

## Chapter 2

### Limits and Continuity

#### Section 2.1 Rates of Change and Limits (pp. 59–69)

##### Quick Review 2.1

1.  $f(2) = 2(2^3) - 5(2)^2 + 4 = 0$

2.  $f(2) = \frac{4(2)^2 - 5}{2^3 + 4} = \frac{11}{12}$

3.  $f(2) = \sin\left(\pi \cdot \frac{2}{2}\right) = \sin \pi = 0$

4.  $f(2) = \frac{1}{2^2 - 1} = \frac{1}{3}$

5.  $|x| < 4$   
 $-4 < x < 4$

6.  $|x| < c^2$   
 $-c^2 < x < c^2$

7.  $|x-2| < 3$   
 $-3 < x-2 < 3$   
 $-1 < x < 5$

8.  $|x-c| < d^2$   
 $-d^2 < x-c < d^2$   
 $-d^2 + c < x < d^2 + c$

9.  $\frac{x^2 - 3x - 18}{x+3} = \frac{(x+3)(x-6)}{x+3} = x-6, x \neq -3$

10.  $\frac{2x^2 - x}{2x^2 + x - 1} = \frac{x(2x-1)}{(2x-1)(x+1)} = \frac{x}{x+1}, x \neq \frac{1}{2}$

#### Section 2.1 Exercises

1.  $\frac{\Delta y}{\Delta t} = \frac{16(3)^2 - 16(0)^2}{3-0} = 48 \text{ ft/sec}$

2.  $\frac{\Delta y}{\Delta t} = \frac{16(4)^2 - 16(0)^2}{4-0} = 64 \text{ ft/sec}$

3.  $\frac{\Delta y}{\Delta t} = \frac{16(3+h)^2 - 16(3)^2}{h}, \text{ say } h = 0.01$   
 $= \frac{16(3+0.01)^2 - 16(9)}{0.01}$   
 $= \frac{16(9.0601) - 16(9)}{0.01}$   
 $= \frac{144.9616 - 144}{0.01}$   
 $= \frac{0.9616}{0.01}$   
 $= 96.16 \text{ ft/sec}$

Confirm Algebraically

$$\frac{\Delta y}{\Delta t} = \frac{16(3+h)^2 - 16(3)^2}{h}$$

$$= \frac{16(9+6h+h^2) - 144}{h}$$

$$= \frac{96h+16h^2}{h}$$

$$= (96+16h) \text{ ft/sec}$$

if  $h = 0$ , then  $\frac{\Delta y}{\Delta t} = 96 \frac{\text{ft}}{\text{sec}}$

4.  $\frac{\Delta y}{\Delta t} = \frac{16(4+h)^2 - 16(4)^2}{h}, \text{ say } h = 0.01$   
 $\frac{16(4+0.01)^2 - 16(4)^2}{0.01} = \frac{16(16.0801) - 16(16)}{0.01}$   
 $= \frac{257.2816 - 256}{0.01}$   
 $= \frac{1.2816}{0.01}$   
 $= 128.16 \text{ ft/sec}$

Confirm Algebraically

$$\frac{\Delta y}{\Delta t} = \frac{16(4+h)^2 - 16(4)^2}{h}$$

$$= \frac{16(16+8h+h^2) - 256}{h}$$

$$= \frac{128h+16h^2}{h}$$

$$= (128+16h) \text{ ft/sec}$$

if  $h = 0$ , then  $\frac{\Delta y}{\Delta t} = 128 \frac{\text{ft}}{\text{sec}}$

5.  $\lim_{x \rightarrow c} (2x^3 - 3x^2 + x - 1) = 2c^3 - 3c^2 + c - 1$

6.  $\lim_{x \rightarrow c} \frac{x^4 - x^3 + 1}{x^2 + 9} = \frac{c^4 - c^3 + 1}{c^2 + 9}$

$$\begin{aligned}
 7. \quad \lim_{x \rightarrow -1/2} 3x^2(2x-1) &= 3\left(-\frac{1}{2}\right)^2 \left[2\left(-\frac{1}{2}\right) - 1\right] \\
 &= 3\left(\frac{1}{4}\right)(-2) \\
 &= -\frac{3}{2}
 \end{aligned}$$

$$8. \quad \lim_{x \rightarrow -4} (x+3)^{2016} = (-4+3)^{2016} = (-1)^{2016} = 1$$

$$\begin{aligned}
 9. \quad \lim_{x \rightarrow 1} (x^3 + 3x^2 - 2x - 17) \\
 &= (1)^3 + 3(1)^2 - 2(1) - 17 \\
 &= 1 + 3 - 2 - 17 \\
 &= -15
 \end{aligned}$$

$$10. \quad \lim_{y \rightarrow 2} \frac{y^2 + 5y + 6}{y + 2} = \frac{2^2 + 5(2) + 6}{2 + 2} = \frac{20}{4} = 5$$

$$11. \quad \lim_{y \rightarrow -3} \frac{y^2 + 4y + 3}{y^2 - 3} = \frac{(-3)^2 + 4(-3) + 3}{(-3)^2 - 3} = \frac{0}{6} = 0$$

$$12. \quad \lim_{x \rightarrow 1/2} \text{int } x = \text{int } \frac{1}{2} = 0$$

Note that substitution cannot always be used to find limits of the int function. Its use here can be justified by the Squeeze Theorem, using  $g(x) = h(x) = 0$  on the interval  $(0, 1)$ .

$$\begin{aligned}
 13. \quad \lim_{x \rightarrow -2} (x-6)^{2/3} &= (-2-6)^{2/3} \\
 &= \sqrt[3]{(-8)^2} \\
 &= \sqrt[3]{64} \\
 &= 4
 \end{aligned}$$

$$14. \quad \lim_{x \rightarrow 2} \sqrt{x+3} = \sqrt{2+3} = \sqrt{5}$$

<b>15. (a)</b>	$x$	-0.1	-0.01	-0.001	-0.0001
	$f(x)$	1.566667	1.959697	1.995997	1.999600

<b>(b)</b>	$x$	0.1	0.01	0.001	0.0001
	$f(x)$	2.372727	2.039703	2.003997	2.000400

The limit appears to be 2.

<b>16. (a)</b>	$x$	-0.1	-0.01	-0.001	-0.0001
	$f(x)$	-1.1	-1.01	-1.001	-1.0001

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(b)	$x$	0.1	0.01	0.001	0.0001
	$f(x)$	-0.9	-0.99	-0.999	-0.9999

The limit appears to be  $-1$ .

17. (a)	$x$	-0.1	-0.01	-0.001	-0.0001
	$f(x)$	-0.054402	-0.005064	-0.000827	-0.000031

(b)	$x$	0.1	0.01	0.001	0.0001
	$f(x)$	-0.054402	-0.005064	-0.000827	-0.000031

The limit appears to be  $0$ .

18. (a)	$x$	-0.1	-0.01	-0.001	-0.0001
	$f(x)$	0.5440	0.5064	-0.8269	0.3056

(b)	$x$	0.1	0.01	0.001	0.0001
	$f(x)$	-0.5440	-0.5064	0.8269	-0.3056

There is no clear indication of a limit.

19. (a)	$x$	-0.1	-0.01	-0.001	-0.0001
	$f(x)$	2.0567	2.2763	2.2999	2.3023

(b)	$x$	0.1	0.01	0.001	0.0001
	$f(x)$	2.5893	2.3293	2.3052	2.3029

The limit appears to be approximately  $2.3$ .

20. (a)	$x$	-0.1	-0.01	-0.001	-0.0001
	$f(x)$	0.074398	-0.009943	0.000585	0.000021

(b)	$x$	0.1	0.01	0.001	0.0001
	$f(x)$	-0.074398	0.009943	-0.000585	-0.000021

The limit appears to be  $0$ .

21. You cannot use substitution because the expression  $\sqrt{x-2}$  is not defined at  $x = -2$ . Since the expression is not defined at points near  $x = -2$ , the limit does not exist.

22. You cannot use substitution because the expression  $\frac{1}{x^2}$  is not defined at  $x = 0$ . Since  $\frac{1}{x^2}$  becomes arbitrarily large as  $x$  approaches  $0$  from either side, there is no (finite) limit. (As we shall see in Section 2.2, we may write  $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$ .)

23. You cannot use substitution because the expression  $\frac{|x|}{x}$  is not defined at  $x = 0$ . Since

$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$  and  $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$ , the left- and right-hand limits are not equal and so the limit does not exist.

24. You cannot use substitution because the expression  $\frac{(4+x)^2 - 16}{x}$  is not defined at

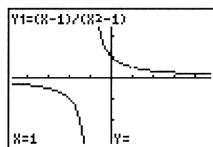
$x = 0$ . Since

$$\frac{(4+x)^2 - 16}{x} = \frac{8x + x^2}{x} = 8 + x \text{ for all } x \neq 0,$$

the limit exists and is equal to

$$\lim_{x \rightarrow 0} (8 + x) = 8 + 0 = 8.$$

25.



