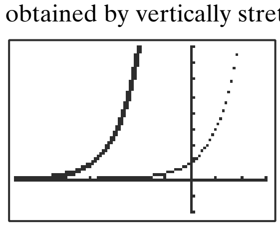
### Section 3.1 Exercises

1. Not an exponential function because the base is variable and the exponent is constant. It is a monomial function.
2. Exponential function, with an initial value of 1 and base of 3.
3. Exponential function, with an initial value of 1 and base of 5.
4. Not an exponential function because the exponent is constant. It is a constant function.
5. Not an exponential function because the base is variable.
6. Not an exponential function because the base is variable. It is a power function.
7. $f(0) = 3 \cdot 5^0 = 3 \cdot 1 = 3$
8. $f(-2) = 6 \cdot 3^{-2} = \frac{6}{9} = \frac{2}{3}$
9. $f\left(\frac{1}{3}\right) = -2 \cdot 3^{1/3} = -2\sqrt[3]{3}$
10. $f(-\frac{3}{2}) = 8 \cdot 4^{-3/2} = \frac{8}{(2^2)^{3/2}} = \frac{8}{2^3} = \frac{8}{8} = 1$
11. $f(x) = \frac{3}{2} \cdot \left(\frac{1}{2}\right)^x$
12. $g(x) = 12 \cdot \left(\frac{1}{3}\right)^x$
13. $f(x) = 3 \cdot (\sqrt{2})^x = 3 \cdot 2^{x/2}$
14. $g(x) = 2 \cdot \left(\frac{1}{e}\right)^x = 2e^{-x}$
15. Translate $f(x) = 2^x$ by 3 units to the right. Alternatively, $g(x) = 2^{x-3} = 2^{-3} \cdot 2^x = \frac{1}{8} \cdot 2^x = \frac{1}{8} \cdot f(x)$, so it can be obtained from $f(x)$ using a vertical shrink by a factor of $\frac{1}{8}$.



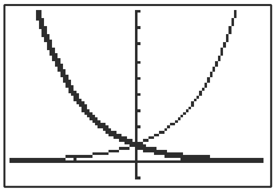
$[-3, 7]$ by $[-2, 8]$

16. Translate $f(x) = 3^x$ by 4 units to the left. Alternatively, $g(x) = 3^{x+4} = 3^4 \cdot 3^x = 81 \cdot 3^x = 81 \cdot f(x)$, so it can be obtained by vertically stretching $f(x)$ by a factor of 81.



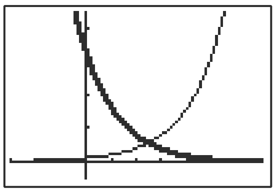
$[-7, 3]$ by $[-2, 8]$

**17.** Reflect $f(x) = 4^x$ over the $y$-axis.



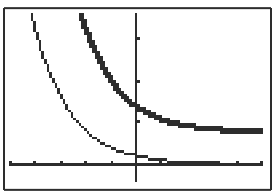
$[-2, 2]$ by $[-1, 9]$

**18.** Reflect $f(x) = 2^x$ over the $y$-axis and then shift by 5 units to the right.



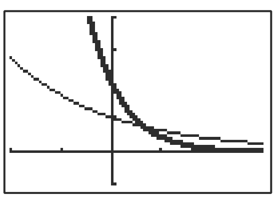
$[-3, 7]$ by $[-5, 45]$

**19.** Vertically stretch $f(x) = 0.5^x$ by a factor of 3 and then shift 4 units up.



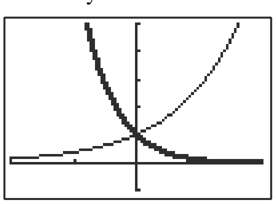
$[-5, 5]$ by $[-2, 18]$

**20.** Vertically stretch $f(x) = 0.6^x$ by a factor of 2 and then horizontally shrink by a factor of 3.



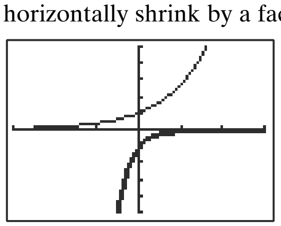
$[-2, 3]$ by $[-1, 4]$

**21.** Reflect $f(x) = e^x$ across the $y$-axis and horizontally shrink by a factor of 2.



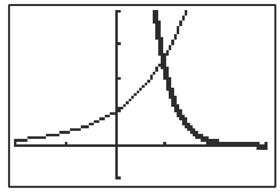
$[-2, 2]$ by $[-1, 5]$

**22.** Reflect $f(x) = e^x$ across the $x$-axis and $y$-axis. Then, horizontally shrink by a factor of 3.



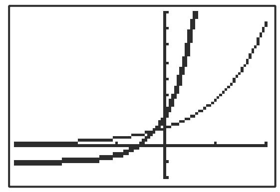
$[-3, 3]$ by $[-5, 5]$

**23.** Reflect $f(x) = e^x$ across the $y$-axis, horizontally shrink by a factor of 3, translate 1 unit to the right, and vertically stretch by a factor of 2.



$[-2, 3]$ by $[-1, 4]$

**24.** Horizontally shrink $f(x) = e^x$ by a factor of 2, vertically stretch by a factor of 3, and shift down 1 unit.



$[-3, 3]$ by $[-2, 8]$

**25.** Graph (a) is the only graph shaped and positioned like the graph of $y = b^x, b > 1$.

**26.** Graph (d) is the reflection of $y = 2^x$ across the $y$-axis.

**27.** Graph (c) is the reflection of $y = 2^x$ across the $x$-axis.

**28.** Graph (e) is the reflection of $y = 0.5^x$ across the $x$-axis.

**29.** Graph (b) is the graph of $y = 3^{-x}$ translated down 2 units.

**30.** Graph (f) is the graph of $y = 1.5^x$ translated down 2 units.

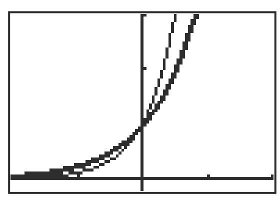
**31.** Exponential decay; $\lim_{x\to\infty} f(x) = 0$; $\lim_{x\to-\infty} f(x) = \infty$

**32.** Exponential decay; $\lim_{x\to\infty} f(x) = 0$; $\lim_{x\to-\infty} f(x) = \infty$

**33.** Exponential decay: $\lim_{x\to\infty} f(x) = 0$; $\lim_{x\to-\infty} f(x) = \infty$

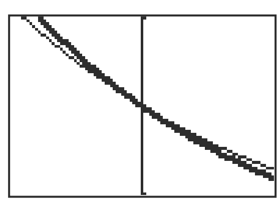
**34.** Exponential growth: $\lim_{x\to\infty} f(x) = \infty$; $\lim_{x\to-\infty} f(x) = 0$

**35.** $x < 0$



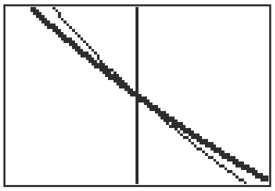
$[-2, 2]$ by $[-0.2, 3]$

**36.** $x > 0$



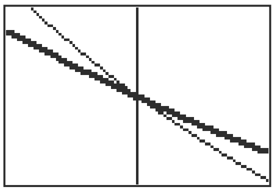
$[-0.25, 0.25]$ by $[0.5, 1.5]$

**37.** $x < 0$



[–0.25, 0.25] by [0.75, 1.25]

**38.** $x > 0$

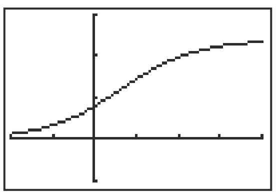


[–0.25, 0.25] by [0.75, 1.25]

**39.** $y_1 = y_3$, since $3^{2x+4} = 3^{2(x+2)} = (3^2)^{x+2} = 9^{x+2}$.

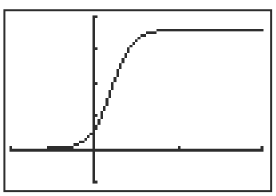
**40.** $y_2 = y_3$, since $2 \cdot 2^{3x-2} = 2^1\, 2^{3x-2} = 2^{1+3x-2} = 2^{3x-1}$.

**41.** $y$-intercept: $(0, 4)$. Horizontal asymptotes: $y = 0$, $y = 12$.



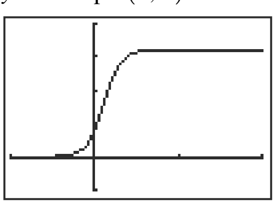
[–10, 20] by [–5, 15]

**42.** $y$-intercept: $(0, 3)$. Horizontal asymptotes: $y = 0$, $y = 18$.



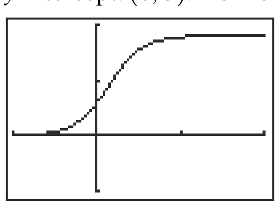
[–5, 10] by [–5, 20]

**43.** $y$-intercept: $(0, 4)$. Horizontal asymptotes: $y = 0$, $y = 16$.



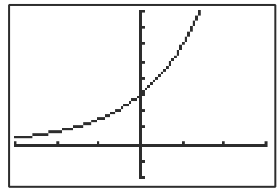
[–5, 10] by [–5, 20]

**44.** $y$-intercept: $(0, 3)$. Horizontal asymptotes: $y = 0$, $y = 9$.



[–5, 10] by [–5, 10]

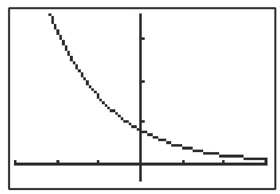
**45.**



[−3, 3] by [−2, 8]

Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Continuous
Always increasing
Not symmetric
Bounded below by $y = 0$, which is also the only asymptote
No local extrema
$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 0$

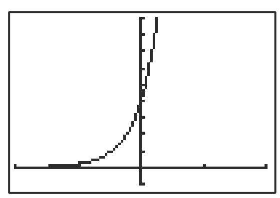
**46.**



[−3, 3] by [−2, 18]

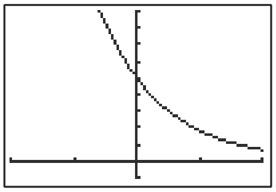
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Continuous
Always decreasing
Not symmetric
Bounded below by $y = 0$, which is the only asymptote
No local extrema
$\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to-\infty} f(x) = \infty$

**47.**



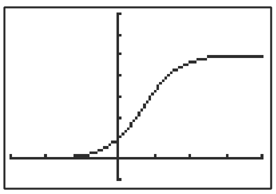
[−2, 2] by [−1, 9]

Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Continuous
Always increasing
Not symmetric
Bounded below by $y = 0$, which is the only asymptote
No local extrema
$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 0$

**48.** 

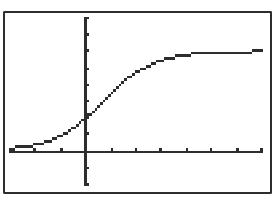
$[-2, 2]$ by $[-1, 9]$

Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Continuous
Always decreasing
Not symmetric
Bounded below by $y = 0$, which is also the only asymptote
No local extrema
$\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to-\infty} f(x) = \infty$

**49.** 

$[-3, 4]$ by $[-1, 7]$

Domain: $(-\infty, \infty)$
Range: $(0, 5)$
Continuous
Always increasing
Symmetric about $(0.69, 2.5)$
Bounded below by $y = 0$ and above by $y = 5$; both are asymptotes
No local extrema
$\lim_{x\to\infty} f(x) = 5$, $\lim_{x\to-\infty} f(x) = 0$

**50.** 

$[-3, 7]$ by $[-2, 8]$

Domain: $(-\infty, \infty)$
Range: $(0, 6)$
Continuous
Always increasing
Symmetric about $(0.69, 3)$
Bounded below by $y = 0$ and above by $y = 6$; both are asymptotes
No local extrema
$\lim_{x\to\infty} f(x) = 6$, $\lim_{x\to-\infty} f(x) = 0$

For #51 and 52, refer to Example 7 on page 260 in the text.

**51.** Let $P(t)$ be Austin's population $t$ years after 1990. Then with exponential growth, $P(t) = P_0 \cdot b^t$ where $P_0 = 465{,}622$. From Table 3.7, $P(21) = 465{,}622 \cdot b^{21} = 820{,}611$. So,
$b = \sqrt[21]{\frac{820{,}611}{465{,}622}} \approx 1.0274.$
Solving graphically, we find that the curve $y = 465{,}622(1.0274)^t$ intersects the line $y = 800{,}000$ at $t \approx 20.02$. Austin's population will pass 800,000 in 2010.

**52.** Let $P(t)$ be Columbus's population $t$ years after 1990. Then with exponential growth, $P(t) = P_0 \cdot b^t$ where $P_0 = 632{,}910$. From Table 3.7, $P(21) = 632{,}910 \cdot b^{21} = 7797{,}434$. So,
$b = \sqrt[21]{\frac{797{,}434}{632{,}910}} \approx 1.0111.$
Solving graphically, we find that the curve $y = 632{,}910$ $(1.0111)^t$ intersects the line $y = 800{,}000$ at $t \approx 21.22$. Columbus's population will pass 800,000 in 2011.

**53.** Using the results from Exercises 51 and 52, we represent Austin's population as $y = 465{,}622(1.0274)^t$ and Columbus's population as $y = 632{,}910(1.0111)^t$. Solving graphically, we find that the curves intersect at $t \approx 19.19$. The two populations were equal, at 782,273, in 2009.

**54.** Let $P(t)$ be Austin's population $t$ years after 1990. Then with exponential growth, $P(t) = P_0\, b^t$ where $P_0 = 465{,}622$. From Table 3.7, $P(21) = 465{,}622\, b^{21} = 820{,}611$. So,
$b = \sqrt[21]{\frac{820{,}611}{465{,}622}} \approx 1.0274.$
Solving graphically, we find that the curve $y = 465{,}622(1.0274)^t$ intersects the line $y = 1{,}000{,}000$ at $t \approx 28.28$. Austin's population passed 1,000,000 in 2018.

Let $P(t)$ be Columbus's population $t$ years after 1990. Then with exponential growth, $P(t) = P_0\, b^t$ where $P_0 = 632{,}910$. From Table 3.7, $P(21) = 632{,}910\, b^{21} = 7797{,}434$. So,
$b = \sqrt[21]{\frac{797{,}434}{632{,}910}} \approx 1.0111.$
Solving graphically, we find that the curve $y = 632{,}910$ $(1.0111)^t$ intersects the line $y = 1{,}000{,}000$ at $t \approx 41.44$. Columbus's population will pass 1,000,000 in 2031.

From the results of above, Austin's population will reach 1,000,00 first, in 2018.

**55.** Solving graphically, we find that the curve $y = \frac{12.79}{1 + 2.402e^{-0.0309x}}$ intersects the line $y = 10$ when $t \approx 69.67$. Ohio's population stood at 10 million in 1969.

**56.** **(a)** $P(50) = \frac{19.875}{1 + 57.993e^{-0.035005(50)}} \approx 1.794558$ or 1,794,558 people.

**(b)** $P(215) = \frac{19.875}{1 + 57.993e^{-0.035005(215)}} \approx 19.272737$ or 19,272,737 people.

**(c)** $\lim_{x\to\infty} P(t) = 19.875$ or 19,875,000 people.

**57.** **(a)** When $t = 0$, $B = 100$.

**(b)** When $t = 6$, $B \approx 6394$.

**58.** **(a)** When $t = 0$, $C = 20$ grams.

**(b)** When $t = 10{,}400$, $C \approx 5.647$. After about 5700.22 years, 10 grams remain.

**59.** False. If $a > 0$ and $0 < b < 1$, or if $a < 0$ and $b > 1$, then $f(x) = a \cdot b^x$ is decreasing.

**60.** True. For $f(x) = \frac{c}{1 + a \cdot b^x}$ the horizontal asymptotes are $y = 0$ and $y = c$, where $c$ is the limit of growth.

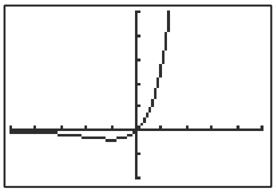
**61.** Only $8^x$ has the form $a \cdot b^x$ with $a$ nonzero and $b$ positive but not equal to 1. The answer is E.

**62.** For $b > 0, f(0) = b^0 = 1$. The answer is C.

**63.** The growth factor of $f(x) = a \cdot b^x$ is the base $b$. The answer is A.

**64.** With $x > 0, a^x > b^x$ requires $a > b$ (regardless of whether $x < 1$ or $x > 1$). The answer is B.

**65. (a)**



$[-5, 5]$ by $[-2, 5]$

Domain: $(-\infty, \infty)$

Range: $\left[-\frac{1}{e}, \infty\right)$

Intercept: $(0, 0)$

Decreasing on $(-\infty, -1]$. Increasing on $[-1, \infty)$

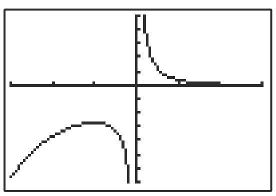
Bounded below by $y = -\frac{1}{e}$

Local minimum at $\left(-1, -\frac{1}{e}\right)$

Asymptote: $y = 0$

$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 0$

**(b)**



$[-3, 3]$ by $[-7, 5]$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -e] \cup (0, \infty)$
No intercepts
Increasing on $(-\infty, -1]$;
Decreasing on $[-1, 0) \cup (0, \infty)$
Not bounded
Local maxima at $(-1, -e)$
Asymptotes: $x = 0, y = 0$
$\lim_{x\to\infty} g(x) = 0$, $\lim_{x\to-\infty} g(x) = -\infty$

**66. (a)** $2^x = (2^2)^2 = 2^4$, so $x = 4$.

**(b)** $3^x = 3^3$, so $x = 3$.

**(c)** $8^{x/2} = 4^{x+1}, (2^2)^{x/2} = (2^2)^{x+1} \cdot \frac{3x}{2} = 2x + 2,$
$3x = 4x + 4, x = -4.$

**(d)** $9^x = 3^{x+1}, (3^2)^x = 3^{x+1}, 2x = x + 1, x = 1.$

**67. (a)** You have $2^4 = 16$ 4th great grandparents and they are in the 4th generation.

**(b)** $y = 2^x$ with domain 0, 1, 2, 3 etc.

**(c)** You have $2^6 = 64$ 6th great grandparents.

**(d)** You have $2^{25} = 33{,}554{,}432$ 25th great grandparents.

**(e)** It would take about $25 \times 30 = 750$ years to span 25 generations. You would be directly related to about $\frac{33{,}554{,}432}{400{,}000{,}000} = 0.084$ or 8.4% of the world's population in 1250.

**68. (a)** $y_1$—$f(x)$ decreases less rapidly as $x$ increases.

**(b)** $y_3$—as $x$ increases, $g(x)$ decreases ever more rapidly.

**69.** $c = 2^a$: To the graph of $(2^a)^x$ apply a vertical stretch by $2^b$, since $f(ax + b) = 2^{ax+b} = 2^{ax}2^b = (2^b)(2^a)^x$.

**70.** $a \neq 0, c = 2$.

**71.** $a < 0, c = 1$.

**72.** $a > 0$ and $b > 1$, or $a < 0$ and $0 < b < 1$.

**73.** $a > 0$ and $0 < b < 1$, or $a < 0$ and $b > 1$.

**74.** Since $0 < b < 1$, $\lim_{x\to-\infty} (1 + a \cdot b^x) = \infty$ and $\lim_{x\to\infty} (1 + a \cdot b^x) = 1$. Thus, $\lim_{x\to-\infty} \frac{c}{1 + a \cdot b^x} = 0$ and $\lim_{x\to\infty} \frac{c}{1 + a \cdot b^x} = c$.

## ■ Section 3.2 Exponential and Logistic Modeling

### Quick Review 3.2

**1.** 0.15

**2.** 4%

**3.** (1.07)(23)

**4.** (0.96)(52)

**5.** $b^2 = \frac{160}{40} = 4$, so $b = \pm\sqrt{4} = \pm 2$.

**6.** $b^3 = \frac{9}{243}$, so $b = \sqrt[3]{\frac{9}{243}} = \sqrt[3]{\frac{1}{27}} = \frac{1}{3}$.

**7.** $b = \sqrt[6]{\frac{838}{782}} \approx 1.01$

**8.** $b = \sqrt[5]{\frac{521}{93}} \approx 1.41$

**9.** $b = \sqrt[4]{\frac{91}{672}} \approx 0.61$

**10.** $b = \sqrt[7]{\frac{56}{127}} \approx 0.89$

### Section 3.2 Exercises

For #1–20, use the model $P(t) = P_0 (1 + r)^t$.

**1.** $r = 0.09$, so $P(t)$ is an exponential growth function of 9%.

**2.** $r = 0.018$, so $P(t)$ is an exponential growth function of 1.8%.

**3.** $r = -0.032$, so $f(x)$ is an exponential decay function of 3.2%.

**4.** $r = -0.0032$, so $f(x)$ is an exponential decay function of 0.32%.

**5.** $r = 1$, so $g(t)$ is an exponential growth function of 100%.

**6.** $r = -0.95$, so $g(t)$ is an exponential decay function of 95%.

**7.** $f(x) = 5 \cdot (1 + 0.17)^x = 5 \cdot 1.17^x$ ($x$ = years)

**8.** $f(x) = 52 \cdot (1 + 0.023)^x = 52 \cdot 1.023^x$ ($x$ = days)

**9.** $f(x) = 16 \cdot (1 - 0.5)^x = 16 \cdot 0.5^x$ ($x$ = months)

**10.** $f(x) = 5 \cdot (1 - 0.0059) = 5 \cdot 0.9941^x$ ($x$ = weeks)

**11.** $f(x) = 28{,}900 \cdot (1 - 0.026)^x = 28{,}900 \cdot 0.974^x$ ($x$ = years)

**12.** $f(x) = 502{,}000 \cdot (1 + 0.017)^x = 502{,}000 \cdot 1.017^x$ ($x$ = years)

**13.** $f(x) = 18 \cdot (1 + 0.052)^x = 18 \cdot 1.052^x$ ($x$ = weeks)

**14.** $f(x) = 15 \cdot (1 - 0.046)^x = 15 \cdot 0.954^x$ ($x$ = days)

**15.** $f(x) = 0.6 \cdot 2^{x/3}$ ($x$ = days)

**16.** $f(x) = 250 \cdot 2^{x/7.5} = 250 \cdot 2^{2x/15}$ ($x$ = hours)

**17.** $f(x) = 592 \cdot 2^{-x/6}$ ($x$ = years)

**18.** $f(x) = 17 \cdot 2^{-x/32}$ ($x$ = hours)

**19.** $f_0 = 2.3, \dfrac{2.875}{2.3} = 1.25 = r + 1$, so

$f(x) = 2.3 \cdot 1.25^x$ (Growth Model).

**20.** $g_0 = -5.8, \dfrac{-4.64}{-5.8} = 0.8 = r + 1$, so

$g(x) = -5.8 \cdot 0.8^x$ (Decay Model).

For #21 and 22, use $f(x) = f_0 \cdot b^x$.

**21.** $f_0 = 4$, so $f(x) = 4 \cdot b^x$. Since $f(5) = 4 \cdot b^5 = 8.05$,

$b^5 = \dfrac{8.05}{4}, b = \sqrt[5]{\dfrac{8.05}{4}} \approx 1.15.\ f(x) \approx 4 \cdot 1.15^x$.

**22.** $f_0 = 3$, so $f(x) = 3 \cdot b^x$. Since $f(4) = 3 \cdot b^4 = 1.49$

$b^4 = \dfrac{1.49}{3}, b = \sqrt[4]{\dfrac{1.49}{3}} \approx 0.84.\ f(x) \approx 3 \cdot 0.84^x$.

For #23–28, use the model $f(x) = \dfrac{c}{1 + a \cdot b^x}$.

**23.** $c = 40, a = 3$, so $f(1) = \dfrac{40}{1 + 3b} = 20, 20 + 60b = 40$,

$60b = 20, b = \dfrac{1}{3}$, thus $f(x) = \dfrac{40}{1 + 3 \cdot \left(\frac{1}{3}\right)^x}$.

**24.** $c = 60, a = 4$, so $f(1) = \dfrac{60}{1 + 4b} = 24, 60 = 24 + 96b$,

$96b = 36, b = \dfrac{3}{8}$, thus $f(x) = \dfrac{60}{1 + 4\left(\frac{3}{8}\right)^x}$.

**25.** $c = 128, a = 7$, so $f(5) = \dfrac{128}{1 + 7b^5} = 32$,

$128 = 32 + 224b^5, 224b^5 = 96, b^5 = \dfrac{96}{224}$,

$b = \sqrt[5]{\dfrac{96}{224}} \approx 0.844$, thus $f(x) \approx \dfrac{128}{1 + 7 \cdot 0.844^x}$.

**26.** $c = 30, a = 5$, so $f(3) = \dfrac{30}{1 + 5b^3} = 15, 30 = 15 + 75b^3$,

$75b^3 = 15, b^3 = \dfrac{15}{75} = \dfrac{1}{5}, b = \sqrt[3]{\dfrac{1}{5}} \approx 0.585$,

thus $f(x) \approx \dfrac{30}{1 + 5 \cdot 0.585^x}$.

**27.** $c = 20, a = 3$, so $f(2) = \dfrac{20}{1 + 3b^2} = 10, 20 = 10 + 30b^2$,

$30b^2 = 10, b^2 = \dfrac{1}{3}, b = \sqrt{\dfrac{1}{3}} \approx 0.58$,

thus $f(x) = \dfrac{20}{1 + 3 \cdot 0.58^x}$.

**28.** $c = 60, a = 3$, so $f(8) = \dfrac{60}{1 + 3b^8} = 30, 60 = 30 + 90b^8$,

$90b^8 = 30, b^8 = \dfrac{1}{3}, b = \sqrt[8]{\dfrac{1}{3}} \approx 0.87$,

thus $f(x) = \dfrac{60}{1 + 3 \cdot 0.87^x}$.

**29.** $P(t) = 736{,}000(1.0149)^t$; $P(t) = 1{,}000{,}000$ when $t \approx 20.73$ years, or in the year 2020.

**30.** $P(t) = 478{,}000(1.0628)^t$; $P(t) = 1{,}000{,}000$ when $t \approx 12.12$ years, or in the year 2012.

**31.** The model is $P(t) = 6250(1.0275)^t$.

**(a)** In 1915: about $P(25) \approx 12{,}315$. In 1940: about $P(50) \approx 24{,}265$.

**(b)** $P(\text{t}) = 50{,}000$ when $t \approx 76.65$ years after 1890 — in 1966.

**32.** The model is $P(t) = 4200(1.0225)^t$.

**(a)** In 1930: about $P(20) \approx 6554$. In 1945: about $P(35) \approx 9151$.

**(b)** $P(t) = 20{,}000$ when $t \approx 70.14$ years after 1910: about 1980.

**33. (a)** $y = 6.6\left(\dfrac{1}{2}\right)^{t/14}$, where $t$ is time in days.

**(b)** After 38.11 days.

**34. (a)** $y = 3.5\left(\dfrac{1}{2}\right)^{t/65}$, where $t$ is time in days.

**(b)** After 117.48 days.

**35.** One possible answer: Exponential and linear functions are similar in that they are always increasing or always decreasing. However, the two functions vary in how *quickly* they increase or decrease. While a linear function will increase or decrease at a steady rate over a given interval, the rate at which exponential functions increase or decrease over a given interval will vary.

**36.** One possible answer: Exponential functions and logistic functions are similar in the sense that they are always increasing or always decreasing. They differ, however, in the sense that logistic functions have both an upper and a lower limit to their growth (or decay), while exponential functions generally have only a lower limit. (Exponential functions just keep growing.)

**37.** One possible answer: From the graph we see that the doubling time for this model is 4 years. This is the time required to double from 50,000 to 100,000, from 100,000 to 200,000, or from any population size to twice that size. Regardless of the population size, it takes 4 years for it to double.

**38.** One possible answer: The number of atoms of a radioactive substance that change to a nonradioactive state in a given time is a fixed percentage of the number of radioactive atoms initially present. So the time it takes for half of the atoms to change state (the half-life) does not depend on the initial amount.

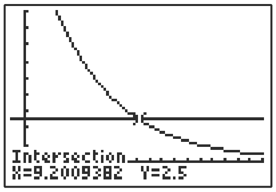
**39.** When $t = 1, B \approx 200$ — the population doubles every hour.

**40.** The half-life is about 5700 years.

For #41 and 42, use the formula $P(h) = 14.7 \cdot 0.5^{h/3.6}$, where $h$ is miles above sea level.

**41.** $P(10) = 14.7 \cdot 0.5^{10/3.6} = 2.14 \text{ lb/in}^2$

**42.** $P(h) = 14.7 \cdot 0.5^{h/3.6}$ intersects $y = 2.5$ when $h \approx 9.20$ miles above sea level.




$[-1, 19]$ by $[-1, 9]$

**43.** The exponential regression model is $P(t) = 1149.61904(1.012133)^t$, where $P(t)$ is measured in thousands of people and $t$ is years since 1900. The predicted population for Los Angeles for 2011 is $P(111) \approx 4384.5$, or 4,384,500 people. This is an overestimate of 565,000 people,

an error of $\dfrac{565{,}000}{3{,}820{,}000} \approx 0.15 = 15\%$.

**44.** The exponential regression model using 1950–2000 data is $P(t) = 20.84002(1.04465)^t$, where $P(t)$ is measured in thousands of people and $t$ is years since 1900. The predicted population for Phoenix for 2011 is $P(111) \approx 2658.6$, or 2,658,600 people. This is an overestimate of 1,189,200 people,

an error of $\dfrac{1{,}189{,}200}{1{,}469{,}4000} \approx 0.81 = 81\%$.

The equations in #45 and 46 can be solved either algebraically or graphically; the latter approach is generally faster.

**45.** **(a)** $P(0) = 16$ students.

**(b)** $P(t) = 200$ when $t \approx 13.97$ — about 14 days.

**(c)** $P(t) = 300$ when $t \approx 16.90$ — about 17 days.

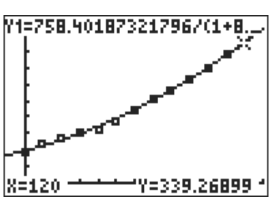
**46.** **(a)** $P(0) = 11$.

**(b)** $P(t) = 600$ when $t \approx 24.51$ — after 24 or 25 years.

**(c)** As $t \to \infty$, $P(t) \to 1001$—the population never rises above this level.

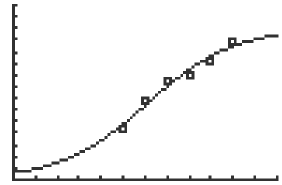
**47.** The logistic regression model is

$$P(x) = \frac{758.4018732}{1 + 8.69834271e^{-0.01626448x}},$$ where $x$ is the number of years since 1900 and $P(x)$ is measured in millions of people. In the year 2020, $x = 120$, so the model predicts a population of $P(120) \approx 339.3$ or about 339.3 million people.



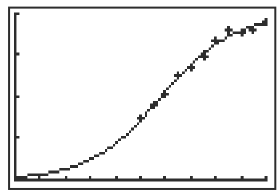

[–10, 130] by [–20, 400]

**48.** $P(t) \approx \dfrac{1{,}301{,}642}{1 + 21.602\,e^{-0.05054t}}$, which is the same model as the solution in Example 8 of Section 3.1. Note that $t$ represents the number of years since 1900.



[0,120] by [0,1500000]

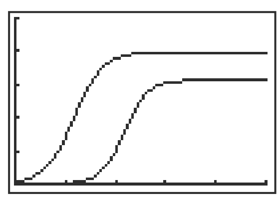
**49.** $P(x) \approx \dfrac{19.875}{1 + 57.993e^{-0.035005x}}$ where $x$ is the number of years after 1800 and $P$ is measured in millions. Our model is the same as the model in Exercise 56 of Section 3.1.



[0, 200] by [0, 20]

**50.** $P(x) \approx \dfrac{15.64}{1 + 11799.36e^{-0.043241x}}$, where $x$ is the number of years since 1800 and $P$ is measured in millions.

As $x \to \infty$, $P(x) \to 15.64$, or nearly 16 million, which is significantly less than New York's population limit of 20 million. The population of Arizona, according to our models, will not surpass the population of New York. Our graph confirms this.



[0, 500] by [0, 25]

**51.** False. This is true for *logistic* growth, not for exponential growth.

**52.** False. When $r < 0$, the base of the function, $1 + r$, is merely less than 1.

**53.** The base is $1.049 = 1 + 0.049$, so the constant percentage growth rate is $0.049 = 4.9\%$. The answer is C.

**54.** The base is $0.834 = 1 - 0.166$, so the constant percentage decay rate is $0.166 = 16.6\%$. The answer is B.

**55.** The growth can be modeled as $P(t) = 1 \cdot 2^{t/4}$. Solve $P(t) = 1000$ to find $t \approx 39.86$. The answer is D.

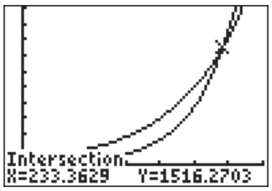
**56.** Check $S(0), S(2), S(4), S(6)$, and $S(8)$. The answer is E.

**57.** **(a)** $P(x) \approx \dfrac{694.27}{1 + 7.90e^{-0.017x}}$, where $x$ is the number of years since 1900 and $P$ is measured in millions. $P(111) \approx 308.9$, or about 308,900,000 people.

**(b)** The logistic model underestimates the 2000 population by about 2.7 million, an error of around 0.8%.

**(c)** The logistic model predicted a value closer to the actual value than the exponential model, perhaps indicating a better fit.

**58. (a)** Using the exponential growth model and the data from 1900–2011, Mexico's population can be represented by $M(x) \approx 11.59 \times 1.021^x$, where $x$ is the number of years since 1900 and $M$ is measured in millions. Using the 1900–2011 data for the United States, and the exponential growth model, the population of the United States can be represented by $P(x) \approx 81.27 \times 1.013^x$, where $x$ is the number of years since 1900 and $P$ is measured in millions. Since Mexico's rate of growth outpaces the United States' rate of growth, the model predicts that Mexico will eventually have a larger population. Our graph indicates that this will occur at $x \approx 233$, or 2133.



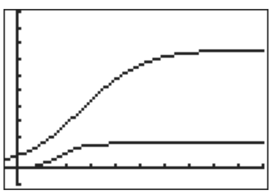

[–20, 280] by [–200, 2000]

**(b)** Using logistic growth models and the same data,

$$M(x) = \frac{159.68}{1 + 43.13e^{-0.0429x}}$$

$$P(x) = \frac{758.40}{1 + 8.698e^{-0.016x}}.$$

Using this model, Mexico's population will not exceed that of the United States, confirmed by our graph.



[–20, 500] by [–100, 1000]

**(c)** According to the logistic growth models, the maximum sustainable populations are:
Mexico—160 million people.
United States—758 million people.

**(d)** Answers will vary. However, a logistic model acknowledges that there is a limit to how much a country's population can grow.

**59.** $\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2}$
$= -\left(\frac{e^x - e^{-x}}{2}\right) = -\sinh(x)$, so the function is odd.

**60.** $\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = \frac{e^x + e^{-x}}{2}$
$= \cosh(x)$, so the function is even.

**61. (a)** $\frac{\sinh(x)}{\cosh(x)} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}}$
$= \frac{e^x - e^{-x}}{2} \cdot \frac{2}{e^x + e^{-x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh(x)$.

**(b)** $\tanh(-x) = \frac{e^{-x} - e^{-(-x)}}{e^{-x} + e^{-(-x)}} = \frac{e^{-x} - e^{x}}{e^{-x} + e^{x}}$
$= -\frac{e^x - e^{-x}}{e^x + e^{-x}} = -\tanh(x)$, so the function is odd.

**(c)** $f(x) = 1 + \tanh(x) = 1 + \frac{e^x - e^{-x}}{e^x + e^{-x}}$
$= \frac{e^x + e^{-x} + e^x - e^{-x}}{e^x + e^{-x}} = \frac{2e^x}{e^x + e^{-x}}$
$= \frac{e^x}{e^x}\left(\frac{2}{1 + e^{-x}e^{-x}}\right) = \frac{2}{1 + e^{-2x}}$,
which is a logistic function of $c = 2, a = 1$, and $k = 2$.

## ■ Section 3.3 Logarithmic Functions and Their Graphs

### Exploration 1

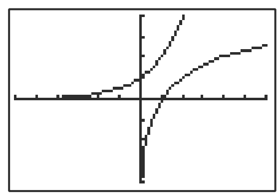
**1.**

| T | X1T | Y1T |
|---|---|---|
| -3 | -3 | .125 |
| -2 | -2 | .25 |
| -1 | -1 | .5 |
| 0 | 0 | 1 |
| 1 | 1 | 2 |
| 2 | 2 | 4 |
| 3 | 3 | 8 |

T= -3

| T | X2T | Y2T |
|---|---|---|
| -3 | .125 | -3 |
| -2 | .25 | -2 |
| -1 | .5 | -1 |
| 0 | 1 | 0 |
| 1 | 2 | 1 |
| 2 | 4 | 2 |
| 3 | 8 | 3 |

X2T =.125



[−6, 6] by [−4, 4]

**2.** Same graph as part 1.

### Quick Review 3.3

**1.** $\frac{1}{25} = 0.04$

**2.** $\frac{1}{1000} = 0.001$

**3.** $\frac{1}{5} = 0.2$

**4.** $\frac{1}{2} = 0.5$

**5.** $\frac{2^{33}}{2^{28}} = 2^5 = 32$

**6.** $\frac{3^{26}}{3^{24}} = 3^2 = 9$

**7.** $5^{1/2}$

**8.** $10^{1/3}$

**9.** $\left(\frac{1}{e}\right)^{1/2} = e^{-1/2}$

**10.** $\left(\frac{1}{e^2}\right)^{1/3} = e^{-2/3}$

### Section 3.3 Exercises

**1.** $\log_4 4 = 1$ because $4^1 = 4$

**2.** $\log_6 1 = 0$ because $6^0 = 1$

**3.** $\log_2 32 = 5$ because $2^5 = 32$

**4.** $\log_3 81 = 4$ because $3^4 = 81$

**5.** $\log_5 \sqrt[3]{25} = \frac{2}{3}$ because $5^{2/3} = \sqrt[3]{25}$

**6.** $\log_6 \frac{1}{\sqrt[5]{36}} = -\frac{2}{5}$ because $6^{-2/5} = \frac{1}{6^{2/5}} = \frac{1}{\sqrt[5]{36}}$

**7.** $\log 10^3 = 3$

**8.** $\log 10{,}000 = \log 10^4 = 4$

**9.** $\log 100{,}000 = \log 10^5 = 5$

**10.** $\log 10^{-4} = -4$

**11.** $\log \sqrt[3]{10} = \log 10^{1/3} = \dfrac{1}{3}$

**12.** $\log \dfrac{1}{\sqrt{1000}} = \log 10^{-3/2} = -\dfrac{3}{2}$

**13.** $\ln e^3 = 3$

**14.** $\ln e^{-4} = -4$

**15.** $\ln \dfrac{1}{e} = \ln e^{-1} = -1$

**16.** $\ln 1 = \ln e^0 = 0$

**17.** $\ln \sqrt[4]{e} = \ln e^{1/4} = \dfrac{1}{4}$

**18.** $\ln \dfrac{1}{\sqrt{e^7}} = \ln e^{-7/2} = -\dfrac{7}{2}$

**19.** 3, because $b^{\log_b 3} = 3$ for any $b > 0$

**20.** 8, because $b^{\log_b 8} = 8$ for any $b > 0$

**21.** $10^{\log(0.5)} = 10^{\log_{10}(0.5)} = 0.5$

**22.** $10^{\log 14} = 10^{\log_{10} 14} = 14$

**23.** $e^{\ln 6} = e^{\log_e 6} = 6$

**24.** $e^{\ln(1/5)} = e^{\log_e(1/5)} = 1/5$

**25.** $\log 9.43 \approx 0.9745 \approx 0.975$ and $10^{0.9745} \approx 9.43$

**26.** $\log 0.908 \approx -0.042$ and $10^{-0.042} \approx 0.908$

**27.** $\log(-14)$ is undefined because $-14 < 0$.

**28.** $\log(-5.14)$ is undefined because $-5.14 < 0$.

**29.** $\ln 4.05 \approx 1.399$ and $e^{1.399} \approx 4.05$

**30.** $\ln 0.733 \approx -0.311$ and $e^{-0.311} \approx 0.733$

**31.** $\ln(-0.49)$ is undefined because $-0.49 < 0$.

**32.** $\ln(-3.3)$ is undefined because $-3.3 < 0$.

**33.** $x = 10^2 = 100$

**34.** $x = 10^4 = 10{,}000$

**35.** $x = 10^{-1} = \dfrac{1}{10} = 0.1$

**36.** $x = 10^{-3} = \dfrac{1}{1000} = 0.001$

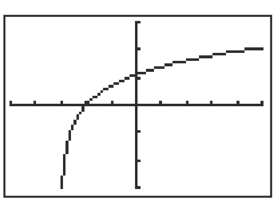
**37.** $f(x)$ is undefined for $x > 1$. The answer is (d).