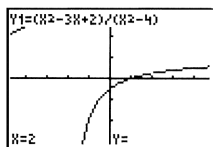
$[-4.7, 4.7]$  by  $[-3.1, 3.1]$

$$\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

Algebraic confirmation:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x-1}{x^2-1} &= \lim_{x \rightarrow 1} \frac{x-1}{(x+1)(x-1)} \\ &= \lim_{x \rightarrow 1} \frac{1}{x+1} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2} \end{aligned}$$

26.



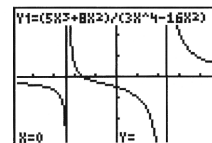
$[-4.7, 4.7]$  by  $[-3.1, 3.1]$

$$\lim_{t \rightarrow 2} \frac{t^2 - 3t + 2}{t^2 - 4} = \frac{1}{4}$$

Algebraic confirmation:

$$\begin{aligned} \lim_{t \rightarrow 2} \frac{t^2 - 3t + 2}{t^2 - 4} &= \lim_{t \rightarrow 2} \frac{(t-2)(t-1)}{(t-2)(t+2)} \\ &= \lim_{t \rightarrow 2} \frac{t-1}{t+2} \\ &= \frac{2-1}{2+2} \\ &= \frac{1}{4} \end{aligned}$$

27.



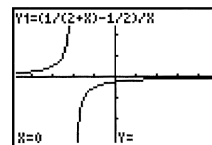
$[-4.7, 4.7]$  by  $[-3.1, 3.1]$

$$\lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} = -\frac{1}{2}$$

Algebraic confirmation:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} &= \lim_{x \rightarrow 0} \frac{x^2(5x+8)}{x^2(3x^2-16)} \\ &= \lim_{x \rightarrow 0} \frac{5x+8}{3x^2-16} \\ &= \frac{5(0)+8}{3(0)^2-16} \\ &= \frac{8}{-16} \\ &= -\frac{1}{2} \end{aligned}$$

28.

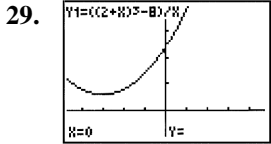


$[-4.7, 4.7]$  by  $[-3.1, 3.1]$

$$\lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = -\frac{1}{4}$$

Algebraic confirmation:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} &= \lim_{x \rightarrow 0} \frac{2 - (2+x)}{x(2+x)(2)} \\ &= \lim_{x \rightarrow 0} \frac{-x}{x(2+x)(2)} \\ &= \lim_{x \rightarrow 0} \frac{-1}{2(2+x)} \\ &= \frac{-1}{2(2+0)} \\ &= -\frac{1}{4} \end{aligned}$$

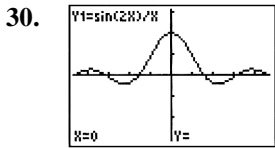


$[-4.7, 4.7]$  by  $[-5, 20]$

$$\lim_{x \rightarrow 0} \frac{(2+x)^3 - 8}{x} = 12$$

Algebraic confirmation:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{(2+x)^3 - 8}{x} &= \lim_{x \rightarrow 0} \frac{12x + 6x^2 + x^3}{x} \\ &= \lim_{x \rightarrow 0} (12 + 6x + x^2) \\ &= 12 + 6(0) + (0)^2 \\ &= 12 \end{aligned}$$

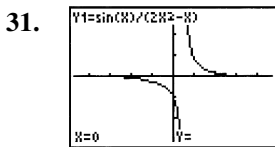


$[-4.7, 4.7]$  by  $[-3.1, 3.1]$

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2$$

Algebraic confirmation:

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2 \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} = 2(1) = 2$$

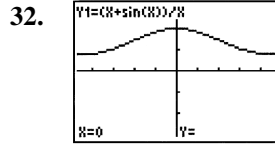


$[-4.7, 4.7]$  by  $[-3.1, 3.1]$

$$\lim_{x \rightarrow 0} \frac{\sin x}{2x^2 - x} = -1$$

Algebraic confirmation:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x}{2x^2 - x} &= \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \cdot \frac{1}{2x-1} \right) \\ &= \left( \lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \left( \lim_{x \rightarrow 0} \frac{1}{2x-1} \right) \\ &= (1) \left( \frac{1}{2(0)-1} \right) \\ &= -1 \end{aligned}$$

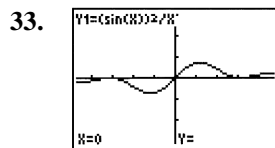


$[-4.7, 4.7]$  by  $[-3.1, 3.1]$

$$\lim_{x \rightarrow 0} \frac{x + \sin x}{x} = 2$$

Algebraic confirmation:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x + \sin x}{x} &= \lim_{x \rightarrow 0} \left( 1 + \frac{\sin x}{x} \right) \\ &= \left( \lim_{x \rightarrow 0} 1 \right) + \left( \lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

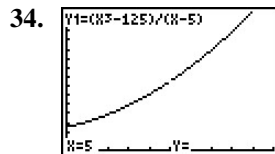


$[-4.7, 4.7]$  by  $[-3.1, 3.1]$

$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = 0$$

Algebraic confirmation:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin^2 x}{x} &= \lim_{x \rightarrow 0} \left( \sin x \cdot \frac{\sin x}{x} \right) \\ &= \left( \lim_{x \rightarrow 0} \sin x \right) \cdot \left( \lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \\ &= (\sin 0)(1) \\ &= 0 \end{aligned}$$



$[0, 10]$  by  $[0, 150]$

$$\lim_{x \rightarrow 5} \frac{x^3 - 125}{x - 5} = 75$$

Algebraic confirmation:

$$\begin{aligned} \lim_{x \rightarrow 5} \frac{x^3 - 125}{x - 5} &= \lim_{x \rightarrow 5} \frac{(x-5)(x^2 + 5x + 25)}{x - 5} \\ &= \lim_{x \rightarrow 5} (x^2 + 5x + 25) \\ &= (5)^2 + 5(5) + 25 \\ &= 75 \end{aligned}$$

35. Answers will vary. One possible graph is given by the window  $[-4.7, 4.7]$  by  $[-15, 15]$  with  $Xscl = 1$  and  $Yscl = 5$ .

36. Answers will vary. One possible graph is given by the window  $[-4.7, 4.7]$  by  $[-15, 15]$  with  $Xscl = 1$  and  $Yscl = 5$ .
37. Since  $\text{int } x = 0$  for  $x$  in  $(0, 1)$ ,  $\lim_{x \rightarrow 0^+} \text{int } x = 0$ .
38. Since  $\text{int } x = -1$  for  $x$  in  $(-1, 0)$ ,  $\lim_{x \rightarrow 0^-} \text{int } x = -1$ .
39. Since  $\text{int } x = 0$  for  $x$  in  $(0, 1)$ ,  $\lim_{x \rightarrow 0.01} \text{int } x = 0$ .
40. Since  $\text{int } x = 1$  for  $x$  in  $(1, 2)$ ,  $\lim_{x \rightarrow 2^-} \text{int } x = 1$ .
41. Since  $\frac{x}{|x|} = 1$  for  $x > 0$ ,  $\lim_{x \rightarrow 0^+} \frac{x}{|x|} = 1$ .
42. Since  $\frac{x}{|x|} = -1$  for  $x < 0$ ,  $\lim_{x \rightarrow 0^-} \frac{x}{|x|} = -1$ .
43. (a) True  
 (b) True  
 (c) False, since  $\lim_{x \rightarrow 0^-} f(x) = 0$ .  
 (d) True, since both are equal to 0.  
 (e) True, since (d) is true.  
 (f) True  
 (g) False, since  $\lim_{x \rightarrow 0} f(x) = 0$ .  
 (h) False,  $\lim_{x \rightarrow 1^-} f(x) = 1$ , but  $\lim_{x \rightarrow 1} f(x)$  is undefined.  
 (i) False,  $\lim_{x \rightarrow 1^+} f(x) = 0$ , but  $\lim_{x \rightarrow 1} f(x)$  is undefined.  
 (j) False, since  $\lim_{x \rightarrow 2^-} f(x) = 0$ .
44. (a) True  
 (b) False, since  $\lim_{x \rightarrow 2} f(x) = 1$ .  
 (c) False, since  $\lim_{x \rightarrow 2} f(x) = 1$ .
- (d) True  
 (e) True  
 (f) True, since  $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$ .  
 (g) True, since both are equal to 0.  
 (h) True  
 (i) True, since  $\lim_{x \rightarrow c} f(x) = 1$  for all  $c$  in  $(1, 3)$ .
45. (a)  $\lim_{x \rightarrow 3^-} f(x) = 3$   
 (b)  $\lim_{x \rightarrow 3^+} f(x) = -2$   
 (c)  $\lim_{x \rightarrow 3} f(x)$  does not exist, because the left- and right-hand limits are not equal.  
 (d)  $f(3) = 1$
46. (a)  $\lim_{t \rightarrow -4^-} g(t) = 5$   
 (b)  $\lim_{t \rightarrow -4^+} g(t) = 2$   
 (c)  $\lim_{t \rightarrow -4} g(t)$  does not exist, because the left- and right-hand limits are not equal.  
 (d)  $g(-4) = 2$
47. (a)  $\lim_{h \rightarrow 0^-} f(h) = -4$   
 (b)  $\lim_{h \rightarrow 0^+} f(h) = -4$   
 (c)  $\lim_{h \rightarrow 0} f(h) = -4$   
 (d)  $f(0) = -4$
48. (a)  $\lim_{s \rightarrow -2^-} p(s) = 3$   
 (b)  $\lim_{s \rightarrow -2^+} p(s) = 3$   
 (c)  $\lim_{s \rightarrow -2} p(s) = 3$   
 (d)  $p(-2) = 3$

49. (a)  $\lim_{x \rightarrow 0^-} F(x) = 4$

(b)  $\lim_{x \rightarrow 0^+} F(x) = -3$

(c)  $\lim_{x \rightarrow 0} F(x)$  does not exist, because the left- and right- hand limits are not equal.