**38.** $f(x)$ is undefined for $x < -1$. The answer is (b).

**39.** $f(x)$ is undefined for $x < 3$. The answer is (a).

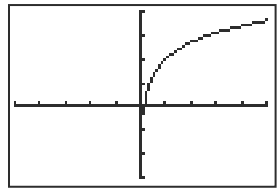
**40.** $f(x)$ is undefined for $x > 4$. The answer is (c).

**41.** Starting from $y = \ln x$: translate left 3 units.



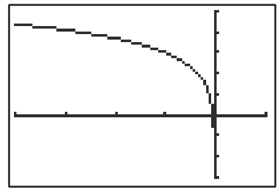
[–5, 5] by [–3, 3]

**42.** Starting from $y = \ln x$: translate up 2 units.



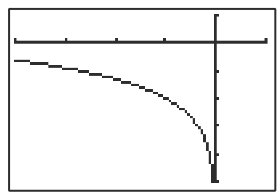
[–5, 5] by [–3, 4]

**43.** Starting from $y = \ln x$: reflect across the $y$-axis and translate up 3 units.



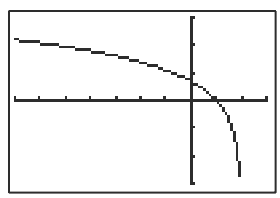
[−4, 1] by [−3, 5]

**44.** Starting from $y = \ln x$: reflect across the $y$-axis and translate down 2 units.



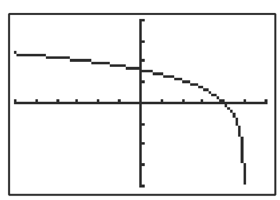
[−4, 1] by [−5, 1]

**45.** Starting from $y = \ln x$: reflect across the $y$-axis and translate right 2 units.



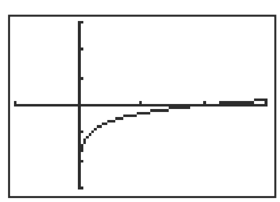
[−7, 3] by [−3, 3]

**46.** Starting from $y = \ln x$: reflect across the $y$-axis and translate right 5 units.



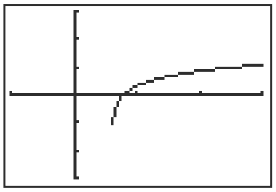
[−6, 6] by [−4, 4]

**47.** Starting from $y = \log x$: translate down 1 unit.



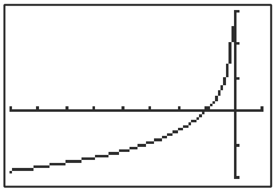
[–5, 15] by [–3, 3]

**48.** Starting from $y = \log x$: translate right 3 units.



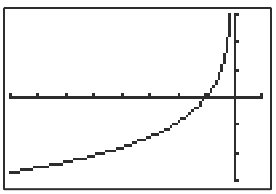
[–5, 15] by [–3, 3]

**49.** Starting from $y = \log x$: reflect across both axes and vertically stretch by 2.



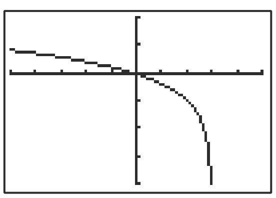
[−8, 1] by [−2, 3]

**50.** Starting from $y = \log x$: reflect across both axes and vertically stretch by 3.



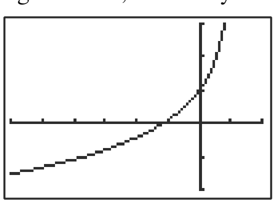
[−8, 7] by [−3, 3]

**51.** Starting from $y = \log x$: reflect across the $y$-axis, translate right 3 units, vertically stretch by 2, translate down 1 unit.



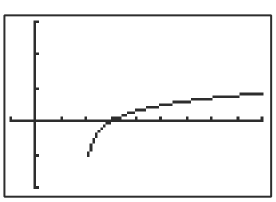
[−5, 5] by [−4, 2]

**52.** Starting from $y = \log x$: reflect across both axes, translate right 1 unit, vertically stretch by 3, translate up 1 unit.



[−6, 2] by [−2, 3]

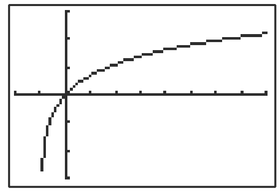
**53.**



[−1, 9] by [−3, 3]

Domain: $(2, \infty)$
Range: $(-\infty, \infty)$
Continuous
Always increasing
Not symmetric
Not bounded
No local extrema
Asymptote at $x = 2$
$\lim_{x\to\infty} f(x) = \infty$

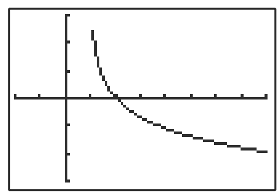
**54.**



[−2, 8] by [−3, 3]

Domain: $(-1, \infty)$
Range: $(-\infty, \infty)$
Continuous
Always increasing
Not symmetric
Not bounded
No local extrema
Asymptote: $x = -1$
$\lim_{x\to\infty} f(x) = \infty$

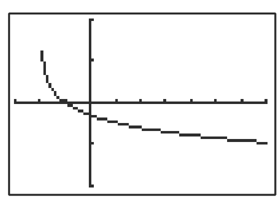
**55.**



[−2, 8] by [−3, 3]

Domain: $(1, \infty)$
Range: $(-\infty, \infty)$
Continuous
Always decreasing
Not symmetric
Not bounded
No local extrema
Asymptote: $x = 1$
$\lim_{x\to\infty} f(x) = -\infty$

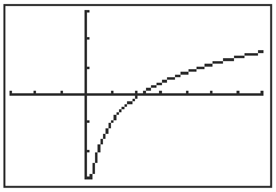
**56.**



[−3, 7] by [−2, 2]

Domain: $(-2, \infty)$
Range: $(-\infty, \infty)$
Continuous
Always decreasing
Not symmetric
Not bounded
No local extrema
Asymptote: $x = -2$
$\lim_{x\to\infty} f(x) = -\infty$

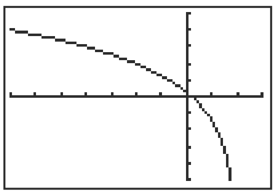
**57.**



[–3, 7] by [–3, 3]

Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Continuous
Increasing on its domain
No symmetry
Not bounded
No local extrema
Asymptote at $x = 0$
$\lim_{x \to \infty} f(x) = \infty$

**58.**



[–7, 3, 1] by [–10, 10, 2]

Domain: $(-\infty, 2)$
Range: $(-\infty, \infty)$
Continuous
Decreasing on its domain
No symmetry
Not bounded
No local extrema
Asymptote at $x = 2$
$\lim_{x \to -\infty} f(x) = \infty$

**59.** **(a)** $\beta = 10 \log\left(\frac{10^{-11}}{10^{-12}}\right) = 10 \log 10 = 10(1) = 10 \text{ dB}$

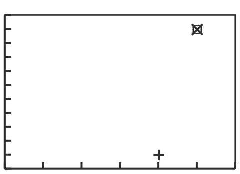
**(b)** $\beta = 10 \log\left(\frac{10^{-5}}{10^{-12}}\right) = 10 \log 10^7 = 10(7) = 70 \text{ dB}$

**(c)** $\beta = 10 \log\left(\frac{10^{3}}{10^{-12}}\right) = 10 \log 10^{15} = 10(15) = 150 \text{ dB}$

**60.** $I = 12 \cdot 10^{-0.0705} \approx 10.2019$ lumens.

**61.** **(a)** A magnitude 3 earthquake is $\frac{1000}{100} = 10$ times more powerful than a magnitude 2 earthquake. A magnitude 5 earthquake is $\frac{100{,}000}{1000} = 100$ times more powerful than a magnitude 2 earthquake.

**(b)**



[0, 6] by [0, 110000]

**(c)** Ground motion $= 10^x$ where $x$ is the magnitude of the earthquake, so $y = 10^x$.

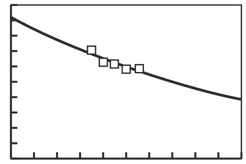
**(d)** $y = 10^x$, so $x = \log y$.

**(e)** Extremely large values can be represented by much smaller values.

**(f)** Yes

**62.** **(a)** The exponential regression model is $2552165025 \cdot 0.995838^x$, where $x$ is the year and $y$ is the population.

**(b)**



[1900, 2100] by [100000, 1000000]

**(c)** Solving graphically, we find that the curve $y = 2552165025 \cdot 0.995838^x$ intersects the line $y = 500{,}000$ at $t = 2047$.

**(d)** Not in most cases as populations will not continue to grow without bound.

**63.** True, by the definition of a logarithmic function.

**64.** True, by the definition of common logarithm.

**65.** $\log 2 \approx 0.30103$. The answer is C.

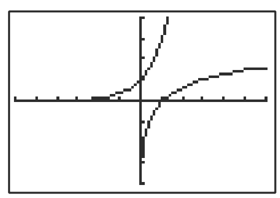
**66.** $\log 5 \approx 0.699$ but $2.5 \log 2 \approx 0.753$. The answer is A.

**67.** The graph of $f(x) = \ln x$ lies entirely to the right of the origin. The answer is B.

**68.** For $f(x) = 2 \cdot 3^x$, $f^{-1}(x) = \log_3(x/2)$
because $f^{-1}(f(x)) = \log_3(2 \cdot 3^x/2)$
$= \log_3 3^x$
$= x.$
The answer is A.

**69.**

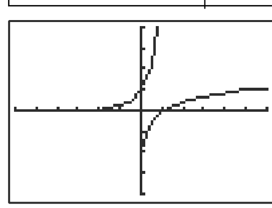
| $f(x)$ | $3^x$ | $\log_3 x$ |
|---|---|---|
| Domain | $(-\infty, \infty)$ | $(0, \infty)$ |
| Range | $(0, \infty)$ | $(-\infty, \infty)$ |
| Intercepts | (0, 1) | (1, 0) |
| Asymptotes | $y = 0$ | $x = 0$ |



$[-6, 6]$ by $[-4, 4]$

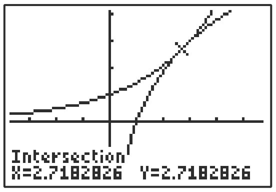
**70.**

| $f(x)$ | $5^x$ | $\log_5 x$ |
|---|---|---|
| Domain | $(-\infty, \infty)$ | $(0, \infty)$ |
| Range | $(0, \infty)$ | $(-\infty, \infty)$ |
| Intercepts | (0, 1) | (1, 0) |
| Asymptotes | $y = 0$ | $x = 0$ |



$[-6, 6]$ by $[-4, 4]$

**71.** $b = \sqrt[e]{e}$. The point that is common to both graphs is $(e, e)$.




$[-3.7, 5.7]$ by $[-2.1, 4.1]$

**72.** 0 is not in the domain of the logarithm functions because 0 is not in the range of exponential functions; that is, $a^x$ is never equal to 0.

**73.** Reflect across the $x$-axis.

**74.** Reflect across the $x$-axis.

## ■ Section 3.4 Properties of Logarithmic Functions

### Exploration 1

**1.** $\log(2 \cdot 4) \approx 0.90309$,
$\log 2 + \log 4 \approx 0.30103 + 0.60206 \approx 0.90309$

**2.** $\log\left(\frac{8}{2}\right) \approx 0.60206, \log 8 - \log 2 \approx 0.90309 - 0.30103$
$\approx 0.60206$

**3.** $\log 2^3 \approx 0.90309, 3 \log 2 \approx 3(0.30103) \approx 0.90309$

**4.** $\log 5 = \log\left(\frac{10}{2}\right) = \log 10 - \log 2 \approx 1 - 0.30103$
$= 0.69897$

**5.** $\log 16 = \log 2^4 = 4 \log 2 \approx 1.20412$
$\log 32 = \log 2^5 = 5 \log 2 \approx 1.50515$
$\log 64 = \log 2^6 = 6 \log 2 \approx 1.80618$

**6.** $\log 25 = \log 5^2 = 2 \log 5 = 2 \log\left(\frac{10}{2}\right)$
$= 2(\log 10 - \log 2) \approx 1.39794$
$\log 40 = \log(4 \cdot 10) = \log 4 + \log 10 \approx 1.60206$
$\log 50 = \log\left(\frac{100}{2}\right) = \log 100 - \log 2 \approx 1.69897$

The list consists of 1, 2, 4, 5, 8, 16, 20, 25, 32, 40, 50, 64, and 80.

### Exploration 2

**1.** False

**2.** False; $\log_3(7x) = \log_3 7 + \log_3 x$

**3.** True

**4.** True

**5.** False; $\log \frac{x}{4} = \log x - \log 4$

**6.** True

**7.** False; $\log_5 x^2 = \log_5 x + \log_5 x = 2 \log_5 x$

**8.** True

### Quick Review 3.4

**1.** $\log 10^2 = 2$

**2.** $\ln e^3 = 3$

**3.** $\ln e^{-2} = -2$

**4.** $\log 10^{-3} = -3$

**5.** $\frac{x^5 y^{-2}}{x^2 y^{-4}} = x^{5-2} y^{-2-(-4)} = x^3 y^2$

**6.** $\frac{u^{-3} v^7}{u^{-2} v^2} = \frac{v^{7-2}}{u^{-2-(-3)}} = \frac{v^5}{u}$

**7.** $(x^6 y^{-2})^{1/2} = (x^6)^{1/2} (y^{-2})^{1/2} = \frac{|x|^3}{|y|}$

**8.** $(x^{-8} y^{12})^{3/4} = (x^{-8})^{3/4} (y^{12})^{3/4} = \frac{|y|^9}{x^6}$

**9.** $\frac{(u^2 v^{-4})^{1/2}}{(27 u^6 v^{-6})^{1/3}} = \frac{|u||v|^{-2}}{3u^2 v^{-2}} = \frac{1}{3|u|}$

**10.** $\frac{(x^{-2} y^3)^{-2}}{(x^3 y^{-2})^{-3}} = \frac{x^4 y^{-6}}{x^{-9} y^6} = \frac{x^{13}}{y^{12}}$

### Section 3.4 Exercises

**1.** $\ln 8x = \ln 8 + \ln x = 3 \ln 2 + \ln x$

**2.** $\ln 9y = \ln 9 + \ln y = 2 \ln 3 + \ln y$

**3.** $\log \frac{3}{x} = \log 3 - \log x$

**4.** $\log \frac{2}{y} = \log 2 - \log y$

**5.** $\log_2 y^5 = 5 \log_2 y$

**6.** $\log_2 x^{-2} = -2 \log_2 x$

**7.** $\log x^3 y^2 = \log x^3 + \log y^2 = 3 \log x + 2 \log y$

**8.** $\log xy^3 = \log x + \log y^3 = \log x + 3 \log y$

**9.** $\ln \frac{x^2}{y^3} = \ln x^2 - \ln y^3 = 2 \ln x - 3 \ln y$

**10.** $\log 1000x^4 = \log 1000 + \log x^4 = 3 + 4 \log x$

**11.** $\log \sqrt[4]{\frac{x}{y}} = \frac{1}{4}(\log x - \log y) = \frac{1}{4} \log x - \frac{1}{4} \log y$

**12.** $\ln \frac{\sqrt[3]{x}}{\sqrt[3]{y}} = \frac{1}{3}(\ln x - \ln y) = \frac{1}{3} \ln x - \frac{1}{3} \ln y$

**13.** $\log x + \log y = \log xy$

**14.** $\log x + \log 5 = \log 5x$

**15.** $\ln y - \ln 3 = \ln(y/3)$

**16.** $\ln x - \ln y = \ln(x/y)$

**17.** $\frac{1}{3} \log x = \log x^{1/3} = \log \sqrt[3]{x}$

**18.** $\frac{1}{5} \log z = \log z^{1/5} = \log \sqrt[5]{z}$

**19.** $2 \ln x + 3 \ln y = \ln x^2 + \ln y^3 = \ln(x^2 y^3)$

**20.** $4 \log y - \log z = \log y^4 - \log z = \log\left(\frac{y^4}{z}\right)$

**21.** $4 \log(xy) - 3 \log(yz) = \log(x^4 y^4) - \log(y^3 z^3)$
$= \log\left(\frac{x^4 y^4}{y^3 z^3}\right) = \log\left(\frac{x^4 y}{z^3}\right)$

**22.** $3 \ln(x^3 y) + 2 \ln(yz^2) = \ln(x^9 y^3) + \ln(y^2 z^4)$
$= \ln(x^9 y^5 z^4)$

In #23–28, natural logarithms are shown, but common (base-10) logarithms would produce the same results.

**23.** $\dfrac{\ln 7}{\ln 2} \approx 2.8074$

**24.** $\dfrac{\ln 19}{\ln 5} \approx 1.8295$

**25.** $\dfrac{\ln 175}{\ln 8} \approx 2.4837$

**26.** $\dfrac{\ln 259}{\ln 12} \approx 2.2362$

**27.** $\dfrac{\ln 12}{\ln 0.5} = -\dfrac{\ln 12}{\ln 2} \approx -3.5850$

**28.** $\dfrac{\ln 29}{\ln 0.2} = -\dfrac{\ln 29}{\ln 5} \approx -2.0922$

**29.** $\log_3 x = \dfrac{\ln x}{\ln 3}$

**30.** $\log_7 x = \dfrac{\ln x}{\ln 7}$

**31.** $\log_2(a + b) = \dfrac{\ln(a + b)}{\ln 2}$

**32.** $\log_5(c - d) = \dfrac{\ln(c - d)}{\ln 5}$

**33.** $\log_2 x = \dfrac{\log x}{\log 2}$

**34.** $\log_4 x = \dfrac{\log x}{\log 4}$

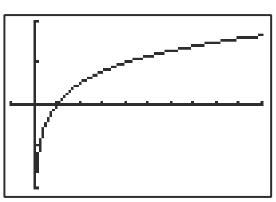
**35.** $\log_{1/2}(x + y) = \dfrac{\log(x + y)}{\log (1/2)} = -\dfrac{\log(x + y)}{\log 2}$

**36.** $\log_{1/3}(x - y) = \dfrac{\log(x - y)}{\log(1/3)} = -\dfrac{\log(x - y)}{\log 3}$

**37.** Let $x = \log_b R$ and $y = \log_b S$.
Then $b^x = R$ and $b^y = S$, so that
$\dfrac{R}{S} = \dfrac{b^x}{b^y} = b^{x-y}$
$\log_b\left(\dfrac{R}{S}\right) = \log_b b^{x-y} = x - y = \log_b R - \log_b S.$

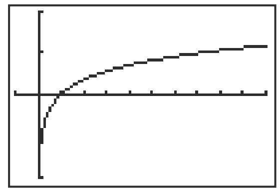
**38.** Let $x = \log_b R$. Then $b^x = R$, so that
$R^c = (b^x)^c = b^{c \cdot x}$
$\log_b R^c = \log_b b^{c \cdot x} = c \cdot x = c \log_b R.$

**39.** Starting from $g(x) = \ln x$: vertically shrink by a factor $1/\ln 4 \approx 0.72$.



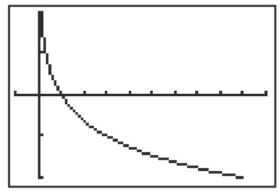
[−1, 10] by [−2, 2]

**40.** Starting from $g(x) = \ln x$: vertically shrink by a factor $1/\ln 7 \approx 0.51$.



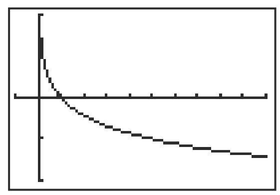
[–1, 10] by [–2, 2]

**41.** Starting from $g(x) = \ln x$: reflect across the $x$-axis, then vertically shrink by a factor $1/\ln 3 \approx 0.91$.



[–1, 10] by [–2, 2]

**42.** Starting from $g(x) = \ln x$: reflect across the $x$-axis, then shrink vertically by a factor of $1/\ln 5 \approx 0.62$.



[–1, 10] by [–2, 2]

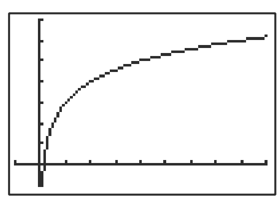
**43.** (b): $[-5, 5]$ by $[-3, 3]$, with Xscl = 1 and Yscl = 1 (graph $y = \ln(2 - x)/\ln 4$).

**44.** (c): $[-2, 8]$ by $[-3, 3]$, with Xscl = 1 and Yscl = 1 (graph $y = \ln(x - 3)/\ln 6$).

**45.** (d): $[-2, 8]$ by $[-3, 3]$, with Xscl = 1 and Yscl = 1 (graph $y = \ln(x - 2)/\ln 0.5$).

**46.** (a): $[-8, 4]$ by $[-8, 8]$, with Xscl = 1 and Yscl = 1 (graph $y = \ln(3 - x)/\ln 0.7$).

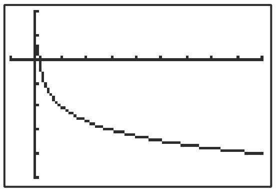
**47.**



[−1, 9] by [−1, 7]

Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Continuous
Always increasing
Asymptote: $x = 0$
$\lim_{x \to \infty} f(x) = \infty$
$f(x) = \log_2 (8x) = \dfrac{\ln (8x)}{\ln (2)}$

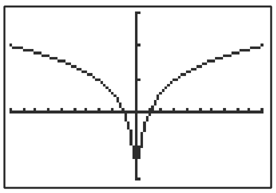
**48.**



$[-1, 9]$ by $[-5, 2]$

Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Continuous
Always decreasing
Asymptote: $x = 0$
$\lim_{x \to \infty} f(x) = -\infty$
$f(x) = \log_{1/3}(9x) = \dfrac{\ln(9x)}{\ln\left(\frac{1}{3}\right)}$

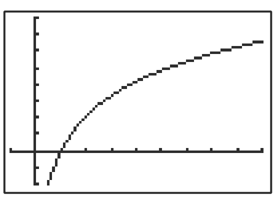
**49.**



$[-10, 10]$ by $[-2, 3]$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, \infty)$
Discontinuous at $x = 0$
Decreasing on interval $(-\infty, 0)$; increasing on interval $(0, \infty)$
Asymptote: $x = 0$
$\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = \infty$,

**50.**



$[-1, 9]$ by $[-2, 8]$

Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Continuous
Always increasing
Asymptote: $x = 0$
$\lim_{x \to \infty} f(x) = \infty$

**51.** In each case, take the exponent of 10, add 12, and multiply the result by 10.

**(a)** 0 **(b)** 10
**(c)** 60 **(d)** 80
**(e)** 100 **(f)** 120 $(1 = 10^0)$

**52.** **(a)** $R = \log \dfrac{250}{2} + 4.25 = \log 125 + 4.25 \approx 6.3469$

**(b)** $R = \log \dfrac{300}{4} + 3.5 = \log 75 + 3.5 \approx 5.3751$

**53.** $\log \dfrac{I}{12} = -0.00235(40) = -0.094$, so
$I = 12 \cdot 10^{-0.094} \approx 9.6645$ lumens.

**54.** $\log \dfrac{I}{12} = -0.0125(10) = -0.125$, so
$I = 12 \cdot 10^{-0.125} \approx 8.9987$ lumens.

**55.** From the change-of-base formula, we know that
$f(x) = \log_3 x = \dfrac{\ln x}{\ln 3} = \dfrac{1}{\ln 3} \cdot \ln x \approx 0.9102 \ln x$.
$f(x)$ can be obtained from $g(x) = \ln x$ by vertically stretching by a factor of approximately 0.9102.

**56.** From the change-of-base formula, we know that
$f(x) = \log_{0.8} x = \dfrac{\log x}{\log 0.8} = \dfrac{1}{\log 0.8} \cdot \log x \approx -10.32 \log x$.
$f(x)$ can be obtained from $g(x) = \log x$ by reflecting across the $x$-axis and vertically stretching by a factor of approximately 10.32.

**57.** True. This is the product rule for logarithms.

**58.** False. The logarithm of a positive number less than 1 is negative. For example, $\log 0.01 = -2$.

**59.** $\log 12 = \log(3 \cdot 4) = \log 3 + \log 4$ by the product rule. The answer is B.

**60.** $\log_9 64 = (\ln 64)/(\ln 9)$ by the change-of-base formula. The answer is C.

**61.** $\ln x^5 = 5 \ln x$ by the power rule. The answer is A.

**62.** $\log_{1/2} x^2 = 2 \log_{1/2} |x|$

$$= 2\frac{\ln|x|}{\ln(1/2)}$$
$$= 2\frac{\ln|x|}{\ln 1 - \ln 2}$$
$$= -2\frac{\ln|x|}{\ln 2}$$
$$= -2\log_2|x|.$$

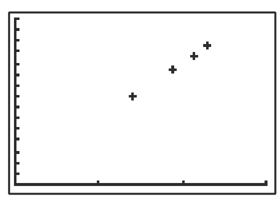
The answer is E.

**63.** **(a)** $f(x) = 2.75 \cdot x^{5.0}$

**(b)** $f(7.1) \approx 49{,}616$

**(c)**

| $\ln(x)$ | 1.39 | 1.87 | 2.14 | 2.30 |
|---|---|---|---|---|
| $\ln(y)$ | 7.94 | 10.37 | 11.71 | 12.53 |



$[0, 3]$ by $[0, 15]$

**(d)** $\ln(y) = 5.00 \ln x + 1.01$

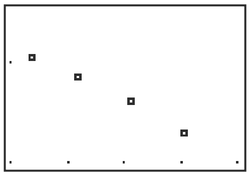
**(e)** $a \approx 5, b \approx 1$ so $f(x) = e^1 x^5 = ex^5 \approx 2.72x^5$. The two equations are the same.

**64.** **(a)** $f(x) = 8.095 \cdot x^{-0.113}$

**(b)** $f(9.2) = 8.095 \cdot (9.2)^{-0.113} \approx 6.30$

**(c)**

| $\ln(x)$ | 0.69 | 1.10 | 1.57 | 2.04 |
|---|---|---|---|---|
| $\ln(y)$ | 2.01 | 1.97 | 1.92 | 1.86 |



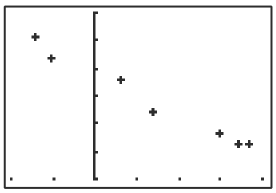
[0.5, 2.5] by [1.8, 2.1]

**(d)** $\ln(y) = -0.113 \ln(x) + 2.091$

**(e)** $a \approx -0.113, b \approx 2.1$, so $f(x) = e^{2.1} \cdot x^{-0.113}$ $\approx 8.09x^{-0.113}$.

**65. (a)**

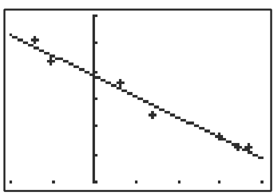
| $\log(w)$ | −0.70 | −0.52 | 0.30 | 0.70 | 1.48 | 1.70 | 1.85 |
|---|---|---|---|---|---|---|---|
| $\log(r)$ | 2.62 | 2.48 | 2.31 | 2.08 | 1.93 | 1.85 | 1.86 |



$[-1, 2]$ by $[1.6, 2.8]$

**(b)** $\log r = (-0.30) \log w + 2.36$

**(c)**



$[-1, 2]$ by $[1.6, 2.8]$

**(d)** $\log r = (-0.30) \log (450) + 2.36 \approx 1.58, r \approx 37.69$, very close.

**(e)** One possible answer: Consider the power function $y = a \cdot x^b$. Then:

$$\begin{aligned}\log y &= \log (a \cdot x^b)\\ &= \log a + \log x^b\\ &= \log a + b \log x\\ &= b(\log x) + \log a\end{aligned}$$

which is clearly a linear function of the form $f(t) = mt + c$ where $m = b, c = \log a, f(t) = \log y$ and $t = \log x$. As a result, there is a linear relationship between $\log y$ and $\log x$.

**66.** $\log 4 = 2\log 2$

$\log 5 = \log(10/2) = \log 10 - \log 2 = 1 - \log 2$

$\log 6 = \log 2 + \log 3$

$\log 8 = \log(2^3) = 3\log 2$

$\log 9 = \log(3^2) = 2\log 3$

$\log 12 = \log 3 + \log 4 = \log 3 + 2\log 2$

$\log 15 = \log\left(\frac{10 \times 3}{2}\right) = \log 10 - \log 2 + \log 3$

$= 1 - \log 2 + \log 3$

$\log 16 = \log(2^4) = 4\log 2$

$\log 18 = \log 2 + \log 9 = \log 2 + 2\log 3$

$\log 20 = \log(10 \times 2) = \log 10 + \log 2 = 1 + \log 2$

$\log 24 = \log(3 \times 8) = 3\log 2 + \log 3$

$\log 25 = \log\left(\frac{100}{4}\right) = \log 100 - \log 4 = 2 - 2\log 2$

$\log 27 = \log(3^3) = 3\log 3$

$\log 30 = \log(10 \times 3) = \log 10 + \log 3 = 1 + \log 3$

$\log 32 = \log(2^5) = 5\log 2$

$\log 36 = \log 4 + \log 9 = 2\log 2 + 2\log 3$

$\log 40 = \log(10 \times 4) = \log 10 + \log 4 = 1 + 2\log 2$

$\log 45 = \log 5 + \log 9 = 1 - \log 2 + 2\log 3$

$\log 48 = \log 3 + \log 16 = \log 3 + 4\log 2$

$\log 50 = \log\left(\frac{100}{2}\right) = \log 100 - \log 2 = 2 - \log 2$

$\log 54 = \log 2 + \log 27 = \log 2 + 3\log 3$

$\log 60 = \log(10 \times 6) = \log 10 + \log 6 = 1 + \log 2 + \log 3$

$\log 64 = \log(2^6) = 6\log 2$

$\log 72 = \log 8 + \log 9 = 3\log 2 + 2\log 3$

$\log 75 = \log\left(\frac{100 \times 3}{4}\right) = 2 - 2\log 2 + \log 3$

$\log 80 = \log(10 \times 8) = \log 10 + \log 8 = 1 + 3\log 2$

$\log 81 = \log(3^4) = 4\log 3$

$\log 90 = \log(10 \times 9) = \log 10 + \log 9 = 1 + 2\log 3$

$\log 96 = \log 3 + \log 32 = \log 3 + 5\log 2$

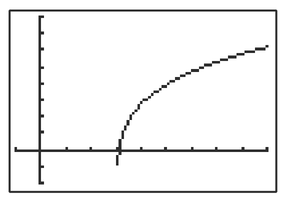
**67.** Since $\sqrt[5]{4581} = 4581^{1/5}$ and $\log (4581^{1/5}) = \frac{1}{5} \log 4581$, we need to find $\log 4581 \approx 3.66096$ (using tables, before calculators were available) and then find $\frac{3.66096}{5} \approx 0.732192$, so $\sqrt[5]{4581} \approx 10^{0.732192} \approx 5.40$.

For #68 and 69, solve graphically.

**68.** $\approx 6.41 < x < 93.35$

**69.** $\approx 1.26 \le x \le 14.77$

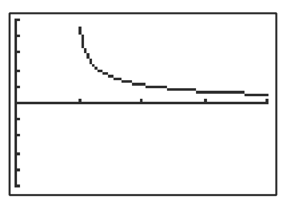
**70. (a)**



$[-1, 9]$ by $[-2, 8]$

Domain of $f$ and $g$: $(3, \infty)$

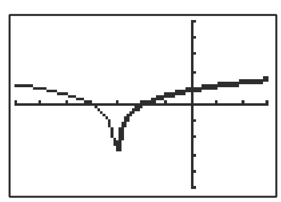
**(b)**



$[0, 20]$ by $[-2, 8]$

Domain of $f$ and $g$: $(5, \infty)$

**(c)**



$[-7, 3]$ by $[-5, 5]$

Domain of $f$: $(-\infty, -3) \cup (-3, \infty)$
Domain of $g$: $(-3, \infty)$
Answers will vary.

**71.** Recall that $y = \log_a x$ can be written as $x = a^y$.

Let $y = \log_a b$

$a^y = b$

$\log a^y = \log b$

$y \log a = \log b$

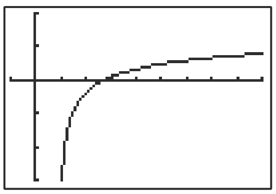
$y = \dfrac{\log b}{\log a} = \log_a b.$

**72.** Let $y = \dfrac{\log x}{\ln x}$. By the change-of-base formula,

$$y = \frac{\log x}{\dfrac{\log x}{\log e}} = \log x \cdot \frac{\log e}{\log x} = \log e \approx 0.43.$$

Thus, $y$ is a constant function.

**73.**



$[-1, 9]$ by $[-3, 2]$

Domain: $(1, \infty)$
Range: $(-\infty, \infty)$
Continuous
Increasing
Not symmetric
Vertical asymptote: $x = 1$
$\lim_{x\to\infty} f(x) = \infty$
One-to one, hence invertible $(f^{-1}(x) = e^{e^x})$

## ■ Section 3.5 Equation Solving and Modeling

### Exploration 1

1. $\log(4 \cdot 10) \approx 1.60206$
   $\log(4 \cdot 10^2) \approx 2.60206$
   $\log(4 \cdot 10^3) \approx 3.60206$
   $\log(4 \cdot 10^4) \approx 4.60206$
   $\log(4 \cdot 10^5) \approx 5.60206$
   $\log(4 \cdot 10^6) \approx 6.60206$
   $\log(4 \cdot 10^7) \approx 7.60206$
   $\log(4 \cdot 10^8) \approx 8.60206$
   $\log(4 \cdot 10^9) \approx 9.60206$
   $\log(4 \cdot 10^{10}) \approx 10.60206$
2. The integers increase by 1 for every increase in a power of 10.
3. The decimal parts are exactly equal.
4. $4 \cdot 10^{10}$ is nine orders of magnitude greater than $4 \cdot 10$.

### Quick Review 3.5

In #1–4, graphical support (i.e., graphing both functions on a square window) is also useful.

1. $f(g(x)) = e^{2\ln(x^{1/2})} = e^{\ln x} = x$ and $g(f(x)) = \ln(e^{2x})^{1/2} = \ln(e^x) = x.$
2. $f(g(x)) = 10^{(\log x^2)/2} = 10^{\log x} = x$ and $g(f(x)) = \log(10^{x/2})^2 = \log(10^x) = x.$
3. $f(g(x)) = \frac{1}{3}\ln(e^{3x}) = \frac{1}{3}(3x) = x$ and $g(f(x)) = e^{3(1/3 \ln x)} = e^{\ln x} = x.$
4. $f(g(x)) = 3\log(10^{x/6})^2 = 6\log(10^{x/6}) = 6(x/6) = x$ and $g(f(x)) = 10^{(3\log x^2)/6} = 10^{(6\log x)/6} = 10^{\log x} = x.$
5. $7.783 \times 10^8$ km
6. $1 \times 10^{-15}$ m
7. 602,000,000,000,000,000,000,000
8. 0.000 000 000 000 000 000 000 000 001 66 (26 zeros between the decimal point and the 1).
9. $(1.86 \times 10^5)(3.1 \times 10^7) = (1.86)(3.1) \times 10^{5+7} = 5.766 \times 10^{12}$
10. $\dfrac{8 \times 10^{-7}}{5 \times 10^{-6}} = \dfrac{8}{5} \times 10^{-7-(-6)} = 1.6 \times 10^{-1}$

### Section 3.5 Exercises

For #1–18, take a logarithm of both sides of the equation, when appropriate.

1. $36\left(\frac{1}{3}\right)^{x/5} = 4$
   $\left(\frac{1}{3}\right)^{x/5} = \frac{1}{9}$
   $\left(\frac{1}{3}\right)^{x/5} = \left(\frac{1}{3}\right)^2$
   $\frac{x}{5} = 2$
   $x = 10$
2. $32\left(\frac{1}{4}\right)^{x/3} = 2$
   $\left(\frac{1}{4}\right)^{x/3} = \frac{1}{16}$
   $\left(\frac{1}{4}\right)^{x/3} = \left(\frac{1}{4}\right)^2$
   $\frac{x}{3} = 2$
   $x = 6$
3. $2 \cdot 5^{x/4} = 250$
   $5^{x/4} = 125$
   $5^{x/4} = 5^3$
   $\frac{x}{4} = 3$
   $x = 12$
4. $3 \cdot 4^{x/2} = 96$
   $4^{x/2} = 32$
   $4^{x/2} = 4^{5/2}$
   $\frac{x}{2} = \frac{5}{2}$
   $x = 5$
5. $10^{-x/3} = 10$, so $-x/3 = 1$, and therefore $x = -3$.
6. $5^{-x/4} = 5$, so $-x/4 = 1$, and therefore $x = -4$.
7. $x = 10^4 = 10{,}000$
8. $x = 2^5 = 32$
9. $x - 5 = 4^{-1}$, so $x = 5 + 4^{-1} = 5.25$.