(d)  $F(0) = 4$

50. (a)  $\lim_{x \rightarrow 2^-} G(x) = 1$

(b)  $\lim_{x \rightarrow 2^+} G(x) = 1$

(c)  $\lim_{x \rightarrow 2} G(x) = 1$

(d)  $G(2) = 3$

51.  $y_1 = \frac{x^2 + x - 2}{x - 1} = \frac{(x - 1)(x + 2)}{x - 1} = x + 2, x \neq 1$

(c)

52.  $y_1 = \frac{x^2 - x - 2}{x - 1} = \frac{(x + 1)(x - 2)}{x - 1}$

(b)

53.  $y_1 = \frac{x^2 - 2x + 1}{x - 1} = \frac{(x - 1)^2}{x - 1} = x - 1, x \neq 1$

(d)

54.  $y_1 = \frac{x^2 + x - 2}{x + 1} = \frac{(x - 1)(x + 2)}{x + 1}$

(a)

55. (a)  $\lim_{x \rightarrow 4} (g(x) + 3) = \left( \lim_{x \rightarrow 4} g(x) \right) + \left( \lim_{x \rightarrow 4} 3 \right)$   
 $= 3 + 3$   
 $= 6$

(b)  $\lim_{x \rightarrow 4} x f(x) = \left( \lim_{x \rightarrow 4} x \right) \left( \lim_{x \rightarrow 4} f(x) \right)$   
 $= 4 \cdot 0$   
 $= 0$

(c)  $\lim_{x \rightarrow 4} g^2(x) = \left( \lim_{x \rightarrow 4} g(x) \right)^2 = 3^2 = 9$

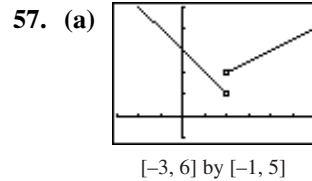
(d)  $\lim_{x \rightarrow 4} \frac{g(x)}{f(x) - 1} = \frac{\lim_{x \rightarrow 4} g(x)}{\left( \lim_{x \rightarrow 4} f(x) \right) - \left( \lim_{x \rightarrow 4} 1 \right)}$   
 $= \frac{3}{0 - 1}$   
 $= -3$

56. (a)  $\lim_{x \rightarrow b} (f(x) + g(x))$   
 $= \left( \lim_{x \rightarrow b} f(x) \right) + \left( \lim_{x \rightarrow b} g(x) \right)$   
 $= 7 + (-3)$   
 $= 4$

(b)  $\lim_{x \rightarrow b} (f(x) \cdot g(x))$   
 $= \left( \lim_{x \rightarrow b} f(x) \right) \left( \lim_{x \rightarrow b} g(x) \right)$   
 $= (7) (-3)$   
 $= -21$

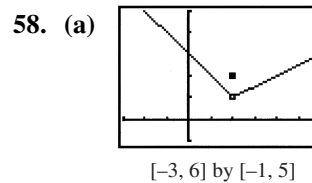
(c)  $\lim_{x \rightarrow b} 4g(x) = 4 \lim_{x \rightarrow b} g(x) = 4(-3) = -12$

(d)  $\lim_{x \rightarrow b} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow b} f(x)}{\lim_{x \rightarrow b} g(x)} = \frac{7}{-3} = -\frac{7}{3}$



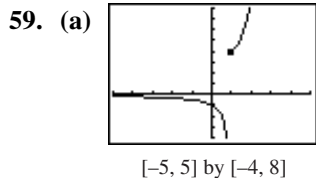
(b)  $\lim_{x \rightarrow 2^+} f(x) = 2; \lim_{x \rightarrow 2^-} f(x) = 1$

(c) No, because the two one-sided limits are different.

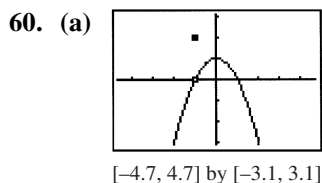


(b)  $\lim_{x \rightarrow 2^+} f(x) = 1; \lim_{x \rightarrow 2^-} f(x) = 1$

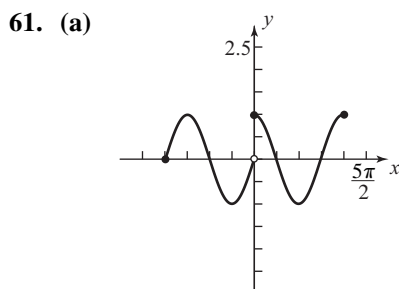
(c) Yes; the limit is 1.



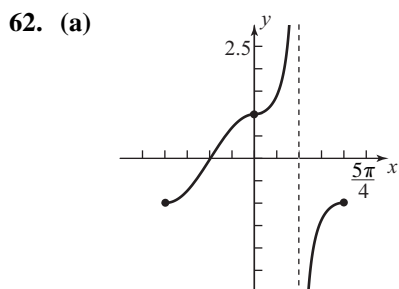
- (b)  $\lim_{x \rightarrow 1^+} f(x) = 4$ ;  $\lim_{x \rightarrow 1^-} f(x)$  does not exist.  
 (c) No, because the left-hand limit does not exist.



- (b)  $\lim_{x \rightarrow -1^+} f(x) = 0$ ;  $\lim_{x \rightarrow -1^-} f(x) = 0$   
 (c) Yes; the limit is 0.

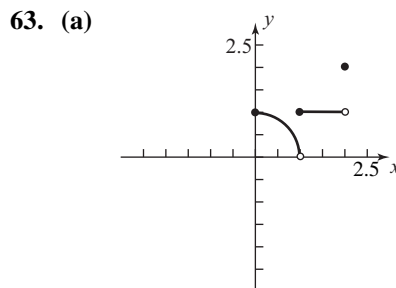


- (b)  $(-2\pi, 0) \cup (0, 2\pi)$   
 (c)  $c = 2\pi$   
 (d)  $c = -2\pi$

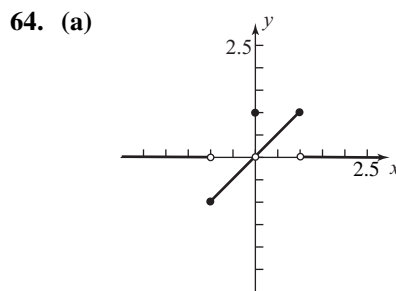


- (b)  $\left(-\pi, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

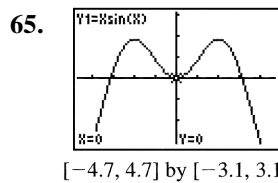
- (c)  $c = \pi$   
 (d)  $c = -\pi$



- (b)  $(0, 1) \cup (1, 2)$   
 (c)  $c = 2$   
 (d)  $c = 0$



- (b)  $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$   
 (c) None  
 (d) None



$\lim_{x \rightarrow 0} (x \sin x) = 0$

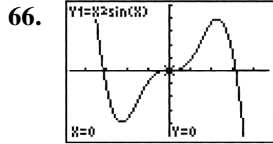
Confirm using the Squeeze Theorem, with  $g(x) = -|x|$  and  $h(x) = |x|$ .

$|x \sin x| = |x| \cdot |\sin x| \leq |x| \cdot 1 = |x|$   
 $-|x| \leq x \sin x \leq |x|$

Because  $\lim_{x \rightarrow 0} (-|x|) = \lim_{x \rightarrow 0} |x| = 0$ , the Squeeze

Theorem gives  $\lim_{x \rightarrow 0} (x \sin x) = 0$ .