**10.** $1 - x = 4^1$, so $x = -3$.

**11.** $x = \dfrac{\ln 4.1}{\ln 1.06} = \log_{1.06} 4.1 \approx 24.2151$

**12.** $x = \dfrac{\ln 1.6}{\ln 0.98} = \log_{0.98} 1.6 \approx -23.2644$

**13.** $e^{0.035x} = 4$, so $0.035x = \ln 4$, and therefore $x = \dfrac{1}{0.035}\ln 4 \approx 39.6084$.

**14.** $e^{0.045x} = 3$, so $0.045x = \ln 3$, and therefore $x = \dfrac{1}{0.045}\ln 3 \approx 24.4136$.

**15.** $e^{-x} = \dfrac{3}{2}$, so $-x = \ln\dfrac{3}{2}$, and therefore $x = -\ln\dfrac{3}{2} \approx -0.4055$.

**16.** $e^{-x} = \dfrac{5}{3}$, so $-x = \ln\dfrac{5}{3}$, and therefore $x = -\ln\dfrac{5}{3} \approx -0.5108$.

**17.** $\ln(x - 3) = \dfrac{1}{3}$, so $x - 3 = e^{1/3}$, and therefore $x = 3 + e^{1/3} \approx 4.3956$.

**18.** $\log(x + 2) = -2$, so $x + 2 = 10^{-2}$, and therefore $x = -2 + 10^{-2} = -1.99$.

**19.** We must have $x(x + 1) > 0$, so $x < -1$ or $x > 0$. Domain: $(-\infty, -1) \cup (0, \infty)$; graph (e).

**20.** We must have $x > 0$ and $x + 1 > 0$, so $x > 0$. Domain: $(0, \infty)$; graph (f).

**21.** We must have $\dfrac{x}{x + 1} > 0$, so $x < -1$ or $x > 0$. Domain: $(-\infty, -1) \cup (0, \infty)$; graph (d).

**22.** We must have $x > 0$ and $x + 1 > 0$, so $x > 0$. Domain: $(0, \infty)$; graph (c).

**23.** We must have $x > 0$. Domain: $(0, \infty)$; graph (a).

**24.** We must have $x^2 > 0$, so $x \neq 0$. Domain: $(-\infty, 0) \cup (0, \infty)$; graph (b).

For #25–38, algebraic solutions are shown (and are generally the only way to get *exact* answers). In many cases solving graphically would be faster; graphical support is also useful.

**25.** Write both sides as powers of 10, leaving $10^{\log x^2} = 10^6$, or $x^2 = 1{,}000{,}000$. Then $x = 1000$ or $x = -1000$.

**26.** Write both sides as powers of $e$, leaving $e^{\ln x^2} = e^4$, or $x^2 = e^4$. Then $x = e^2 \approx 7.389$ or $x = -e^2 \approx -7.389$.

**27.** Write both sides as powers of 10, leaving $10^{\log x^4} = 10^2$, or $x^4 = 100$. Then $x^2 = 10$, and $x = \pm\sqrt{10}$.

**28.** Write both sides as powers of $e$, leaving $e^{\ln x^6} = e^{12}$, or $x^6 = e^{12}$. Then $x^2 = e^4$, and $x = \pm e^2$.

**29.** Multiply both sides by $3 \cdot 2^x$, leaving $(2^x)^2 - 1 = 12 \cdot 2^x$, or $(2^x)^2 - 12 \cdot 2^x - 1 = 0$. This is quadratic in $2^x$, leading to $2^x = \dfrac{12 \pm \sqrt{144 + 4}}{2} = 6 \pm \sqrt{37}$. Only $6 + \sqrt{37}$ is positive, so the only answer is $x = \dfrac{\ln(6 + \sqrt{37})}{\ln 2} = \log_2(6 + \sqrt{37}) \approx 3.5949$.

**30.** Multiply both sides by $2 \cdot 2^x$, leaving $(2^x)^2 + 1 = 6 \cdot 2^x$, or $(2^x)^2 - 6 \cdot 2^x + 1 = 0$. This is quadratic in $2^x$, leading to $2^x = \dfrac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}$. Then $x = \dfrac{\ln(3 \pm 2\sqrt{2})}{\ln 2} = \log_2(3 \pm 2\sqrt{2}) \approx \pm 2.5431$.

**31.** Multiply both sides by $2e^x$, leaving $(e^x)^2 + 1 = 8e^x$, or $(e^x)^2 - 8e^x + 1 = 0$. This is quadratic in $e^x$, leading to $e^x = \dfrac{8 \pm \sqrt{64 - 4}}{2} = 4 \pm \sqrt{15}$. Then $x = \ln(4 \pm \sqrt{15}) \approx \pm 2.0634$.

**32.** This is quadratic in $e^x$, leading to $e^x = \dfrac{-5 \pm \sqrt{25 + 24}}{4} = \dfrac{-5 \pm 7}{4}$. Of these two numbers, only $\dfrac{-5 + 7}{4} = \dfrac{1}{2}$ is positive, so $x = \ln\dfrac{1}{2} \approx -0.6931$.

**33.** $\dfrac{500}{200} = 1 + 25e^{0.3x}$, so $e^{0.3x} = \dfrac{3}{50} = 0.06$, and therefore $x = \dfrac{1}{0.3}\ln 0.06 \approx -9.3780$.

**34.** $\dfrac{400}{150} = 1 + 95e^{-0.6x}$, so $e^{-0.6x} = \dfrac{1}{57}$, and therefore $x = \dfrac{1}{-0.6}\ln\dfrac{1}{57} \approx 6.7384$.

**35.** Multiply by 2, then combine the logarithms to obtain $\ln\dfrac{x + 3}{x^2} = 0$. Then $\dfrac{x + 3}{x^2} = e^0 = 1$, so $x + 3 = x^2$. The solutions to this quadratic equation are $x = \dfrac{1 \pm \sqrt{1 + 12}}{2} = \dfrac{1}{2} \pm \dfrac{1}{2}\sqrt{13} \approx 2.3028$.

**36.** Multiply by 2, then combine the logarithms to obtain $\log\dfrac{x^2}{x + 4} = 2$. Then $\dfrac{x^2}{x + 4} = 10^2 = 100$, so $x^2 = 100(x + 4)$. The solutions to this quadratic equation are $x = \dfrac{100 \pm \sqrt{10000 + 1600}}{2} = 50 \pm 10\sqrt{29}$. The original equation requires that $x > 0$, so $50 - 10\sqrt{29}$ is extraneous; the only actual solution is $x = 50 + 10\sqrt{29} \approx 103.8517$.

**37.** $\ln[(x - 3)(x + 4)] = 3\ln 2$, so $(x - 3)(x + 4) = 8$, or $x^2 + x - 20 = 0$. This factors to $(x - 4)(x + 5) = 0$, so $x = 4$ (an actual solution) or $x = -5$ (extraneous, since $x - 3$ and $x + 4$ must be positive).

**38.** $\log[(x - 2)(x + 5)] = 2\log 3$, so $(x - 2)(x + 5) = 9$, or $x^2 + 3x - 19 = 0$. Then $x = \dfrac{-3 \pm \sqrt{9 + 76}}{2} = -\dfrac{3}{2} \pm \dfrac{1}{2}\sqrt{85}$. The actual solution is $x = -\dfrac{3}{2} + \dfrac{1}{2}\sqrt{85} \approx 3.1098$; since $x - 2$ must be positive, the other algebraic solution, $x = -\dfrac{3}{2} - \dfrac{1}{2}\sqrt{85}$, is extraneous.

**39.** A \$100 bill has the value of 1000, or $10^3$, dimes so they differ by an order of magnitude of 3.

**40.** A 2-kg hen weighs 2000, or $2 \cdot 10^3$, grams while a 20-g canary weighs $2 \cdot 10$ grams. They differ by an order of magnitude of 2.

**41.** $7 - 5.5 = 1.5$. They differ by an order of magnitude of 1.5.

**42.** $4.1 - 2.3 = 1.8$. They differ by an order of magnitude of 1.8.

**43.** Given

$$\beta_1 = 10 \log \frac{I_1}{I_0} = 95$$

$$\beta_2 = 10 \log \frac{I_2}{I_0} = 65,$$

we seek the logarithm of the ratio $I_1/I_2$.

$$10 \log \frac{I_1}{I_0} - 10 \log \frac{I_2}{I_0} = \beta_1 - \beta_2$$

$$10\left(\log \frac{I_1}{I_0} - \log \frac{I_2}{I_0}\right) = 95 - 65$$

$$10 \log \frac{I_1}{I_2} = 30$$

$$\log \frac{I_1}{I_2} = 3$$

The two intensities differ by 3 orders of magnitude.

**44.** Given

$$\beta_1 = 10 \log \frac{I_1}{I_0} = 70$$

$$\beta_2 = 10 \log \frac{I_2}{I_0} = 10,$$

we seek the logarithm of the ratio $I_1/I_2$.

$$10 \log \frac{I_1}{I_0} - 10 \log \frac{I_2}{I_0} = \beta_1 - \beta_2$$

$$10\left(\log \frac{I_1}{I_0} - \log \frac{I_2}{I_0}\right) = 70 - 10$$

$$10 \log \frac{I_1}{I_2} = 60$$

$$\log \frac{I_1}{I_2} = 6$$

The two intensities differ by 6 orders of magnitude.

**45.** Assuming that $T$ and $B$ are the same for the two quakes, we have $7.9 = \log a_1 - \log T + B$ and $6.6 = \log a_2 - \log T + B$, so $7.9 - 6.6 = 1.3 = \log(a_1/a_2)$. Then $a_1/a_2 = 10^{1.3}$, so $a_1 \approx 19.95a_2$ — the Mexico City amplitude was about 20 times greater.

**46.** If $T$ and $B$ were the same, we have $7.2 = \log a_1 - \log T + B$ and $6.6 = \log a_2 - \log T + B$, so $7.2 - 6.6 = 0.6 = \log(a_1/a_2)$. Then $a_1/a_2 = 10^{0.6}$, so $a_1 \approx 3.98a_2$ — Kobe's amplitude was about 4 times greater.

**47.** **(a)** Carbonated water: $-\log [H^+] = 3.9$

$$\log [H^+] = -3.9$$

$$[H^+] = 10^{-3.9} \approx 1.26 \times 10^{-4}$$

Household ammonia: $-\log [H^+] = 11.9$

$$\log [H^+] = -11.9$$

$$[H^+] = 10^{-11.9} \approx 1.26 \times 10^{-12}$$

**(b)** $$\frac{[H^+] \text{ of carbonated water}}{[H^+] \text{ of household ammonia}} = \frac{10^{-3.9}}{10^{-11.9}} = 10^8$$

**(c)** They differ by an order of magnitude of 8.

**48.** **(a)** Stomach acid: $-\log [H^+] = 2.0$

$$\log [H^+] = -2.0$$

$$[H^+] = 10^{-2.0} = 1 \times 10^{-2}$$

Blood: $-\log [H^+] = 7.4$

$$\log [H^+] = -7.4$$

$$[H^+] = 10^{-7.4} \approx 3.98 \times 10^{-8}$$

**(b)** $$\frac{[H^+] \text{ of stomach acid}}{[H^+] \text{ of blood}} = \frac{10^{-2}}{10^{-7.4}} \approx 2.51 \times 10^5$$

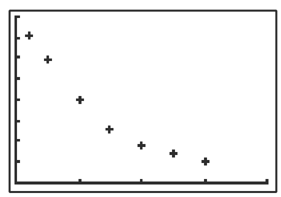
**(c)** They differ by an order of magnitude of 5.4.

The equations in #49 and 50 can be solved either algebraically or graphically; the latter approach is generally faster.

**49.** Substituting known information into $T(t) = T_m + (T_0 - T_m)e^{-kt}$ leaves $T(t) = 22 + 70e^{-kt}$. Using $T(12) = 50 = 22 + 70e^{-12k}$, we have $e^{-12k} = \frac{2}{5}$, so $k = -\frac{1}{12} \ln \frac{2}{5} \approx 0.0764$. Solving $T(t) = 30$ yields $t \approx 28.41$ minutes.

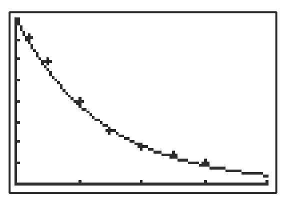
**50.** Substituting known information into $T(t) = T_m + (T_0 - T_m)e^{-kt}$ leaves $T(t) = 65 + 285e^{-kt}$. Using $T(20) = 120 = 65 + 285e^{-20k}$, we have $e^{-20k} = \frac{11}{57}$, so $k = -\frac{1}{20} \ln \frac{11}{57} \approx 0.0823$. Solving $T(t) = 90$ yields $t \approx 29.59$ minutes.

**51.** **(a)**



[0, 40] by [0, 80]

**(b)**

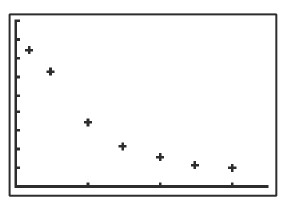


[0, 40] by [0, 80]

$T(x) \approx 79.47 \cdot 0.93^x$

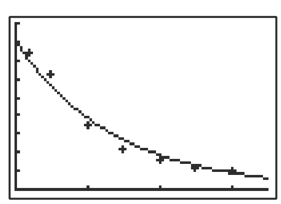
**(c)** $T(0) + 10 = 79.47 + 10 \approx 89.47°C$

**52.** **(a)**



[0, 35] by [0, 90]

**(b)**

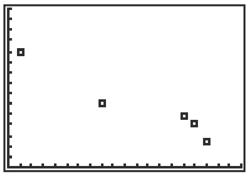


[0, 35] by [0, 90]

$T(x) \approx 79.96 \cdot 0.93^x$

**(c)** $T(0) + 0 \approx 79.96 = 79.96°C$.

**53.** **(a)**

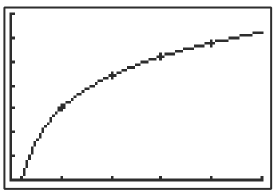


[0, 20] by [0, 15]

**(b)** The scatter plot is better because it accurately represents the times between the measurements. The equal spacing on the bar graph suggests that the measurements were taken at equally spaced intervals, which distorts our perceptions of how the consumption has changed over time.

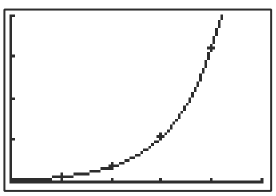
**54.** Answers will vary.

**55.** Logarithmic seems best — the scatterplot of $(x, y)$ looks most logarithmic. (The data can be modeled by $y = 3 + 2 \ln x$.)



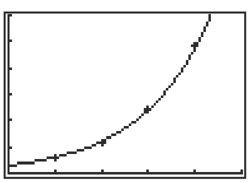
[0, 5] by [0, 7]

**56.** Exponential — the scatterplot of $(x, y)$ is *exactly* exponential. (The data can be modeled by $y = 2 \cdot 3^x$.)



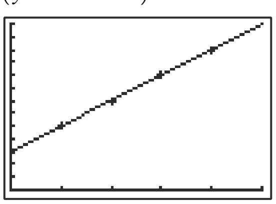
[0, 5] by [0, 200]

**57.** Exponential — the scatterplot of $(x, y)$ is *exactly* exponential. (The data can be modeled by $y = \frac{3}{2} \cdot 2^x$.)



[0, 5] by [0, 30]

**58.** Linear — the scatterplot of $(x, y)$ is *exactly* linear ($y = 2x + 3$.)



[0, 5] by [0, 13]

**59.** False. The order of magnitude of a positive number is its *common* logarithm.

**60.** True. In the formula $T(t) = T_m + (T_0 - T_m)e^{-kt}$, the term $(T_0 - T_m)e^{-kt}$ goes to zero as $t$ gets large, so that $T(t)$ approaches $T_m$.

**61.** $2^{3x-1} = 32$
$2^{3x-1} = 2^5$
$3x - 1 = 5$
$x = 2$
The answer is B.

**62.** $\ln x = -1$
$e^{\ln x} = e^{-1}$
$x = \frac{1}{e}$
The answer is B.

**63.** Given

$$R_1 = \log \frac{a_1}{T} + B = 8.1$$

$$R_2 = \log \frac{a_2}{T} + B = 6.1,$$

we seek the ratio of amplitudes (severities) $a_1/a_2$.

$$\left(\log \frac{a_1}{T} + B\right) - \left(\log \frac{a_2}{T} + B\right) = R_1 - R_2$$

$$\log \frac{a_1}{T} - \log \frac{a_2}{T} = 8.1 - 6.1$$

$$\log \frac{a_1}{a_2} = 2$$

$$\frac{a_1}{a_2} = 10^2 = 100$$

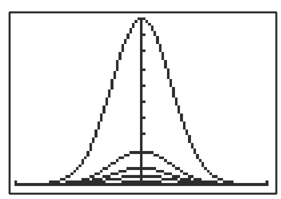
The answer is E.

**64.** As the second term on the right side of the formula $T(t) = T_m + (T_0 - T_m)e^{-kt}$ indicates, and as the graph confirms, the model is exponential.
The answer is A.

**65.** A logistic regression $\left(f(x) = \frac{943}{1 + 59.126e^{-0.04786x}}\right)$ most closely matches the data, and would provide a natural "cap" to the population growth at approx. 945,000 people. (Note: $x$ = number of years since 1900.)

**66.** The logistic regression model $\left(f(x) = \frac{2027}{1 + 12.538e^{-0.0297x}}\right)$ matches the data well and provides a natural cap of 2.0 million people. (Note: $x$ = number of years since 1900.)

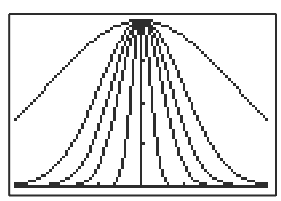
**67.** **(a)**



[−3, 3] by [0, 10]

As $k$ increases, the bell curve stretches vertically. Its height increases and the slope of the curve seems to steepen.

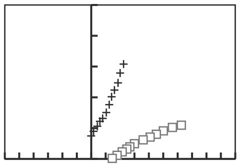
**(b)**



[−3, 3] by [0, 1]

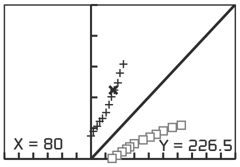
As $c$ increases, the bell curve compresses horizontally. Its slope seems to steepen, increasing more rapidly to (0, 1) and decreasing more rapidly from (0, 1).

**68. (a and b)**

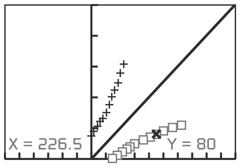


[–302, 502] by [0, 500]

**(c)**



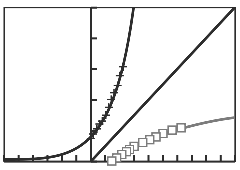

[–302, 502] by [0, 500]




[–302, 502] by [0, 500]

**(d)** $y_1 = 81.274 \cdot 1.013^x$ and $y_2 = -348.540 + 79.320 \ln x$

**(e)**



[–302, 502] by [0, 500]

$$311.6 = 81.274 \cdot 1.013^x$$

$$\frac{311.6}{81.274} = 1.013^x$$

**(f)** $\ln(1.013^x) = \ln\left(\frac{311.6}{81.274}\right) = \ln 311.6 - \ln 81.274$

$$x \ln 1.013 = \ln 311.6 - \ln 81.274$$

$$x = \frac{\ln 311.6 - \ln 81.274}{\ln 1.013} \approx 104.05$$

**69.** Let $\frac{u}{v} = 10^n, u, v > 0$

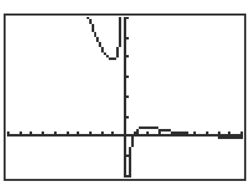
$$\log \frac{u}{v} = \log 10^n$$

$$\log u - \log v = n \log 10$$

$$\log u - \log v = n(1) = n$$

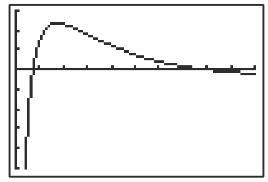
For the initial expression to be true, either $u$ and $v$ are both powers of ten, or they are the same constant $k$ multiplied by powers of 10 (i.e., either $u = 10^k$ and $v = 10^m$ or $u = a \cdot 10^k$ and $v = a \cdot 10^m$, where $a$, $k$, and $m$ are constants). As a result, $u$ and $v$ vary by an order of magnitude $n$. That is, $u$ is $n$ orders of magnitude greater than $v$.