$[-4.7, 4.7]$  by  $[-5, 5]$

$$\lim_{x \rightarrow 0} (x^2 \sin x) = 0$$

Confirm using the Squeeze Theorem, with

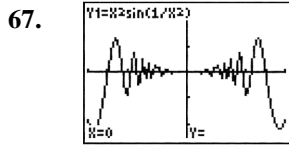
$$g(x) = -x^2 \text{ and } h(x) = x^2.$$

$$|x^2 \sin x| = |x^2| \cdot |\sin x| \leq |x^2| \cdot 1 = x^2.$$

$$-x^2 \leq x^2 \sin x \leq x^2$$

Because  $\lim_{x \rightarrow 0} (-x^2) = \lim_{x \rightarrow 0} x^2 = 0$ , the

Squeeze Theorem gives  $\lim_{x \rightarrow 0} (x^2 \sin x) = 0$



$[-0.5, 0.5]$  by  $[-0.25, 0.25]$

$$\lim_{x \rightarrow 0} \left( x^2 \sin \frac{1}{x^2} \right) = 0$$

Confirm using the Squeeze Theorem, with

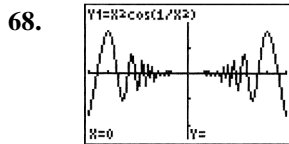
$$g(x) = -x^2 \text{ and } h(x) = x^2.$$

$$\left| x^2 \sin \frac{1}{x^2} \right| = |x^2| \cdot \left| \sin \frac{1}{x^2} \right| \leq |x^2| \cdot 1 = x^2.$$

$$-x^2 \leq x^2 \sin \frac{1}{x^2} \leq x^2$$

Because  $\lim_{x \rightarrow 0} (-x^2) = \lim_{x \rightarrow 0} x^2 = 0$ , the Squeeze

Theorem gives  $\lim_{x \rightarrow 0} \left( x^2 \sin \frac{1}{x^2} \right) = 0$ .



$[-0.5, 0.5]$  by  $[-0.25, 0.25]$

$$\lim_{x \rightarrow 0} \left( x^2 \cos \frac{1}{x^2} \right) = 0$$

Confirm using the Squeeze Theorem, with

$$g(x) = -x^2 \text{ and } h(x) = x^2.$$

$$\left| x^2 \cos \frac{1}{x^2} \right| = |x^2| \cdot \left| \cos \frac{1}{x^2} \right| \leq |x^2| \cdot 1 = x^2.$$

$$-x^2 \leq x^2 \cos \frac{1}{x^2} \leq x^2$$

Because  $\lim_{x \rightarrow 0} (-x^2) = \lim_{x \rightarrow 0} x^2 = 0$ , the Squeeze

Theorem gives  $\lim_{x \rightarrow 0} \left( x^2 \cos \frac{1}{x^2} \right) = 0$ .

69. (a) In three seconds, the ball falls

$4.9(3)^2 = 44.1$  m, so its average speed is

$$\frac{44.1}{3} = 14.7 \text{ m/sec.}$$

(b) The average speed over the interval from time  $t = 3$  to time  $3 + h$  is

$$\begin{aligned} \frac{\Delta y}{\Delta t} &= \frac{4.9(3+h)^2 - 4.9(3)^2}{(3+h) - 3} \\ &= \frac{4.9(6h + h^2)}{h} \\ &= 29.4 + 4.9h \end{aligned}$$

Since  $\lim_{h \rightarrow 0} (29.4 + 4.9h) = 29.4$ , the

instantaneous speed is 29.4 m/sec.

70. (a)  $y = gt^2$

$$20 = g(4^2)$$

$$g = \frac{20}{16} = \frac{5}{4} \text{ or } 1.25$$

(b) Average speed  $= \frac{20}{4} = 5$  m/sec

(c) If the rock had not been stopped, its average speed over the interval from time  $t = 4$  to time  $t = 4 + h$  is

$$\begin{aligned} \frac{\Delta y}{\Delta t} &= \frac{1.25(4+h)^2 - 1.25(4)^2}{(4+h) - 4} \\ &= \frac{1.25(8h + h^2)}{h} \\ &= 10 + 1.25h \end{aligned}$$

Since  $\lim_{h \rightarrow 0} (10 + 1.25h) = 10$ , the

instantaneous speed is 10 m/sec.

71. True; the definition of a limit.

72. True

$$\begin{aligned}\lim_{x \rightarrow 0} \left( \frac{x + \sin x}{x} \right) &= \lim_{x \rightarrow 0} \left( 1 + \frac{\sin x}{x} \right) \\ &= 1 + \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= 2\end{aligned}$$

 $\sin x \approx x$  as  $x \rightarrow 0$ .

73. C

74. B

75. E

76. C

77. (a) Because the right-hand limit at zero depends only on the values of the function for positive  $x$ -values near zero.

$$\begin{aligned}\text{(b) Area of } \triangle OAP &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(1)(\sin \theta) \\ &= \frac{\sin \theta}{2}\end{aligned}$$

$$\begin{aligned}\text{Area of sector } OAP &= \frac{(\text{angle})(\text{radius})^2}{2} \\ &= \frac{\theta(1)^2}{2} \\ &= \frac{\theta}{2}\end{aligned}$$

$$\begin{aligned}\text{Area of } \triangle OAT &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(1)(\tan \theta) \\ &= \frac{\tan \theta}{2}\end{aligned}$$

(c) This is how the areas of the three regions compare.

(d) Multiply by 2 and divide by  $\sin \theta$ .

(e) Take reciprocals, remembering that all of the values involved are positive.

(f) The limits for  $\cos \theta$  and 1 are both equal to 1. Since  $\frac{\sin \theta}{\theta}$  is between them, it must also have a limit of 1.

$$\text{(g) } \frac{\sin(-\theta)}{-\theta} = \frac{-\sin \theta}{-\theta} = \frac{\sin \theta}{\theta}$$

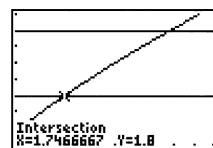
(h) If the function is symmetric about the  $y$ -axis, and the right-hand limit at zero is 1, then the left-hand limit at zero must also be 1.

(i) The two one-sided limits both exist and are equal to 1.

78. (a) The limit can be found by substitution.

$$\lim_{x \rightarrow 2} f(x) = f(2) = \sqrt{3(2) - 2} = \sqrt{4} = 2$$

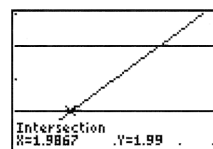
(b) The graphs of  $y_1 = f(x)$ ,  $y_2 = 1.8$ , and  $y_3 = 2.2$  are shown.



[1.5, 2.5] by [1.5, 2.3]

The intersections of  $y_1$  with  $y_2$  and  $y_3$  are at  $x = 1.7467$  and  $x = 2.28$ , respectively, so we may choose any value of  $a$  in  $[1.7467, 2)$  (approximately) and any value of  $b$  in  $(2, 2.28]$ . One possible answer:  $a = 1.75$ ,  $b = 2.28$ .

(c) The graphs of  $y_1 = f(x)$ ,  $y_2 = 1.99$ , and  $y_3 = 2.01$  are shown.

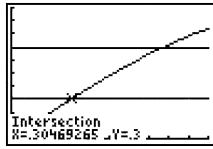


[1.97, 2.03] by [1.98, 2.02]

The intersections of  $y_1$  with  $y_2$  and  $y_3$  are at  $x = 1.9867$  and  $x = 2.0134$ , respectively, so we may choose any value of  $a$  in  $[1.9867, 2)$ , and any value of  $b$  in  $(2, 2.0134]$  (approximately). One possible answer:  $a = 1.99$ ,  $b = 2.01$ .

$$\text{79. (a) } f\left(\frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2}$$

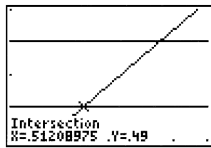
- (b) The graphs of  $y_1 = f(x)$ ,  $y_2 = 0.3$ , and  $y_3 = 0.7$  are shown.



$[0, 1]$  by  $[0, 1]$

The intersections of  $y_1$  with  $y_2$  and  $y_3$  are at  $x = 0.3047$  and  $x = 0.7754$ , respectively, so we may choose any value of  $a$  in  $\left[0.3047, \frac{\pi}{6}\right)$ , and any value of  $b$  in  $\left[\frac{\pi}{6}, 0.7754\right]$ , where the interval endpoints are approximate. One possible answer:  $a = 0.305$ ,  $b = 0.775$ .

- (c) The graphs of  $y_1 = f(x)$ ,  $y_2 = 0.49$ , and  $y_3 = 0.51$  are shown.



$[0.49, 0.55]$  by  $[0.48, 0.52]$

The intersections of  $y_1$  with  $y_2$  and  $y_3$  are at  $x = 0.5121$  and  $x = 0.5352$ , respectively, so we may choose any value of  $a$  in  $\left[0.5121, \frac{\pi}{6}\right)$ , and any value of  $b$  in  $\left[\frac{\pi}{6}, 0.5352\right]$ , where the interval endpoints are approximate. One possible answer:  $a = 0.513$ ,  $b = 0.535$ .

80. Line segment  $OP$  has endpoints  $(0, 0)$  and  $(a, a^2)$ , so its midpoint is

$$\left(\frac{0+a}{2}, \frac{0+a^2}{2}\right) = \left(\frac{a}{2}, \frac{a^2}{2}\right)$$
 and its slope is

$$\frac{a^2 - 0}{a - 0} = a.$$
 The perpendicular bisector is the

line through  $\left(\frac{a}{2}, \frac{a^2}{2}\right)$  with slope  $-\frac{1}{a}$ , so its

equation is  $y = -\frac{1}{a}\left(x - \frac{a}{2}\right) + \frac{a^2}{2}$ , which is

equivalent to  $y = -\frac{1}{a}x + \frac{1+a^2}{2}$ . Thus the

$y$ -intercept is  $b = \frac{1+a^2}{2}$ . As the point  $P$

approaches the origin along the parabola, the value of  $a$  approaches zero. Therefore,

$$\lim_{P \rightarrow O} b = \lim_{a \rightarrow 0} \frac{1+a^2}{2} = \frac{1+0^2}{2} = \frac{1}{2}.$$

**Section 2.2** Limits Involving Infinity  
(pp. 70–77)

**Exploration 1** Exploring Theorem 5

1. Neither  $\lim_{x \rightarrow \infty} f(x)$  or  $\lim_{x \rightarrow \infty} g(x)$  exist. In this case, we can describe the behavior of  $f$  and  $g$  as  $x \rightarrow \infty$  by writing  $\lim_{x \rightarrow \infty} f(x) = \infty$  and

$\lim_{x \rightarrow \infty} g(x) = \infty$ . We cannot apply the quotient

rule because both limits must exist. However, from Example 5,

$$\lim_{x \rightarrow \infty} \frac{5x + \sin x}{x} = \lim_{x \rightarrow \infty} \left(5 + \frac{\sin x}{x}\right) = 5 + 0 = 5,$$

so the limit of the quotient exists.

2. Both  $f$  and  $g$  oscillate between 0 and 1 as  $x \rightarrow \infty$ , taking on each value infinitely often. We cannot apply the sum rule because neither limit exists. However,

$$\lim_{x \rightarrow \infty} (\sin^2 x + \cos^2 x) = \lim_{x \rightarrow \infty} (1) = 1,$$

so the limit of the sum exists.

3. The limit of  $f$  and  $g$  as  $x \rightarrow \infty$  do not exist, so we cannot apply the difference rule to  $f - g$ . We can say that  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty$ .

We can write the difference as

$$f(x) - g(x) = \ln(2x) - \ln(x+1) = \ln \frac{2x}{x+1}.$$
 We

can use graphs or tables to convince ourselves that this limit is equal to  $\ln 2$ .

4. The fact that the limits of  $f$  and  $g$  as  $x \rightarrow \infty$  do not exist does not necessarily mean that the limits of  $f + g$ ,  $f - g$  or  $\frac{f}{g}$  do not exist, just that Theorem 5 cannot be applied.

**Quick Review 2.2**

1.  $y = 2x - 3$

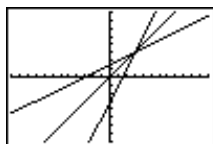
$y + 3 = 2x$

$\frac{y + 3}{2} = x$

Interchange  $x$  and  $y$ .

$\frac{x + 3}{2} = y$

$f^{-1}(x) = \frac{x + 3}{2}$



[-12, 12] by [-8, 8]

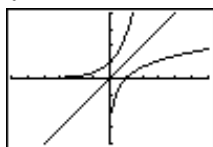
2.  $y = e^x$

$\ln y = x$

Interchange  $x$  and  $y$ .

$\ln x = y$

$f^{-1}(x) = \ln x$



[-6, 6] by [-4, 4]

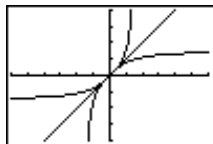
3.  $y = \tan^{-1} x$

$\tan y = x, -\frac{\pi}{2} < y < \frac{\pi}{2}$

Interchange  $x$  and  $y$ .

$\tan x = y, -\frac{\pi}{2} < x < \frac{\pi}{2}$

$f^{-1}(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$



[-6, 6] by [-4, 4]

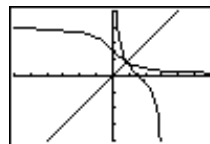
4.  $y = \cot^{-1} x$

$\cot y = x, 0 < y < \pi$

Interchange  $x$  and  $y$ .

$\cot x = y, 0 < x < \pi$

$f^{-1}(x) = \cot x, 0 < x < \pi$



[-6, 6] by [-4, 4]

$$5. \begin{array}{r} 3x^3 + 4x - 5 \overline{) 2x^3 - 3x^2 + x - 1} \\ \underline{2x^3 + 0x^2 + \frac{8}{3}x - \frac{10}{3}} \\ -3x^2 - \frac{5}{3}x + \frac{7}{3} \end{array}$$

$q(x) = \frac{2}{3}$

$r(x) = -3x^2 - \frac{5}{3}x + \frac{7}{3}$

$$6. \begin{array}{r} x^3 - x^2 + 1 \overline{) 2x^5 + 0x^4 - x^3 + 0x^2 + x - 1} \\ \underline{2x^5 - 2x^4 + 0x^3 + 2x^2} \\ 2x^4 - x^3 - 2x^2 + x - 1 \\ \underline{2x^4 - 2x^3 + 0x^2 + 2x} \\ x^3 - 2x^2 - x - 1 \\ \underline{x^3 - x^2 + 0x + 1} \\ -x^2 - x - 2 \end{array}$$

$q(x) = 2x^2 + 2x + 1$

$r(x) = -x^2 - x - 2$

7. (a)  $f(-x) = \cos(-x) = \cos x$

(b)  $f\left(\frac{1}{x}\right) = \cos\left(\frac{1}{x}\right)$

8. (a)  $f(-x) = e^{-(-x)} = e^x$

(b)  $f\left(\frac{1}{x}\right) = e^{-1/x}$

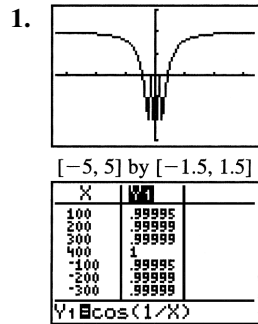
9. (a)  $f(-x) = \frac{\ln|-x|}{-x} = -\frac{\ln|x|}{x}$

(b)  $f\left(\frac{1}{x}\right) = \frac{\ln\left|\frac{1}{x}\right|}{\frac{1}{x}} = x \ln\left|\frac{1}{x}\right| = -x \ln|x|$

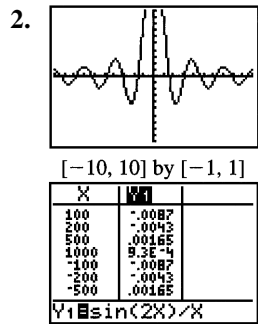
10. (a)  $f(-x) = \left(-x + \frac{1}{-x}\right) \sin(-x)$   
 $= -\left(x + \frac{1}{x}\right) (-\sin x)$   
 $= \left(x + \frac{1}{x}\right) \sin x$

(b)  $f\left(\frac{1}{x}\right) = \left(\frac{1}{x} + \frac{1}{\frac{1}{x}}\right) \sin\left(\frac{1}{x}\right)$   
 $= \left(\frac{1}{x} + x\right) \sin\left(\frac{1}{x}\right)$

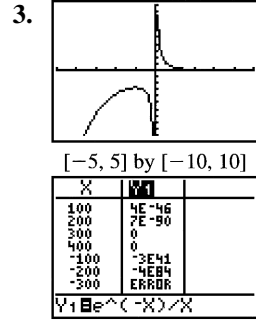
Section 2.2 Exercises



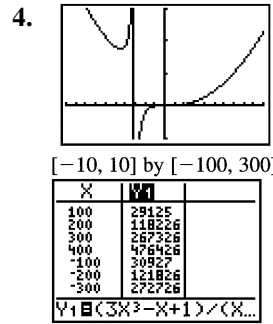
- (a)  $\lim_{x \rightarrow \infty} f(x) = 1$
- (b)  $\lim_{x \rightarrow -\infty} f(x) = 1$
- (c)  $y = 1$



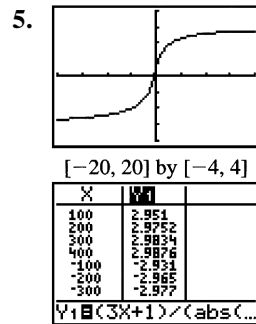
- (a)  $\lim_{x \rightarrow \infty} f(x) = 0$
- (b)  $\lim_{x \rightarrow -\infty} f(x) = 0$
- (c)  $y = 0$



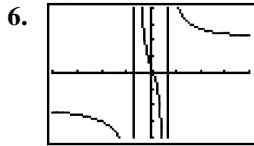
- (a)  $\lim_{x \rightarrow \infty} f(x) = 0$
- (b)  $\lim_{x \rightarrow -\infty} f(x) = -\infty$
- (c)  $y = 0$



- (a)  $\lim_{x \rightarrow \infty} f(x) = \infty$
- (b)  $\lim_{x \rightarrow -\infty} f(x) = \infty$
- (c) No horizontal asymptotes.