**70. (a)** $r$ cannot be negative since it is a distance.



[–10, 10] by [–10, 30]

**(b)** [0, 10] by [−5, 3] is a good choice. The maximum energy, approximately 2.3807, occurs when $r \approx 1.729$.



[0, 10] by [−5, 3]

**71.** Since $T_0 \approx 66.156$ and $T_m = 4.5$, we have

$$(66.156 - 4.5)e^{-kt} = 61.656 \times (0.92770)^t$$
$$61.656e^{-kt} = 61.656 \times (0.92770)^t$$
$$e^{-kt} = \frac{61.656}{61.656} \times (0.92770)^t$$
$$e^{-kt} = 1 \times (0.92770)^t$$
$$\ln e^{-kt} = \ln(1 \cdot (0.92770)^t)$$
$$-kt = \ln(1) + \ln(0.92770)^t$$
$$-kt = 0 + t \ln(0.92770)$$
$$k = -\ln(0.92770)$$
$$\approx 0.075.$$

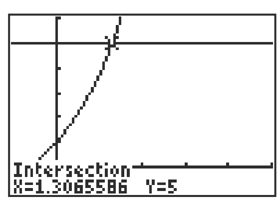
**72.** One possible answer: We "map" our data so that all points $(x, y)$ are plotted as $(\ln x, y)$. If these "new" points are linear — and thus can be represented by some standard linear regression $y = ax + b$ — we make the same substitution $(x \to \ln x)$ and find $y = a \ln x + b$, a logarithmic regression.

**73.** One possible answer: We "map" our data so that all points $(x, y)$ are plotted as $(\ln x, \ln y)$. If these "new" points are linear — and thus can be represented by some standard linear regression $y = ax + b$ — we make the same "mapping" $(x \to \ln x, y \to \ln y)$ and find $\ln y = a \ln x + b$. Using algebra and the properties of algorithms, we have:

$$\ln y = a \ln x + b$$
$$e^{\ln y} = e^{a \ln x + b}$$
$$y = e^{a \ln x} \cdot e^b$$
$$= e^{\ln x^a} \cdot e^b$$
$$= e^b \cdot x^a$$
$$= c\,x^a, \text{ where } c = e^b, \text{ exactly the power regression.}$$

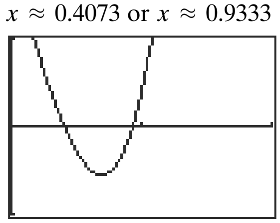
The equations and inequalities in #73–76 must be solved graphically — they cannot be solved algebraically. For #77 and 78, algebraic solution is possible, although a graphical approach may be easier.

**74.** $x \approx 1.3066$



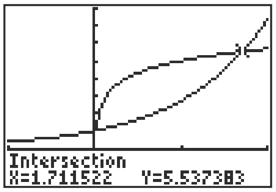

[–1, 5] by [–1, 6]

**75.** $x \approx 0.4073$ or $x \approx 0.9333$



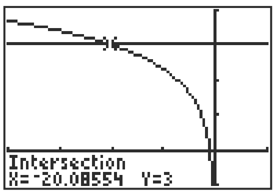
[0, 2] by [–1, 1]

**76.** $0 < x < 1.7115$ (approx.)




[–1, 2] by [–2, 8]

**77.** $x \le -20.0855$ (approx.)




[–40, 10] by [–1, 4]

**78.** $\log x - 2\log 3 > 0$, so $\log(x/9) > 0$. Then $\frac{x}{9} > 10^0 = 1$, so $x > 9$.

**79.** $\log(x + 1) - \log 6 < 0$, so $\log\frac{x+1}{6} < 0$.

Then $\frac{x+1}{6} < 10^0 = 1$, so $x + 1 < 6$, or $x < 5$. The original equation also requires that $x + 1 > 0$, so the solution is $-1 < x < 5$.

## Section 3.6 Mathematics of Finance

### Exploration 1

**1.**

| $k$ | $A$ |
|---|---|
| 10 | 1104.6 |
| 20 | 1104.9 |
| 30 | 1105 |
| 40 | 1105 |
| 50 | 1105.1 |
| 60 | 1105.1 |
| 70 | 1105.1 |
| 80 | 1105.1 |
| 90 | 1105.1 |
| 100 | 1105.1 |

$A$ approaches a limit of about 1105.1.

**2.** $y = 1000e^{0.1} \approx 1105.171$ is an upper bound (and asymptote) for $A(x)$. $A(x)$ approaches, but never equals, this bound.

### Quick Review 3.6

**1.** $200 \cdot 0.035 = 7$

**2.** $150 \cdot 0.025 = 3.75$

**3.** $\frac{1}{4} \cdot 7.25\% = 1.8125\%$

**4.** $\frac{1}{12} \cdot 6.5\% \approx 0.5417\%$

**5.** $\frac{78}{120} = 0.65 = 65\%$

**6.** $\frac{28}{80} = 0.35 = 35\%$

**7.** $0.32x = 48$ gives $x = 150$

**8.** $0.84x = 176.4$ gives $x = 210$

**9.** $300\,(1 + 0.05) = 315$ dollars

**10.** $500\,(1 + 0.45) = 522.50$ dollars

### Section 3.6 Exercises

**1.** Compound interest: $A = 1500(1 + 0.07)^6 \approx \$2251.10$; simple interest: $A = 1500(1 + 0.07(6)) \approx \$2130$.

**2.** Compound interest: $A = 3200(1 + 0.08)^4 \approx \$4353.56$; simple interest: $A = 3200(1 + 0.08(4)) \approx \$4224$.

**3.** Compound interest: $A = 12{,}000(1 + 0.075)^7 \approx \$19{,}908.59$; simple interest: $A = 12{,}000(1 + 0.075(7)) \approx \$18{,}300$.

**4.** Compound interest: $A = 15{,}500(1 + 0.095)^{12} \approx \$46{,}057.58$; simple interest: $A = 15{,}500(1 + 0.095(12)) \approx \$33{,}170$.

**5.** $A = 1500\left(1 + \frac{0.07}{4}\right)^{20} \approx \$2122.17$

**6.** $A = 3500\left(1 + \frac{0.05}{4}\right)^{40} \approx \$5752.67$

**7.** $A = 40{,}500\left(1 + \frac{0.038}{12}\right)^{240} \approx \$86{,}496.26$

**8.** $A = 25{,}300\left(1 + \frac{0.045}{12}\right)^{300} \approx \$77{,}765.69$

**9.** $A = 1250e^{(0.054)(6)} \approx \$1728.31$

**10.** $A = 3350e^{(0.062)(8)} \approx \$5501.17$

**11.** $A = 21{,}000e^{(0.037)(10)} \approx \$30{,}402.43$

**12.** $A = 8875e^{(0.044)(25)} \approx \$26{,}661.97$

**13.** $FV = 500 \cdot \dfrac{\left(1 + \frac{0.07}{4}\right)^{24} - 1}{\frac{0.07}{4}} \approx \$14{,}755.51$

**14.** $FV = 300 \cdot \dfrac{\left(1 + \frac{0.06}{4}\right)^{48} - 1}{\frac{0.06}{4}} \approx \$20{,}869.57$

**15.** $FV = 450 \cdot \dfrac{\left(1 + \frac{0.0525}{12}\right)^{120} - 1}{\frac{0.0525}{12}} \approx \$70{,}819.63$

**16.** $FV = 610 \cdot \dfrac{\left(1 + \frac{0.065}{12}\right)^{300} - 1}{\frac{0.065}{12}} \approx \$456{,}790.28$

**17.** $PV = 815.37 \cdot \dfrac{1 - \left(1 + \frac{0.047}{12}\right)^{-60}}{\frac{0.047}{12}} \approx \$43{,}523.31$

**18.** $PV = 1856.82 \cdot \dfrac{1 - \left(1 + \frac{0.065}{12}\right)^{-360}}{\frac{0.065}{12}} \approx \$293{,}769.01$

**19.** $R = \dfrac{PV \cdot i}{1-(1+i)^{-n}} = \dfrac{(18{,}000)\left(\frac{0.054}{12}\right)}{1-\left(1+\frac{0.054}{12}\right)^{-72}} \approx \$293.24$

**20.** $R = \dfrac{PV \cdot i}{1-(1+i)^{-n}} = \dfrac{(154{,}000)\left(\frac{0.072}{12}\right)}{1-\left(1+\frac{0.072}{12}\right)^{-180}} \approx \$1401.47$

In #21–24, the time must be rounded up to the end of the next compounding period.

**21.** Solve $2300\left(1+\frac{0.09}{4}\right)^{4t} = 4150$: $(1.0225)^{4t} = \frac{83}{46}$, so $t = \frac{1}{4}\frac{\ln(83/46)}{\ln 1.0225} \approx 6.63$ years — round to 6 years 9 months (the next full compounding period).

**22.** Solve $8000\left(1+\frac{0.09}{12}\right)^{12t} = 16{,}000$: $(1.0075)^{12t} = 2$, so $t = \frac{1}{12}\frac{\ln 2}{\ln 1.0075} \approx 7.73$ years — round to 7 years 9 months (the next full compounding period).

**23.** Solve $15{,}000\left(1+\frac{0.08}{12}\right)^{12t} = 45{,}000$: $(1.0067)^{12t} = 3$, so $t = \frac{1}{12}\frac{\ln 3}{\ln 1.0067} \approx 13.71$ years — round to 13 years 9 months (the next full compounding period). Note: A graphical solution provides $t \approx 13.78$ years — round to 13 years 10 months.

**24.** Solve $1.5\left(1+\frac{0.08}{4}\right)^{4t} = 3.75$: $(1.02)^{4t} = 2.5$, so $t = \frac{1}{4}\frac{\ln 2.5}{\ln 1.02} \approx 11.57$ years — round to 11 years 9 months (the next full compounding period).

**25.** Solve $22{,}000\left(1+\frac{r}{365}\right)^{(365)(5)} = 36{,}500$:
$1+\frac{r}{365} = \left(\frac{73}{44}\right)^{1/1825}$, so $r \approx 10.13\%$.

**26.** Solve $8500\left(1+\frac{r}{12}\right)^{(12)(5)} = 3 \cdot 8500$:
$1+\frac{r}{12} = 3^{1/60}$, so $r \approx 22.17\%$.

**27.** Solve $14.6\,(1+r)^6 = 22$: $1+r = \left(\frac{110}{73}\right)^{1/6}$, so $r \approx 7.07\%$.

**28.** Solve $18\,(1+r)8 = 25$: $1+r = \left(\frac{25}{18}\right)^{1/8}$, so $r \approx 4.19\%$.

In #29 and 30, the time must be rounded up to the end of the next compounding period.

**29.** Solve $\left(1+\frac{0.0575}{4}\right)^{4t} = 2$: $t = \frac{1}{4}\frac{\ln 2}{\ln 1.014375} \approx 12.14$ — round to 12 years 3 months.

**30.** Solve $\left(1+\frac{0.0625}{12}\right)^{12t} = 3$:
$t = \frac{1}{12}\frac{\ln 3}{\ln(1+0.0625/12)} \approx 17.62$ — round to 17 years 8 months.

For #31–34, use the formula $S = Pe^{rt}$.

**31.** Time to double: Solve $2 = e^{0.09t}$, leading to $t = \frac{1}{0.09}\ln 2 \approx 7.7016$ years. After 15 years: $S = 12{,}500e^{(0.09)(15)} \approx \$48{,}217.82$.

**32.** Time to double: Solve $2 = e^{0.08t}$, leading to $t = \frac{1}{0.08}\ln 2 \approx 8.6643$ years. After 15 years: $S = 32{,}500e^{(0.08)(15)} \approx \$107{,}903.80$.

**33.** APR: Solve $2 = e^{4r}$, leading to $r = \frac{1}{4}\ln 2 \approx 17.33\%$.
After 15 years: $S = 9500e^{(0.1733)(15)} \approx \$127{,}816.26$ (using the "exact" value of $r$).

**34.** APR: Solve $2 = e^{6r}$, leading to $r = \frac{1}{6}\ln 2 \approx 11.55\%$.
After 15 years: $S = 16{,}800e^{(0.1155)(15)} \approx \$95{,}035.15$ (using the "exact" value of $r$).

In #35–41, the time must be rounded up to the end of the next compounding period (except in the case of continuous compounding).

**35.** Solve $\left(1+\frac{0.04}{4}\right)^{4t} = 2$: $t = \frac{1}{4}\frac{\ln 2}{\ln 1.01} \approx 17.42$, which rounds to 17 years 6 months.

**36.** Solve $\left(1+\frac{0.08}{4}\right)^{4t} = 2$: $t = \frac{1}{4}\frac{\ln 2}{\ln 1.02} \approx 8.751$, which rounds to 9 years (*almost* by 8 years 9 months).

**37.** Solve $1 + 0.07t = 2$: $t = \frac{1}{0.07} \approx 14.29$, which rounds to 15 years.

**38.** Solve $1.07^t = 2$: $t = \frac{\ln 2}{\ln 1.07} \approx 10.24$, which rounds to 11 years.

**39.** Solve $\left(1+\frac{0.07}{4}\right)^{4t} = 2$: $t = \frac{1}{4}\frac{\ln 2}{\ln 1.0175} \approx 9.99$, which rounds to 10 years.

**40.** Solve $\left(1+\frac{0.07}{12}\right)^{12t} = 2$: $t = \frac{1}{12}\frac{\ln 2}{\ln(1+0.07/12)} \approx 9.93$, which rounds to 10 years.

**41.** Solve $e^{0.07t} = 2$: $t = \frac{1}{0.07}\ln 2 \approx 9.90$ years.

For #42–45, observe that the initial balance has no effect on the APY.

**42.** $\text{APY} = \left(1+\frac{0.06}{4}\right)^4 - 1 \approx 6.14\%$

**43.** $\text{APY} = \left(1+\frac{0.0575}{365}\right)^{365} - 1 \approx 5.92\%$

**44.** $\text{APY} = e^{0.063} - 1 \approx 6.50\%$

**45.** $\text{APY} = \left(1+\frac{0.047}{12}\right)^{12} - 1 \approx 4.80\%$

**46.** The APYs are $\left(1+\frac{0.05}{12}\right)^{12} - 1 \approx 5.1162\%$ and $\left(1+\frac{0.051}{4}\right)^4 - 1 \approx 5.1984\%$. So, the better investment is 5.1% compounded quarterly.

**47.** The APYs are $5\frac{1}{8}\% = 5.125\%$ and $e^{0.05} - 1 \approx 5.1271\%$.

So, the better investment is 5% compounded continuously.

For #48–51, use the formula $S = R\frac{(1+i)^n - 1}{i}$.

**48.** $i = \frac{0.0726}{12} = 0.00605$ and $R = 50$, so

$S = 50\frac{(1.00605)^{(12)(25)} - 1}{0.00605} \approx \$42{,}211.46$.

**49.** $i = \frac{0.155}{12} = 0.0129\ldots$ and $R = 50$, so

$S = 50\frac{(1.0129)^{(12)(20)} - 1}{0.0129} \approx \$80{,}367.73$.

**50.** $i = \frac{0.124}{12}$; solve

$250{,}000 = R\frac{\left(1 + \frac{0.124}{12}\right)^{(12\times30)} - \quad - 1}{\frac{0.124}{12}}$ to obtain

$R \approx \$239.42$ per month (round up, since \$239.41 will not be adequate).

**51.** $i = \frac{0.045}{12} = 0.00375$; solve

$120{,}000 = R\frac{(1.00375)^{(12)(30)} - 1}{0.00375}$ to obtain $R \approx \$158.03$ per month (round up, since \$158.02 will not be adequate).

For #52–55, use the formula $A = R\frac{1 - (1+i)^{-n}}{i}$.

**52.** $i = \frac{0.0795}{12} = 0.006625$; solve

$9000 = R\frac{1 - (1.006625)^{-(12)(4)}}{0.006625}$ to obtain $R \approx \$219.51$ per month.

**53.** $i = \frac{0.1025}{12} = 0.0085417$; solve

$4500 = R\frac{1 - (1.0085417)^{-(12)(3)}}{0.0085417}$ to obtain $R \approx \$145.74$ per month (round up, since \$145.73 will not be adequate).

**54.** $i = \frac{0.0875}{12} = 0.0072917$; solve

$86{,}000 = R\frac{1 - (1.0072917)^{-(12)(30)}}{0.0072917}$ to obtain $R \approx \$676.57$ per month (round up, since \$676.56 will not be adequate).

**55.** $i = \frac{0.0925}{12} = 0.0077083$; solve $100{,}000 = R\frac{1 - (1.0077083)^{-(12)(25)}}{0.0077083}$ to obtain $R \approx \$856.39$ per month (round up, since \$856.38 will not be adequate).

**56.** **(a)** With $i = \frac{0.12}{12} = 0.01$, solve

$86{,}000 = 1050\frac{1 - (1.01)^{-n}}{0.01}$; this leads to

$(1.01)^{-n} = 1 - \frac{860}{1050} = \frac{19}{105}$, so $n \approx 171.81$ months, or about 14.32 years. The mortgage will be paid off after 172 months (14 years, 4 months). The last payment will be less than \$1050. A reasonable estimate of the final payment can be found by taking the fractional part of the computed value of $n$ above, 0.81, and multiplying by \$1050, giving about \$850.50. To figure the exact amount of the final payment,

solve $86{,}000 = 1050\frac{1 - (1.01)^{-171}}{0.01} + R\,(1.01)^{-172}$

(the present value of the first 171 payments, plus the present value of a payment of $R$ dollars 172 months from now). This gives a final payment of $R \approx \$846.57$.

**(b)** The total amount of the payments under the original plan is $360 \cdot \$884.61 = \$318{,}459.60$. The total using the higher payments is $172 \cdot \$1050 = \$180{,}660$ (or $171 \cdot \$1050 + \$846.57 = \$180{,}396.57$ if we use the correct amount of the final payment) — a difference of \$137,859.60 (or \$138,063.03 using the correct final payment).

**57.** **(a)** After 10 years, the remaining loan balance is

$86{,}000(1.01)^{120} - 884.61\frac{(1.01)^{120} - 1}{0.01} \approx \$80{,}338.75$

(this is the future value of the initial loan balance, minus the future value of the loan payments). With \$1050 payments, the time required is found by solving

$80{,}338.75 = 1050\frac{1 - (1.01)^{-n}}{0.01}$; this leads to

$(1.01)^{-n} \approx 0.23487$, so $n \approx 145.6$ months, or about 12.13 (additional) years. The mortgage will be paid off after a total of 22 years 2 months, with the final payment being less than \$1050. A reasonable estimate of the final payment is $(0.6)(\$1050) \approx \$630.00$ (see the previous problem); to figure the exact amount, solve

$80{,}338.75 = 1050\frac{1 - (1.01)^{-145}}{0.01} + R(1.01)^{-146}$,

which gives a final payment of $R \approx \$626.93$.

**(b)** The original plan calls for a total of \$318,459.60 in payments; this plan calls for $120 \cdot \$884.61 + 146 \cdot \$1050 = \$259{,}453.20$ (or $120 \cdot \$884.61 + 145 \cdot \$1050 + \$626.93 = \$259{,}030.13$) — a savings of \$59,006.40 (or \$59,429.47).