- (a)  $\lim_{x \rightarrow \infty} f(x) = 3$
- (b)  $\lim_{x \rightarrow -\infty} f(x) = -3$
- (c)  $y = 3, y = -3$



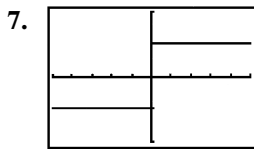
[-20, 20] by [-4, 4]

X	f(X)
100	2.0515
200	2.0254
500	2.0101
1000	2.005
-100	-2.072
-200	-2.036
-500	-2.014

(a)  $\lim_{x \rightarrow \infty} f(x) = 2$

(b)  $\lim_{x \rightarrow -\infty} f(x) = -2$

(c)  $y = 2, y = -2$



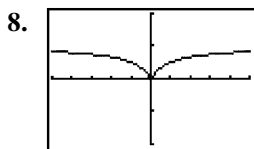
[-5, 5] by [-2, 2]

X	f(X)
100	1
200	1
500	1
1000	1
-100	-1
-200	-1
-500	-1

(a)  $\lim_{x \rightarrow \infty} f(x) = 1$

(b)  $\lim_{x \rightarrow -\infty} f(x) = -1$

(c)  $y = 1, y = -1$



[-5, 5] by [-2, 2]

X	f(X)
100	.9901
200	.99502
500	.998
1000	.999
-100	.9901
-200	.99502
-500	.998

(a)  $\lim_{x \rightarrow \infty} f(x) = 1$

(b)  $\lim_{x \rightarrow -\infty} f(x) = 1$

(c)  $y = 1$

9.  $0 \leq 1 - \cos x \leq 2$ . So, for  $x > 0$  we have

$$0 \leq \frac{1 - \cos x}{x^2} \leq \frac{2}{x^2}.$$

By the Squeeze Theorem,

$$0 = \lim_{x \rightarrow \infty} (0) = \lim_{x \rightarrow \infty} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow \infty} \frac{2}{x^2} = 0.$$

10.  $0 \leq 1 - \cos x \leq 2$ . So, for  $x > 0$  we have

$$0 \leq \frac{1 - \cos x}{x^2} \leq \frac{2}{x^2}.$$
 By the Squeeze Theorem,

$$0 = \lim_{x \rightarrow -\infty} (0) = \lim_{x \rightarrow -\infty} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow -\infty} \frac{2}{x^2} = 0.$$

11.  $-1 \leq \sin x \leq 1$ , so for  $x < 0$ ,  $\frac{1}{x} \leq \frac{\sin x}{x} \leq -\frac{1}{x}$ .

Therefore

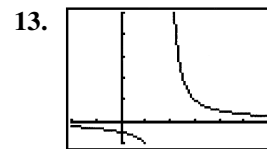
$$\lim_{x \rightarrow -\infty} \frac{1}{x} = \lim_{x \rightarrow -\infty} \frac{\sin x}{x} = \lim_{x \rightarrow -\infty} \left(-\frac{1}{x}\right) = 0$$

by the Squeeze Theorem.

12.  $-1 \leq \sin(x^2) \leq 1$ , so for  $x > 0$ ,

$$-\frac{1}{x} \leq \frac{\sin(x^2)}{x} \leq \frac{1}{x}$$

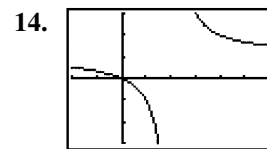
$$\lim_{x \rightarrow \infty} \frac{-1}{x} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0 = \lim_{x \rightarrow \infty} \frac{\sin(x^2)}{x}.$$



[-2, 6] by [-1, 5]

X	f(X)
2.8	1.25
2.4	2.5
2.2	5
2.1	10
2.01	100
2.001	1000
2.0001	10000

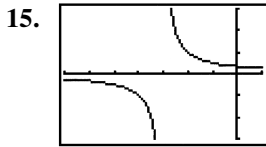
$$\lim_{x \rightarrow 2^+} \frac{1}{x-2} = \infty$$



[-2, 6] by [-3, 3]

X	Y1
1.2	-1.5
1.6	-4
1.8	-9
1.9	-19
1.99	-199
1.999	-1999
1.9999	-19999

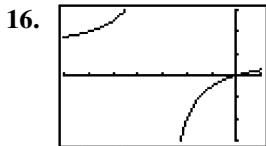
$$\lim_{x \rightarrow 2^-} \frac{x}{x-2} = -\infty$$



[-7, 1] by [-3, 3]

X	Y1
-3.8	-1.25
-3.4	-2.5
-3.2	-3.333
-3.1	-4
-3.01	-100
-3.001	-1000

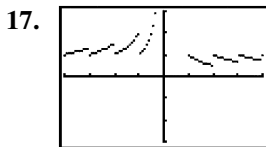
$$\lim_{x \rightarrow -3^-} \frac{1}{x+3} = -\infty$$



[-7, 1] by [-3, 3]

X	Y1
-2.2	-2.25
-2.4	-6.5
-2.6	-9
-2.8	-14
-2.9	-29
-2.99	-299
-2.999	-2999

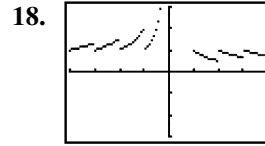
$$\lim_{x \rightarrow -3^+} \frac{x}{x+3} = -\infty$$



[-4, 4] by [-3, 3]

X	Y1
.8	0
.4	0
.2	0
.1	0
.01	0
.001	0
1E-4	0

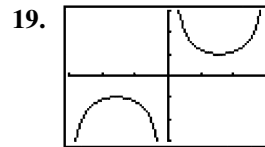
$$\lim_{x \rightarrow 0^+} \frac{\text{int } x}{x} = 0$$



[-4, 4] by [-3, 3]

X	Y1
-.8	1.25
-.4	2.5
-.2	5
-.1	10
-.01	100
-.001	1000
-1E-4	10000

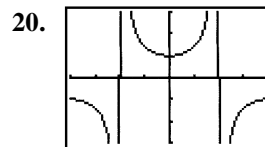
$$\lim_{x \rightarrow 0^-} \frac{\text{int } x}{x} = \infty$$



[-3, 3] by [-3, 3]

X	Y1
.8	1.394
.4	2.5679
.2	5.0338
.1	10.017
.01	100
.001	1000
1E-4	10000

$$\lim_{x \rightarrow 0^+} \text{csc } x = \infty$$



[-pi, pi] by [-3, 3]

X	Y1
1.6	-34.25
1.59	-52.08
1.58	-108.7
1.576	-192.2
1.572	-820.8
1.571	-4810
1.5708	-2.7E5

$$\lim_{x \rightarrow (\frac{\pi}{2})^+} \text{sec } x = -\infty$$

$$\begin{aligned} 21. \quad y &= \left(2 - \frac{x}{x+1}\right) \left(\frac{x^2}{5+x^2}\right) \\ &= \left(\frac{2(x+1) - x}{x+1}\right) \left(\frac{x^2}{5+x^2}\right) \\ &= \left(\frac{x+2}{x+1}\right) \left(\frac{x^2}{5+x^2}\right) \\ &= \frac{x^3 + 2x^2}{x^3 + x^2 + 5x + 5} \end{aligned}$$

An end behavior model for  $y$  is  $\frac{x^3}{x^3} = 1$ .

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} 1 = 1$$

$$\lim_{x \rightarrow -\infty} y = \lim_{x \rightarrow -\infty} 1 = 1$$

$$\begin{aligned} 22. \quad y &= \left(\frac{2}{x} + 1\right) \left(\frac{5x^2 - 1}{x^2}\right) \\ &= \left(\frac{2+x}{x}\right) \left(\frac{5x^2 - 1}{x^2}\right) \\ &= \frac{5x^3 + 10x^2 - x - 2}{x^3} \end{aligned}$$

An end behavior model for  $y$  is  $\frac{5x^3}{x^3} = 5$ .

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} 5 = 5$$

$$\lim_{x \rightarrow -\infty} y = \lim_{x \rightarrow -\infty} 5 = 5$$

23. Use the method of Example 9 in the text.

$$\lim_{x \rightarrow \infty} \frac{\cos\left(\frac{1}{x}\right)}{1 + \frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\cos x}{1 + x} = \frac{\cos(0)}{1 + 0} = \frac{1}{1} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{\cos\left(\frac{1}{x}\right)}{1 + \frac{1}{x}} = \lim_{x \rightarrow 0^-} \frac{\cos x}{1 + x} = \frac{\cos(0)}{1 + 0} = \frac{1}{1} = 1$$

24. Note that  $y = \frac{2x + \sin x}{x} = 2 + \frac{\sin x}{x}$ .

So,  $\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} 2 + \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 2 + 0 = 2$ .

Similarly,  $\lim_{x \rightarrow -\infty} y = 2$ .

25. Note that  $y = \frac{\cos x - 2x^3}{x^3}$

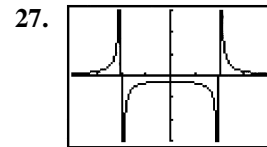
$$y = \frac{\cos x}{x^3} - \frac{2x^3}{x^3}$$

$$y = \frac{\cos x}{x^3} + (-2)$$

So,  $\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \frac{\cos x}{x^3} + \lim_{x \rightarrow \infty} (-2)$   
 $= 0 + (-2)$   
 $= -2$

Similarly,  $\lim_{x \rightarrow -\infty} y = -2$ .

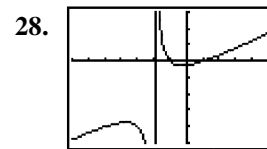
26.  $y = \frac{1}{2} \frac{\sin x}{x} + \frac{1}{x} \frac{\cos x}{x}$   
 So,  $\lim_{x \rightarrow \infty} y = 0$  and  $\lim_{x \rightarrow -\infty} y = 0$ .



[-4, 4] by [-3, 3]

(a)  $x = -2, x = 2$

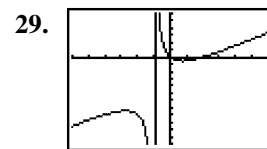
- (b) Left-hand limit at  $-2$  is  $\infty$ .  
 Right-hand limit at  $-2$  is  $-\infty$ .  
 Left-hand limit at  $2$  is  $-\infty$ .  
 Right-hand limit at  $2$  is  $\infty$ .



[-7, 5] by [-5, 3]

(a)  $x = -2$

- (b) Left-hand limit at  $-2$  is  $-\infty$ .  
 Right-hand limit at  $-2$  is  $\infty$ .



[-6, 6] by [-12, 6]

(a)  $x = -1$

- (b) Left-hand limit at  $-1$  is  $-\infty$ .  
 Right-hand limit at  $-1$  is  $\infty$ .



[-2, 4] by [-2, 2]

(a)  $x = -\frac{1}{2}, x = 3$

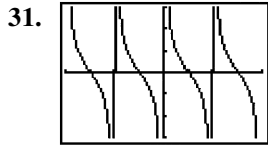


(b) Left-hand limit at  $-\frac{1}{2}$  is  $\infty$ .

Right-hand limit at  $-\frac{1}{2}$  is  $-\infty$ .

Left-hand limit at 3 is  $\infty$ .

Right-hand limit at 3 is  $-\infty$ .



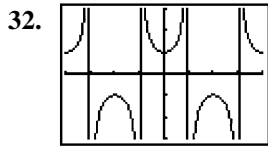
$[-2\pi, 2\pi]$  by  $[-3, 3]$

(a)  $x = k\pi$ ,  $k$  any integer

(b) at each vertical asymptote:

Left-hand limit is  $-\infty$ .

Right-hand limit is  $\infty$ .



$[-2\pi, 2\pi]$  by  $[-3, 3]$

(a)  $x = \frac{\pi}{2} + n\pi$ ,  $n$  any integer

(b) If  $n$  is even:

Left-hand limit is  $\infty$ .

Right-hand limit is  $-\infty$ .

If  $n$  is odd:

Left-hand limit is  $-\infty$ .

Right-hand limit is  $\infty$ .

33.  $f(x) = \frac{\tan x}{\sin x} = \frac{1}{\sin x} \frac{\sin x}{\cos x} = \frac{1}{\cos x}$

$\cos x = 0$  at:  $a = (4k+1)\frac{\pi}{2}$  and  $b = (4k+3)\frac{\pi}{2}$

where  $k$  is any real integer.

$\lim_{x \rightarrow a^-} f(x) = \infty$ ,  $\lim_{x \rightarrow a^+} f(x) = -\infty$ ,

$\lim_{x \rightarrow b^-} f(x) = -\infty$ ,  $\lim_{x \rightarrow b^+} f(x) = \infty$ .

34.  $f(x) = \frac{\cot x}{\cos x} = \frac{\cos x}{\sin x} \frac{1}{\cos x} = \frac{1}{\sin x}$

$\sin x = 0$  at  $a = 2k\pi$  and  $b = (2k+1)\pi$  where  $k$  is any real integer.

$\lim_{x \rightarrow a^-} f(x) = -\infty$ ,  $\lim_{x \rightarrow a^+} f(x) = \infty$ ,

$\lim_{x \rightarrow b^-} f(x) = \infty$ ,  $\lim_{x \rightarrow b^+} f(x) = -\infty$ .

35. An end behavior model is  $\frac{2x^3}{x} = 2x^2$ . (a)

36. An end behavior model is  $\frac{x^5}{2x^2} = 0.5x^3$ . (c)

37. An end behavior model is  $\frac{2x^4}{-x} = -2x^3$ . (d)

38. An end behavior model is  $\frac{x^4}{-x^2} = -x^2$ . (b)

39. (a)  $3x^2$

(b) None

40. (a)  $-4x^3$

(b) None

41. (a)  $\frac{x}{2x^2} = \frac{1}{2x}$

(b)  $y = 0$

42. (a)  $\frac{3x^2}{x^2} = 3$

(b)  $y = 3$

43. (a)  $\frac{4x^3}{x} = 4x^2$

(b) None

44. (a)  $\frac{-x^4}{x^2} = -x^2$

(b) None

45. (a) The function  $y = e^x$  is a right end behavior model because

$$\lim_{x \rightarrow \infty} \frac{e^x - 2x}{e^x} = \lim_{x \rightarrow \infty} \left( 1 - \frac{2x}{e^x} \right) = 1 - 0 = 1.$$

- (b) The function  $y = -2x$  is a left end behavior model because

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{e^x - 2x}{-2x} &= \lim_{x \rightarrow -\infty} \left( -\frac{e^x}{2x} + 1 \right) \\ &= 0 + 1 \\ &= 1.\end{aligned}$$

46. (a) The function  $y = x^2$  is a right end behavior model because

$$\lim_{x \rightarrow \infty} \frac{x^2 + e^{-x}}{x^2} = \lim_{x \rightarrow \infty} \left( 1 + \frac{e^{-x}}{x^2} \right) = 1 + 0 = 1.$$

- (b) The function  $y = e^{-x}$  is a left end behavior model because

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{x^2 + e^{-x}}{e^{-x}} &= \lim_{x \rightarrow -\infty} \left( \frac{x^2}{e^{-x}} + 1 \right) \\ &= \lim_{x \rightarrow -\infty} (x^2 e^x + 1) \\ &= 0 + 1 \\ &= 1.\end{aligned}$$

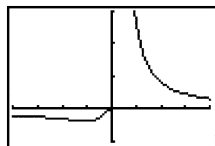
47. (a, b) The function  $y = x$  is both a right end behavior model and a left end behavior model because

$$\begin{aligned}\lim_{x \rightarrow \pm\infty} \left( \frac{x + \ln|x|}{x} \right) &= \lim_{x \rightarrow \pm\infty} \left( 1 + \frac{\ln|x|}{x} \right) \\ &= 1 + 0 \\ &= 1.\end{aligned}$$

48. (a, b) The function  $y = x^2$  is both a right end behavior model and a left end behavior model because

$$\lim_{x \rightarrow \pm\infty} \left( \frac{x^2 + \sin x}{x^2} \right) = \lim_{x \rightarrow \pm\infty} \left( 1 + \frac{\sin x}{x^2} \right) = 1.$$

49.



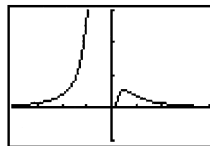
[-4, 4] by [-1, 3]

The graph of  $y = f\left(\frac{1}{x}\right) = \frac{1}{x} e^{1/x}$  is shown.

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow 0^+} f\left(\frac{1}{x}\right) = \infty$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow 0^-} f\left(\frac{1}{x}\right) = 0$$

50.



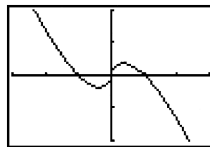
[-4, 4] by [-1, 3]

The graph of  $y = f\left(\frac{1}{x}\right) = \frac{1}{x^2} e^{-1/x}$  is shown.

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow 0^+} f\left(\frac{1}{x}\right) = 0$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow 0^-} f\left(\frac{1}{x}\right) = \infty$$

51.



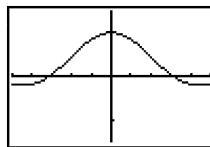
[-3, 3] by [-2, 2]

The graph of  $y = f\left(\frac{1}{x}\right) = x \ln\left|\frac{1}{x}\right|$  is shown.

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow 0^+} f\left(\frac{1}{x}\right) = 0$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow 0^-} f\left(\frac{1}{x}\right) = 0$$

52.



[-5, 5] by [-1.5, 1.5]

The graph of  $y = f\left(\frac{1}{x}\right) = \frac{\sin x}{x}$  is shown.

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow 0^+} f\left(\frac{1}{x}\right) = 1$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow 0^-} f\left(\frac{1}{x}\right) = 1$$

53. (a)  $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \left(\frac{1}{x}\right) = 0$

(b)  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} (-1) = -1$

(c)  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$

(d)  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (-1) = -1$

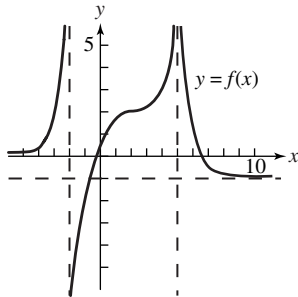
54. (a)  $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x-2}{x-1} = \lim_{x \rightarrow -\infty} \frac{x}{x} = 1$

(b)  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$

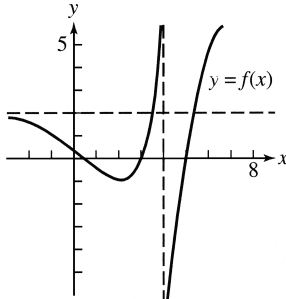
(c)  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{x-2}{x-1} = \frac{0-2}{0-1} = 2$

(d)  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{x^2} = \infty$

55. One possible answer:



56. One possible answer:



57. Note that  $\frac{\frac{f_1(x)}{f_2(x)}}{\frac{g_1(x)}{g_2(x)}} = \frac{f_1(x)g_2(x)}{g_1(x)f_2(x)} = \frac{f_1(x)}{g_2(x)} \cdot \frac{g_2(x)}{f_2(x)}$ .