**58.** One possible answer: The APY is the percentage increase from the initial balance $S(0)$ to the end-of-year balance $S(1)$; specifically, it is $S(1)/S(0) - 1$. Multiplying the initial balance by $P$ results in the end-of-year balance being multiplied by the same amount, so that the ratio remains unchanged. Whether we start with a \$1 investment, or a \$1000 investment, $\text{APY} = \left(1 + \frac{r}{k}\right)^k - 1$.

**59.** One possible answer: The APR will be lower than the APY (except under annual compounding), so the bank's offer looks more attractive when the APR is given. Assuming monthly compounding, the APY is about 4.594%; quarterly and daily compounding give approximately 4.577% and 4.602%, respectively.

**60.** One possible answer: Some of these situations involve counting things (e.g., populations), so that they can only take on whole number values — exponential models which predict, e.g., 439.72 fish, have to be interpreted in light of this fact.

Technically, bacterial growth, radioactive decay, and compounding interest also are "counting problems" — for example, we cannot have fractional bacteria, or fractional atoms of radioactive material, or fractions of pennies. However, because these are generally very large numbers, it is easier to ignore the fractional parts. (This might also apply when one is talking about, e.g., the population of the whole world.)

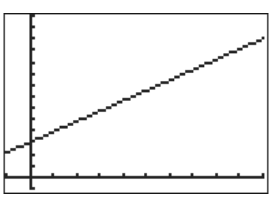
Another distinction: while we often use an exponential model for all these situations, it generally fits better (over long periods of time) for radioactive decay than for most of the others. Rates of growth in populations (esp. human populations) tend to fluctuate more than exponential models suggest. Of course, an exponential model also fits well in compound interest situations where the interest rate is held constant, but there are many cases where interest rates change over time.

**61. (a)** Steve's balance will always remain \$1000, since interest is not added to it. Every year he receives 6% of that \$1000 in interest: 6% in the first year, then another 6% in the second year (for a total of $2 \cdot 6\% = 12\%$), then another 6% (totaling $3 \cdot 6\% = 18\%$), etc. After $t$ years, he has earned $6t\%$ of the \$1000 investment, meaning that altogether he has $1000 + 1000 \cdot 0.06t = 1000(1 + 0.06t)$.

**(b)** The table is shown below; the second column gives values of $1000(1.06)^t$. The effects of annual compounding show up beginning in Year 2.

| Years | Not Compounded | Compounded |
|---|---|---|
| 0 | 1000.00 | 1000.00 |
| 1 | 1060.00 | 1060.00 |
| 2 | 1120.00 | 1123.60 |
| 3 | 1180.00 | 1191.02 |
| 4 | 1240.00 | 1262.48 |
| 5 | 1300.00 | 1338.23 |
| 6 | 1360.00 | 1418.52 |
| 7 | 1420.00 | 1503.63 |
| 8 | 1480.00 | 1593.85 |
| 9 | 1540.00 | 1689.48 |
| 10 | 1600.00 | 1790.85 |

**62. (a)**



[–10, 100] by [–100, 1400]

**(b)** The slope is $P \cdot r$ since that is the coefficient on $t$.

**(c)** The slope of the line increases as the interest rate increases and decreases as the interest rate decreases.

**(d)** $P_{\text{Simple}}(0.5) = 300 + 300 \cdot 0.03 \cdot 0.5 = \$304.50$;
$P_{\text{Compound}}(0.5) = 300(1 + 0.03)^{0.5} = \$304.47$.
So $P_{\text{Simple}}(0.5)$ is greater. In the first half of the year, compound interest grows more slowly than simple interest, so simple interest will earn more money. By the end of the year, compound interest grows more quickly that simple interest, so $P_{\text{Compound}}(1) = P_{\text{Simple}}(1)$.

**63.** False. The limit, with continuous compounding, is $A = Pe^{rt} = 100\, e^{0.05} \approx \$105.13$.

**64.** True. The calculation of interest paid involves compounding, and the compounding effect is greater for longer repayment periods.

**65.** $A = P(1 + r/k)^{kt} = 2250(1 + 0.07/4)^{4(6)} \approx \$3412.00$. The answer is B.

**66.** Let $x$ = APY. Then $1 + x = (1 + 0.06/12)^{12} \approx 1.0617$. So $x \approx 0.0617$. The answer is C.

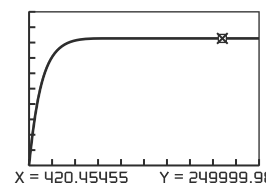
**67.** $FV = R((1 + i)^n - 1)/i = 300((1 + 0.00375)^{240} - 1)/0.00375 \approx \$116{,}437.31$.
The answer is E.

**68.** $R = PV\, i/(1 - (1 + i)^{-n})$
$= 120{,}000(0.0725/12)/(1 - (1 + 0.0725/12)^{-180})$
$\approx \$1095.44$.
The answer is A.

**69.** The last payment will be \$364.38.

**70.** One possible answer:
The answer is (c). This graph shows the loan balance decreasing at a fairly steady rate over time. By contrast, the early payments on a 30-year mortgage go mostly toward interest, while the late payments go mostly toward paying down the debt. So the graph of loan balance versus time for a 30-year mortgage at double the interest rate would start off nearly horizontal and more steeply decrease over time.

**71.** $PV(10) = \$81{,}109$
$PV(50) = \$214{,}823$
$PV(100) = \$254{,}050$
$PV(150) = \$249{,}303$
$PV(200) = \$249{,}902$
$PV(300) = \$249{,}998$




[0, 500] by [0, 300000]

Since $\lim_{n\to\infty}(1+i)^{-n} = \lim_{n\to\infty}\frac{1}{(1+i)^1} = 0$,

$$\lim_{n\to\infty} R\,\frac{1-(1+i)^{-n}}{i} = \frac{R}{i}.$$

For the given annuity, $\frac{R}{i} = \frac{10{,}000}{0.04} = \$250{,}000$.

**72. (a)** Matching up with the formula $S = R\,\frac{(1 + i)^n - 1}{i}$, where $i = r/k$, with $r$ being the rate and $k$ being the number of payments per year, we find $r = 8\%$.

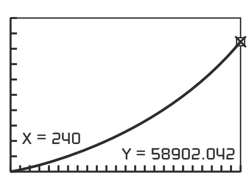
**(b)** $k = 12$ payments per year.

**(c)** Each payment is $R = \$100$.

**(d)** 20 years $= 20 \cdot 12 = 240$, so

$$F(240) = 100\frac{\left(1 + \frac{0.08}{12}\right)^{240} - 1}{\frac{0.08}{12}} \approx 58{,}902.04$$

**(e)**




[0, 240] by [0, 70000]

**73.** **(a)** Matching up with the formula $A = R\dfrac{1-(1+i)^{-n}}{i}$, where $i = r/k$, with $r$ being the rate and $k$ being number of payments per year, we find $r = 8\%$.

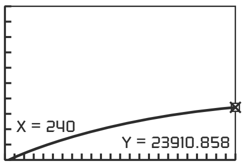
**(b)** $k = 12$ payments per year.

**(c)** Each payment is $R = \$200$.

**(d)** 20 years $= 20 \cdot 12 = 240$, so

$$PV(240) = 200\frac{1-\left(1+\frac{0.08}{12}\right)^{-240}}{\frac{0.08}{12}} \approx \$23{,}910.86$$

**(e)**




[0, 240] by [0, 70000]

**(f)** Since interest is earned on the annuity's remaining balance each month, less money is required to achieve the future value of $48,000. Opinions will vary on which option is better.

## ■ Chapter 3 Review

**1.** $f\left(\dfrac{1}{3}\right) = -3 \cdot 4^{1/3} = -3\sqrt[3]{4}$

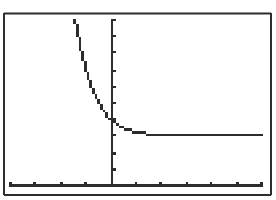
**2.** $f\left(-\dfrac{3}{2}\right) = 6 \cdot 3^{-3/2} = \dfrac{6}{\sqrt{27}} = \dfrac{2}{\sqrt{3}}$ or $\dfrac{2\sqrt{3}}{3}$

For #3 and 4, recall that exponential functions have the form $f(x) = a \cdot b^x$.

**3.** $a = 3$, so $f(2) = 3 \cdot b^2 = 6$, $b^2 = 2$, $b = \sqrt{2}$, $f(x) = 3 \cdot 2^{x/2}$

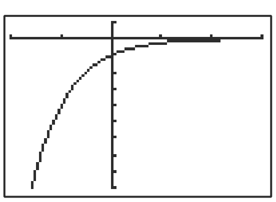
**4.** $a = 2$, so $f(3) = 2 \cdot b^3 = 1$, $b^3 = \dfrac{1}{2}$, $b = 2^{-1/3}$, $f(x) = 2 \cdot 2^{-x/3}$

**5.** $f(x) = 2^{-2x} + 3$ — starting from $2^x$, horizontally shrink by $\dfrac{1}{2}$, reflect across the $y$-axis, and translate up 3 units.



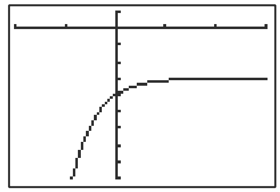
[−4, 6] by [0, 10]

**6.** $f(x) = 2^{-2x}$ — starting from $2^x$, horizontally shrink by $\dfrac{1}{2}$, reflect across the $y$-axis, reflect across $x$-axis.



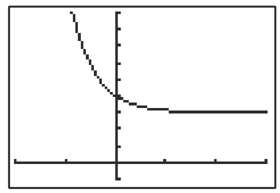
[−2, 3] by [9, −1]

**7.** $f(x) = -2^{-3x} - 3$ — starting from $2^x$, horizontally shrink by $\frac{1}{3}$, reflect across the $y$-axis, reflect across the $x$-axis, translate down 3 units.



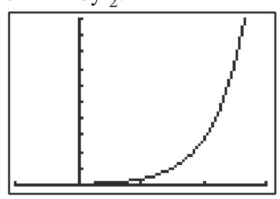
[−2, 3] by [9, −1]

**8.** $f(x) = 2^{-3x} + 3$ — starting from $2^x$, horizontally shrink by $\frac{1}{3}$, reflect across the $y$-axis, translate up 3 units.



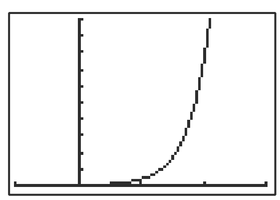
[−2, 3] by [−1, 9]

**9.** Starting from $e^x$, horizontally shrink by $\frac{1}{2}$, then translate right $\frac{3}{2}$ units — or translate right 3 units, then horizontally shrink by $\frac{1}{2}$.



[−1, 3] by [0, 10]

**10.** Starting from $e^x$, horizontally shrink by $\dfrac{1}{3}$, then translate right $\dfrac{4}{3}$ units — or translate right 4 units, then horizontally shrink by $\dfrac{1}{3}$.



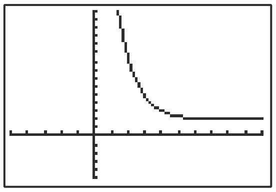
[−1, 3] by [0, 25]

**11.** $f(0) = \dfrac{100}{5+3} = 12.5$, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 20$
$y$-intercept: $(0, 12.5)$.

Asymptotes: $y = 0$ and $y = 20$

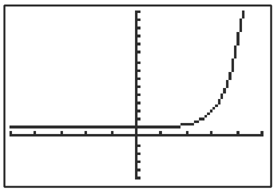
**12.** $f(0) = \dfrac{50}{5+2} = \dfrac{50}{7}$, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 10$
$y$-intercept: $\left(0, \dfrac{50}{7}\right) \approx (0, 7.14)$
Asymptotes: $y = 0$, $y = 10$

**13.** It is an exponential decay function.
$\lim_{x\to\infty} f(x) = 2$, $\lim_{x\to-\infty} f(x) = \infty$



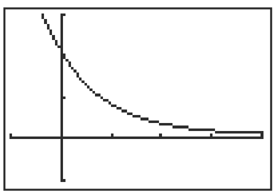
$[-5, 10]$ by $[-5, 15]$

**14.** Exponential growth function
$\lim_{x\to\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = 1$



$[-5, 5]$ by $[-5, 15]$

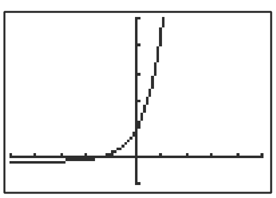
**15.**



$[-1, 4]$ by $[-10, 30]$

Domain: $(-\infty, \infty)$
Range: $(1, \infty)$
Continuous
Always decreasing
Not symmetric
Bounded below by $y = 1$, which is also the only asymptote
No local extrema
$\lim_{x\to\infty} f(x) = 1, \lim_{x\to-\infty} f(x) = \infty$

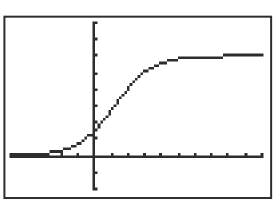
**16.**



$[-5, 5]$ by $[-10, 50]$

Domain: $(-\infty, \infty)$
Range: $(-2, \infty)$
Continuous
Always increasing
Not symmetric
Bounded below by $y = -2$, which is also the only asymptote
No local extrema
$\lim_{x\to\infty} g(x) = \infty, \lim_{x\to-\infty} g(x) = -2$

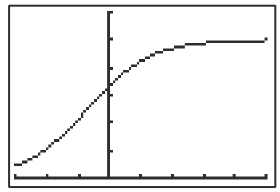
**17.**



$[-5, 10]$ by $[-2, 8]$

Domain: $(-\infty, \infty)$
Range: $(0, 6)$
Continuous
Increasing
Symmetric about $(1.20, 3)$
Bounded above by $y = 6$ and below by $y = 0$, the two asymptotes
No extrema
$\lim_{x\to\infty} f(x) = 6, \lim_{x\to-\infty} f(x) = 0$

**18.**



$[-300, 500]$ by $[0, 30]$

Domain: $(-\infty, \infty)$
Range: $(0, 25)$
Continuous
Always increasing
Symmetric about $(-69.31, 12.5)$
Bounded above by $y = 25$ and below by $y = 0$, the two asymptotes
No local extrema
$\lim_{x\to\infty} g(x) = 25, \lim_{x\to-\infty} g(x) = 0$

For #19–22, recall that exponential functions are of the form $f(x) = a \cdot (1 + r)^{kx}$.

**19.** $a = 24, r = 0.053, k = 1$; so $f(x) = 24 \cdot 1.053^x$, where $x =$ days.

**20.** $a = 67{,}000, r = 0.0167, k = 1$, so $f(x) = 67{,}000 \cdot 1.0167^x$, where $x =$ years.

**21.** $a = 18, r = 1, k = \frac{1}{21}$, so $f(x) = 18 \cdot 2^{x/21}$, where $x =$ days.

**22.** $a = 117, r = -\frac{1}{2}, k = \frac{1}{262}$, so
$f(x) = 117 \cdot \left(\frac{1}{2}\right)^{x/262} = 117 \cdot 2^{-x/262}$, where $x =$ hours.

For #23–26, recall that logistic functions are expressed in $f(x) = \frac{c}{1 + ae^{-bx}}$.

**23.** $c = 30, a = 1.5$, so $f(2) = \frac{30}{1 + 1.5e^{-2b}} = 20$,
$30 = 20 + 30e^{-2b}, 30e^{-2b} = 10, e^{-2b} = \frac{1}{3}$,
$-2b \ln e = \ln \frac{1}{3} \approx -1.0986$, so $b \approx 0.55$.
Thus, $f(x) = \frac{30}{1 + 1.5e^{-0.55x}}$.

**24.** $c = 20, a \approx 2.33$, so $f(3) = \frac{20}{1 + 2.33e^{-3b}} = 15$,
$20 = 15 + 35e^{-3b}, 35e^{-3b} = 5, e^{-3b} = \frac{1}{7}$,
$-3b \ln e = \ln \frac{1}{7} \approx -1.9459$, so $b \approx 0.65$.
Thus, $f(x) = \frac{20}{1 + 2.33e^{-0.65x}}$.

**25.** $c = 20$, $a = 3$, so $f(3) = \dfrac{20}{1 + 3e^{-3b}} = 10$,

$20 = 10 + 30e^{-3b}$, $30e^{-3b} = 10$, $e^{-3b} = \dfrac{10}{30} = \dfrac{1}{3}$,

$-3b \ln e = \ln \dfrac{1}{3} \approx -1.0986$, so $b \approx 0.37$.

Thus, $f(x) \approx \dfrac{20}{1 + 3e^{-0.37x}}$.

**26.** $c = 44$, $a = 3$, so $f(5) = \dfrac{44}{1 + 3e^{-5b}} = 22$,

$44 = 22 + 66e^{-5b}$, $66e^{-5b} = 22$, $e^{-5b} = \dfrac{22}{66} = \dfrac{1}{3}$,

$-5b \ln e = \ln \dfrac{1}{3} \approx -1.0986$, so $b \approx 0.22$.

Thus, $f(x) \approx \dfrac{44}{1 + 3e^{-0.22x}}$.

**27.** $\log_2 32 = \log_2 2^5 = 5 \log_2 2 = 5$

**28.** $\log_3 81 = \log_3 3^4 = 4 \log_3 3 = 4$

**29.** $\log \sqrt[3]{10}\log 10^{\frac{1}{3}} = \dfrac{1}{3} \log 10 = \dfrac{1}{3}$

**30.** $\ln \dfrac{1}{\sqrt{e^7}} = \ln e-\frac{7}{2} = -\dfrac{7}{2} \ln e = -\dfrac{7}{2}$

**31.** $x = 3^5 = 243$

**32.** $x = 2^y$

**33.** $\left(\dfrac{x}{y}\right) = e^{-2}$

$x = \dfrac{y}{e^2}$

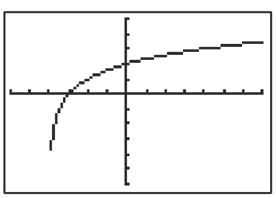
$y = xe^2$

**34.** $\left(\dfrac{a}{b}\right) = 10^{-3}$

$a = \dfrac{b}{1000}$

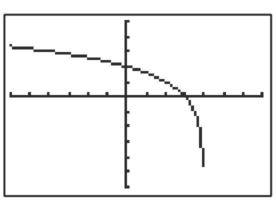
$b = 1000a$

**35.** Translate left 4 units.



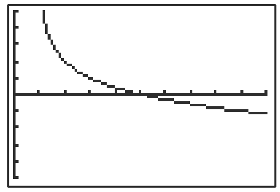
[−6, 7] by [−6, 5]

**36.** Reflect across $y$-axis and translate right 4 units — or translate left 4 units, then reflect across the $y$-axis.



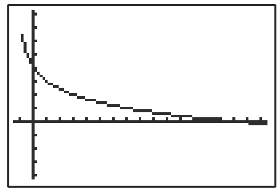
[−6, 7] by [−6, 5]

**37.** Translate right 1 unit, reflect across the $x$-axis, and translate up 2 units.



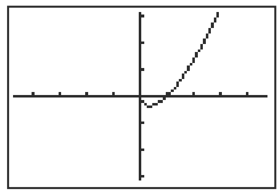
[0, 10] by [−5, 5]

**38.** Translate left 1 unit, reflect across the $x$-axis, and translate up 4 units.



[−1.4, 17.4] by [−4.2, 8.2]

**39.**



[–4.7, 4.7] by [–3.1, 3.1]

Domain: $(0, \infty)$

Range: $\left[-\dfrac{1}{e}, \infty\right) \approx [-0.37, \infty)$

Continuous

Decreasing on $(0, 0.37]$; increasing on $[0.37, \infty)$

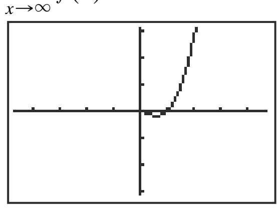
Not symmetric

Bounded below

Local minimum at $\left(\dfrac{1}{e}, -\dfrac{1}{e}\right)$

$\lim_{x\to\infty} f(x) = \infty$

**40.**



[–4.7, 4.7] by [–3.1, 3.1]

Domain: $(0, \infty)$

Range: $[-0.18, \infty)$

Continuous

Decreasing on $(0, 0.61]$; increasing on $[0.61, \infty)$

Not symmetric

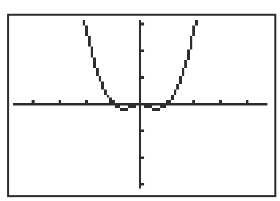
Bounded below

Local minimum at $(0.61, -0.18)$

No asymptotes

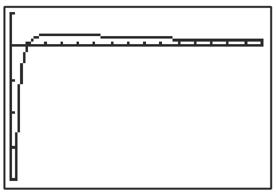
$\lim_{x\to\infty} f(x) = \infty$

**41.**



[–4.7, 4.7] by [–3.1, 3.1]

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $[-0.18, \infty)$
Discontinuous at $x = 0$
Decreasing on $(-\infty, -0.61]$, $(0, 0.61]$;
Increasing on $[-0.61, 0)$, $[0.61, \infty)$
Symmetric across $y$-axis
Bounded below
Local minima at $(-0.61, -0.18)$ and $(0.61, -0.18)$
No asymptotes
$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = \infty$

**42.** 

$[0, 15]$ by $[-4, 1]$

Domain: $(0, \infty)$
Range: $\left(-\infty, \frac{1}{e}\right] \approx (-\infty, 0.37]$
Continuous
Increasing on $(0, e] \approx (0, 2.72]$,
Decreasing $[e, \infty) \approx [2.72, \infty)$
Not symmetric
Bounded above
Local maximum at $\left(e, \frac{1}{e}\right) \approx (2.72, 0.37)$
Asymptotes: $y = 0$ and $x = 0$
$\lim_{x\to\infty} f(x) = 0$

**43.** $x = \log 4 \approx 0.6021$

**44.** $x = \ln 0.25 = -1.3863$

**45.** $x = \dfrac{\ln 3}{\ln 1.05} \approx 22.5171$

**46.** $x = e^{5.4} = 221.4064$

**47.** $x = 10^{-7} = 0.0000001$

**48.** $x = 3 + \dfrac{\ln 5}{\ln 3} \approx 4.4650$

**49.** $\log_2 x = 2$, so $x = 2^2 = 4$

**50.** $\log_3 x = \dfrac{7}{2}$, so $x = 3^{7/2} = 27\sqrt{3} \approx 46.7654$.

**51.** Multiply both sides by $2 \cdot 3^x$, leaving $(3^x)^2 - 1 = 10 \cdot 3^x$, or $(3^x)^2 - 10 \cdot 3^x - 1 = 0$. This is quadratic in $3^x$, leading to $3^x = \dfrac{10 \pm \sqrt{100 + 4}}{2} = 5 \pm \sqrt{26}$. Only $5 + \sqrt{26}$ is positive, so the only answer is $x = \log_3(5 + \sqrt{26}) \approx 2.1049$.

**52.** Multiply both sides by $4 + e^{2x}$, leaving $50 = 44 + 11e^{2x}$, so $11e^{2x} = 6$. Then $x = \dfrac{1}{2}\ln\dfrac{6}{11} \approx -0.3031$.

**53.** $\log[(x + 2)(x - 1)] = 4$, so $(x + 2)(x - 1) = 10^4$. The solutions to this quadratic equation are $x = \dfrac{1}{2}(-1 \pm \sqrt{40{,}009})$, but of these two numbers, only the positive one, $x = \dfrac{1}{2}(\sqrt{40{,}009} - 1) \approx 99.5112$, works in the original equation.

**54.** $\ln\dfrac{3x + 4}{2x + 1} = 5$, so $3x + 4 = e^5(2x + 1)$. Then $x = \dfrac{4 - e^5}{2e^5 - 3} \approx -0.4915$.

**55.** $\log_2 x = \dfrac{\ln x}{\ln 2}$

**56.** $\log_{1/6}(6x^2) = \log_{1/6} 6 + \log_{1/6} x^2 = \log_{1/6} 6 + 2\log_{1/6}|x|$
$= -1 + \dfrac{2\ln|x|}{\ln 1/6} = -1 + \dfrac{2\ln|x|}{\ln 6^{-1}} = -1 - \dfrac{2\ln|x|}{\ln 6}$

**57.** $\log_5 x = \dfrac{\log x}{\log 5}$

**58.** $\log_{1/2}(4x^3) = \log_{1/2} 4 + \log_{1/2} x^3 = -2 + 3\log_{1/2} x$
$= -2 - 3\log_2 x$
$= -2 - \dfrac{3\log x}{\log 2}$

**59.** Increasing, intercept at $(1, 0)$. The answer is (c).

**60.** Decreasing, intercept at $(1, 0)$. The answer is (d).

**61.** Intercept at $(-1, 0)$. The answer is (b).

**62.** Intercept at $(0, 1)$. The answer is (a).

**63.** $A = 450(1 + 0.046)^3 \approx \$515.00$

**64.** $A = 4800\left(1 + \dfrac{0.062}{4}\right)^{(4)(17)} \approx \$13{,}660.81$

**65.** $A = Pe^{rt}$

**66.** $i = \dfrac{r}{k}$, $n = kt$, so $FV = R \cdot \dfrac{\left(1 + \frac{r}{k}\right)^{kt} - 1}{\left(\frac{r}{k}\right)}$

**67.** $PV = \dfrac{550\left(1 - \left(1 + \frac{0.055}{12}\right)^{(-12)(5)}\right)}{\left(\frac{0.055}{12}\right)} \approx \$28{,}794.06$

**68.** $PV = \dfrac{953\left(1 - \left(1 + \frac{0.0725}{26}\right)^{(-26)(15)}\right)}{\left(\frac{0.0725}{26}\right)} \approx \$226{,}396.22$

**69.** $20e^{-3k} = 50$, so $k = -\dfrac{1}{3}\ln\dfrac{5}{2} \approx -0.3054$.

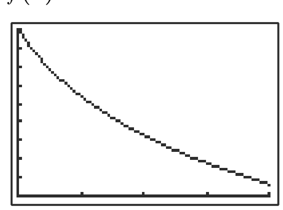
**70.** $20e^{-k} = 30$, so $k = -\ln\dfrac{3}{2} \approx -0.4055$.

**71.** $P(t) = 2.041979 \cdot 1.01296^t$, where $x$ is the number of years since 1900. In 2020, $P(120) = 2.041979 \cdot 1.012960^{120} \approx 9.57$ million.

**72.** $P(t) = \dfrac{14.586176}{1 + 2.043613e^{-0.024325t}}$, where $x$ is the number of years since 1900. In 2020, $P(120) \approx 13.14$ million.

**73.** **(a)** $f(0) = 90$ units.

**(b)** $f(2) \approx 32.8722$ units.

**(c)** 

$[0, 4]$ by $[0, 90]$

**74.** **(a)** $P(t) = 123{,}000(1 - 0.024)^t = 123{,}000(0.976)^t$.

**(b)** $P(t) = 90{,}000$ when $t = \dfrac{\ln(90/123)}{\ln 0.976} \approx 12.86$ years.

**75.** **(a)** $P(t) = 89{,}000(1 - 0.018)^t = 89{,}000(0.982)^t$.

**(b)** $P(t) = 50{,}000$ when $t = \dfrac{\ln(50/89)}{\ln 0.982} \approx 31.74$ years.

**76.** **(a)** $P(0) \approx 5.3959$ — 5 or 6 students.

**(b)** $P(3) \approx 80.6824$ — 80 or 81 students.

**(c)** $P(t) = 100$ when $1 + e^{4-t} = 3$, or $t = 4 - \ln 2$ $\approx 3.3069$ — sometime on the fourth day.

**(d)** As $t \to \infty$, $P(t) \to 300$.

**77.** **(a)** $P(t) = 20 \cdot 2^t$, where $t$ is time in months. (Other possible answers: $20 \cdot 2^{12t}$ if $t$ is in years, or $20 \cdot 2^{t/30}$ if $t$ is in days).

**(b)** $P(12) = 81{,}920$ rabbits after 1 year. $P(60) \approx 2.3058 \times 10^{19}$ rabbits after 5 years.

**(c)** Solve $20 \cdot 2^t = 10{,}000$ to find $t = \log_2 500$ $\approx 8.9658$ months — 8 months and about 29 days.

**78.** **(a)** $P(t) = 4 \cdot 2^t = 2^{t+2}$, where $t$ is time in days.

**(b)** $P(4) = 64$ guppies after 4 days. $P(7) = 512$ guppies after 1 week.

**(c)** Solve $4 \cdot 2^t = 2000$ to find $t = \log_2 500 = 8.9658$ days — 8 days and about 23 hours.

**79.** **(a)** $S(t) = S_0 \cdot \left(\dfrac{1}{2}\right)^{t/1.5}$, where $t$ is time in seconds.

**(b)** $S(1.5) = S_0/2$. $S(3) = S_0/4$.

**(c)** If $1\ g = S(60) = S_0 \cdot \left(\dfrac{1}{2}\right)^{60/1.5} = S0 \cdot \left(\dfrac{1}{2}\right)^{40}$, then

$S_0 = 2^{40} \approx 1.0995 \times 10^{12}\ g = 1.0995 \times 10^9$ kg $= 1{,}099{,}500$ metric tons.

**80.** **(a)** $S(t) = S_0 \cdot \left(\dfrac{1}{2}\right)^{t/2.5}$, where $t$ is time in seconds.

**(b)** $S(2.5) = S_0/2$. $S(7.5) = S_0/8$.

**(c)** If $1\ g = S(60) = S_0 \cdot \left(\dfrac{1}{2}\right)^{60/2.5} = S0 \cdot \left(\dfrac{1}{2}\right)^{24}$, then

$S_0 = 2^{24} = 16{,}777{,}216$ g $= 16{,}777.216$ kg.

**81.** Let $a_1$ = the amplitude of the ground motion of the Feb 4 quake, and let $a_2$ = the amplitude of the ground motion of the May 30 quake. Then:

$$6.1 = \log\frac{a_1}{T} + B \quad\text{and}\quad 6.9 = \log\frac{a_2}{T} + B$$

$$\left(\log\frac{a_2}{T} + B\right) - \left(\log\frac{a_1}{T} + B\right) = 6.9 - 6.1$$

$$\log\frac{a_2}{T} - \log\frac{a_1}{T} = 0.8$$

$$\log\frac{a_2}{a_1} = 0.8$$

$$\frac{a_2}{a_1} = 10^{0.8}$$

$$a_2 \approx 6.31\ a_1.$$

The ground amplitude of the deadlier quake was approximately 6.31 times stronger.

**82.** **(a)** Seawater:

$-\log[\mathrm{H}^+] = 7.6$

$\log[\mathrm{H}^+] = -7.6$

$[\mathrm{H}^+] = 10^{-7.6} \approx 2.51 \times 10^{-8}$

Milk of Magnesia:

$-\log[\mathrm{H}^+] = 10.5$

$\log[\mathrm{H}^+] = -10.5$

$[\mathrm{H}^+] = 10^{-10.5} \approx 3.16 \times 10^{-11}$

**(b)** $\dfrac{[\mathrm{H}^+]\text{ of Seawater}}{[\mathrm{H}^+]\text{ of Milk of Magnesia}} = \dfrac{10^{-7.6}}{10^{-10.5}} \approx 794.33$

**(c)** They differ by an order of magnitude of 2.9.

**83.** Solve $1500\left(1 + \dfrac{0.08}{4}\right)^{4t} = 3750$: $(1.02)^{4t} = 2.5$,

so $t = \dfrac{1}{4}\dfrac{\ln 2.5}{\ln 1.02} \approx 11.5678$ years — round to 11 years 9 months (the next full compounding period).

**84.** Solve $12{,}500e^{0.09t} = 37{,}500$: $e^{0.09t} = 3$,

so $t = \dfrac{1}{0.09}\ln 3 = 12.2068$ years.

**85.** $t = 133.83 \ln\dfrac{700}{250} \approx 137.7940$ — about 11 years 6 months.

**86.** $t = 133.83 \ln\dfrac{500}{50} \approx 308.1550$ — about 25 years 9 months.

**87.** $r = \left(1 + \dfrac{0.0825}{12}\right)^{12} - 1 \approx 8.57\%$

**88.** $r = e^{0.072} - 1 \approx 7.47\%$

**89.** $I = 12 \cdot 10^{(-0.0125)(25)} = 5.84$ lumens

**90.** $\log_b x = \dfrac{\ln x}{\ln b}$. This is a vertical stretch if $e^{-1} < b < e$ (so that $|\ln b| < 1$), and a shrink if $0 < b < e^{-1}$ or $b > e$. (There is also a reflection if $0 < b < 1$.)

**91.** $\log_b x = \dfrac{\log x}{\log b}$. This is a vertical stretch if $\dfrac{1}{10} < b < 10$ (so that $|\log b| < 1$), and a shrink if $0 < b < \dfrac{1}{10}$ or $b > 10$. (There is also a reflection if $0 < b < 1$.)

**92.** $g(x) = \ln[a \cdot b^x] = \ln a + \ln b^x = \ln a + x\ln b$. This has a slope $\ln b$ and $y$-intercept $\ln a$.

**93.** **(a)** $P(0) = 16$ students.

**(b)** $P(t) = 800$ when $1 + 99e^{-0.4t} = 2$, or $e^{0.4t} = 99$,

so $t = \dfrac{1}{0.4}\ln 99 \approx 11.4878$ — about $11\frac{1}{2}$ days.

**(c)** $P(t) = 400$ when $1 + 99e^{-0.4t} = 4$, or $e^{0.4t} = 33$,

so $t = \dfrac{1}{0.4}\ln 33 \approx 8.7413$ — about 8 or 9 days.

**94.** **(a)** $P(0) = 12$ deer.

**(b)** $P(t) = 1000$ when $1 + 99e^{-0.4t} = 1.2$, so

$t = -\dfrac{1}{0.4}\ln\dfrac{0.2}{99} \approx 15.5114$ — about $15\frac{1}{2}$ years.

**(c)** As $t \to \infty$, $P(t) \to 1200$ (and the population never rises above that level).

**95.** The model is $T = 20 + 76e^{-kt}$, and $T(8) = 65$ $= 20 + 76e^{-8k}$. Then $e^{-8k} = \dfrac{45}{76}$, so $k = -\dfrac{1}{8}\ln\dfrac{45}{76}$ $\approx 0.0655$. Finally, $T = 25$ when $25 = 20 + 76e^{-kt}$,

so $t = -\dfrac{1}{k}\ln\dfrac{5}{76} \approx 41.54$ minutes.

**96.** The model is $T = 75 + 145e^{-kt}$, and $T(35) = 150$ $= 75 + 145e^{-35k}$. Then $e^{-35k} = \frac{75}{145}$, so $k = -\frac{1}{35}\ln\frac{15}{29}$ $\approx 0.0188$. Finally, $T = 95$ when $95 = 75 + 145e^{-kt}$, so $t = -\frac{1}{k}\ln\frac{20}{145} \approx 105.17$ minutes.

**97.** **(a)** Matching up with the formula $S = R\frac{(1+i)^n - 1}{i}$, where $i = r/k$, with $r$ being the rate and $k$ being the number of payments per year, we find $r = 9\%$.

**(b)** $k = 4$ payments per year.

**(c)** Each payment is $R = \$100$.

**98.** **(a)** Matching up with the formula $A = R\frac{1 - (1+i)^{-n}}{i}$, where $i = r/k$, with $r$ being the rate and $k$ being the number of payments per year, we find $r = 11\%$.

**(b)** $k = 4$ payments per year.

**(c)** Each payment is $R = \$200$.

**99.** **(a)** Grace's balance will always remain \$1000, since interest is not added to it. Every year she receives 5% of that \$1000 in interest; after $t$ years, she has been paid $5t$% of the \$1000 investment, meaning that altogether she has $1000 + 1000 \cdot 0.05t = 1000(1 + 0.05t)$.

**(b)** The table is shown below; the second column gives values of $1000e^{0.05t}$. The effects of compounding continuously show up immediately.

| Years | Not Compounded | Compounded |
|---|---|---|
| 0 | 1000.00 | 1000.00 |
| 1 | 1050.00 | 1051.27 |
| 2 | 1100.00 | 1105.17 |
| 3 | 1150.00 | 1161.83 |
| 4 | 1200.00 | 1221.40 |
| 5 | 1250.00 | 1284.03 |
| 6 | 1300.00 | 1349.86 |
| 7 | 1350.00 | 1419.07 |
| 8 | 1400.00 | 1491.82 |
| 9 | 1450.00 | 1568.31 |
| 10 | 1500.00 | 1648.72 |

## Chapter 3 Project

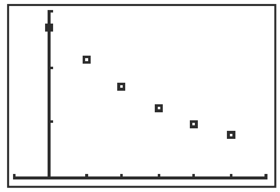
Answers are based on the sample data shown in the table.

**2.** Writing each maximum height as a (rounded) percentage of the previous maximum height produces the following table.

| Bounce Number | Percentage Return |
|---|---|
| 0 | N/A |
| 1 | 79% |
| 2 | 77% |
| 3 | 76% |
| 4 | 78% |
| 5 | 79% |

The average is 77.8%

**3.**



[–1, 6] by [0, 3]

**4.** Each successive height will be predicted by multiplying the previous height by the same percentage of rebound. The rebound height can therefore be predicted by the equation $y = HP^x$ where $x$ is the bounce number. From the sample data, $H = 2.7188$ and $P \approx 0.778$.

**5.** $y = HP^x$ becomes $y \approx 2.7188 \cdot 0.778^x$.

**6.** The regression equation is $y \approx 2.733 \cdot 0.776^x$. Both $H$ and $P$ are close to, though not identical with, the values in the earlier equation.

**7.** A different ball could be dropped from the same original height, but subsequent maximum heights would in general change because the rebound percentage changed. So $P$ would change in the equation.

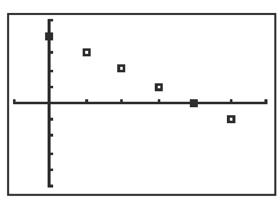
**8.** $H$ would be changed by varying the height from which the ball was dropped. $P$ would be changed by using a different type of ball or a different bouncing surface.

**9.**
$$\begin{aligned} y &= HP^x \\ &= H(e^{\ln P})^x \\ &= He^{(\ln P)x} \\ &= 2.7188\, e^{-0.251x} \end{aligned}$$

**10.**
$$\begin{aligned} \ln y &= \ln(HP^x) \\ &= \ln H + x\ln P \end{aligned}$$
This is a linear equation.

**11.**

| Bounce Number | ln (Height) |
|---|---|
| 0 | 1.0002 |
| 1 | 0.76202 |
| 2 | 0.50471 |
| 3 | 0.23428 |
| 4 | −0.01705 |
| 5 | −0.25125 |



[–1, 6] by [–1.25, 1.25]

The linear regression produces $Y = \ln y \approx -0.253x + 1.005$. Since $\ln y \approx (\ln P)x + \ln H$, the slope of the line is $\ln P$ and the $Y$-intercept (that is, the $\ln y$-intercept) is $\ln H$.