As  $x$  becomes large,  $\frac{f_1}{g_1}$  and  $\frac{f_2}{g_2}$  both approach 1. Therefore, using the above

equation,  $\frac{\frac{f_1}{g_1}}{\frac{f_2}{g_2}}$  must also approach 1.

58. Yes. The limit of  $(f + g)$  will be the same as the limit of  $g$ . This is because adding numbers that are very close to a given real number  $L$  will not have a significant effect on the value of  $(f + g)$  since the values of  $g$  are becoming arbitrarily large.

59. True. For example,  $f(x) = \frac{x}{\sqrt{x^2 + 1}}$  has  $y = \pm 1$  as horizontal asymptotes.

60. False; consider  $f(x) = \frac{1}{x}$ .

61. A;  $\lim_{x \rightarrow 2^-} (x-2)$  approaches zero from the left, so  $\lim_{x \rightarrow 2^-} \frac{x}{x-2}$  approaches  $-\infty$ .

62. E;  $\lim_{x \rightarrow 0} \frac{\cos(2x)}{x}$  is undefined because  $\frac{\cos(2x)}{x}$  has a vertical asymptote at  $x = 0$ .

63. C; let  $t = 3x$ . Then  $t \rightarrow 0$  as  $x \rightarrow 0$ ; and  $\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{t \rightarrow 0} \frac{\sin t}{t/3} = 3 \cdot \lim_{t \rightarrow 0} \frac{\sin t}{t} = 3 \cdot 1 = 3$

64. D;  $\frac{2x^3}{x^3} = 2$

65. (a) Note that  $fg = f(x)g(x) = 1$ .  $f \rightarrow -\infty$  as  $x \rightarrow 0^-$ ,  $f \rightarrow \infty$  as  $x \rightarrow 0^+$ ,  $g \rightarrow 0$ ,  $fg \rightarrow 1$

(b) Note that  $fg = f(x)g(x) = -8$ .  $f \rightarrow \infty$  as  $x \rightarrow 0^-$ ,  $f \rightarrow -\infty$  as  $x \rightarrow 0^+$ ,  $g \rightarrow 0$ ,  $fg \rightarrow -8$

(c) Note that  $fg = f(x)g(x) = 3(x-2)^2$ .  $f \rightarrow -\infty$  as  $x \rightarrow 2^-$ ,  $f \rightarrow \infty$  as  $x \rightarrow 2^+$ ,  $g \rightarrow 0$ ,  $fg \rightarrow 0$

(d) Note that  $fg = f(x)g(x) = \frac{5}{(x-3)^2}$ .  $f \rightarrow \infty$ ,  $g \rightarrow 0$ ,  $fg \rightarrow \infty$

(e) Nothing—you need more information to decide.

66. (a) This follows from  $x - 1 < \text{int } x \leq x$ , which is true for all  $x$ . Dividing by  $x$  gives the result.

(b, c) Since  $\lim_{x \rightarrow \pm\infty} \frac{x-1}{x} = \lim_{x \rightarrow \pm\infty} 1 = 1$ , the

Squeeze Theorem gives

$$\lim_{x \rightarrow \infty} \frac{\text{int } x}{x} = \lim_{x \rightarrow \infty} \frac{\text{int } x}{x} = 1.$$

68. This is because as  $x$  approaches infinity,  $\sin x$  continues to oscillate between 1 and  $-1$  and doesn't approach any given real number.

69.  $\lim_{x \rightarrow \infty} \frac{\ln x^2}{\ln x} = 2$ , because  $\frac{\ln x^2}{\ln x} = \frac{2 \ln x}{\ln x} = 2$ .

70.  $\lim_{x \rightarrow \infty} \frac{\ln x}{\log x} = \ln(10)$ , since

$$\frac{\ln x}{\log x} = \frac{\ln x}{\frac{(\ln x)}{(\ln 10)}} = \ln 10.$$

71.  $\lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\ln x} = 1$

Since

$$\ln(x+1) = \ln \left[ x \left( 1 + \frac{1}{x} \right) \right] = \ln x + \ln \left( 1 + \frac{1}{x} \right),$$

$$\frac{\ln(x+1)}{\ln x} = \frac{\ln x + \ln \left( 1 + \frac{1}{x} \right)}{\ln x} = 1 + \frac{\ln \left( 1 + \frac{1}{x} \right)}{\ln x}$$

But as  $x \rightarrow \infty$ ,  $1 + \frac{1}{x}$  approaches 1, so

$\ln \left( 1 + \frac{1}{x} \right)$  approaches  $\ln(1) = 0$ . Also, as

$x \rightarrow \infty$ ,  $\ln x$  approaches infinity. This means the second term above approaches 0 and the limit is 1.

**Quick Quiz Sections 2.1 and 2.2**

1. D;  $\lim_{x \rightarrow 3} \left( \frac{x^2 - x - 6}{x - 3} \right) = \lim_{x \rightarrow 3} \frac{(x-3)(x+2)}{x-3}$   
 $= \lim_{x \rightarrow 3} (x+2)$   
 $= 3+2$   
 $= 5$

2. A;  $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{5}{x+1} = \frac{5}{2+1} = \frac{5}{3}$

3. E;  $\frac{3x^3}{2x^3} = \frac{3}{2}$

4. (a) Domain:  $(-\infty, 0) \cup (0, \infty)$   
 Range:  $(-\infty, \infty)$

(b)  $f(-x) = \frac{\cos(-x)}{-x} = \frac{\cos x}{-x} = -f(x)$ , so  $f$  is odd.

(c)  $\lim_{x \rightarrow \infty} \frac{\cos x}{x}$ ,  $-1 \leq \cos x \leq 1$

$$\lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0$$

(d) For all  $x > 0$ ,  $-1 \leq \cos x \leq 1$ .

Therefore,  $-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$ .

Since  $\lim_{x \rightarrow \infty} \left( -\frac{1}{x} \right) = \lim_{x \rightarrow \infty} \left( \frac{1}{x} \right) = 0$ , it

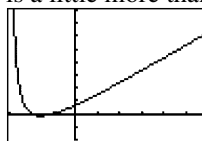
follows by the Squeeze Theorem that

$$\lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0.$$

**Section 2.3 Continuity (pp. 78–86)**

**Exploration 1 Removing a Discontinuity**

- $x^2 - 9 = (x - 3)(x + 3)$ . The domain of  $f$  is  $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$  or all  $x \neq \pm 3$ .
- It appears that the limit of  $f$  as  $x \rightarrow 3$  exists and is a little more than 3.



$[-3, 6]$  by  $[-2, 8]$

3.  $f(3)$  should be defined as  $\frac{10}{3}$ .

4.  $x^3 - 7x - 6 = (x-3)(x+1)(x+2)$ ,  
 $x^2 - 9 = (x-3)(x+3)$ , so  
 $f(x) = \frac{(x+1)(x+2)}{x+3}$  for  $x \neq 3$ .

Thus,  $\lim_{x \rightarrow 3} \frac{(x+1)(x+2)}{x+3} = \frac{20}{6} = \frac{10}{3}$ .

5.  $\lim_{x \rightarrow 3} g(x) = \frac{10}{3} = g(3)$ , so  $g$  is continuous at  $x = 3$ .

**Quick Review 2.3**

1.  $\lim_{x \rightarrow -1} \frac{3x^2 - 2x + 1}{x^3 + 4} = \frac{3(-1)^2 - 2(-1) + 1}{(-1)^3 + 4} = \frac{6}{3} = 2$

2. (a)  $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} \text{int}(x) = -2$

(b)  $\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} f(x) = -1$

(c)  $\lim_{x \rightarrow -1} f(x)$  does not exist, because the left- and right-hand limits are not equal.

(d)  $f(-1) = \text{int}(-1) = -1$

3. (a)  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^2 - 4x + 5)$   
 $= 2^2 - 4(2) + 5$   
 $= 1$

(b)  $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (4 - x) = 4 - 2 = 2$

(c)  $\lim_{x \rightarrow 2} f(x)$  does not exist, because the left- and right-hand limits are not equal.

(d)  $f(2) = 4 - 2 = 2$

4.  $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x} + 1\right)$   
 $= \frac{2\left(\frac{1}{x} + 1\right) - 1}{\left(\frac{1}{x} + 1\right) + 5}$   
 $= \frac{2(1+x) - x}{(1+x) + 5x}$   
 $= \frac{x+2}{6x+1}, x \neq 0$

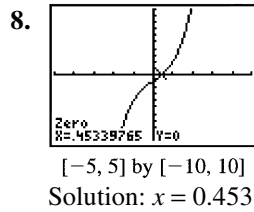
$(g \circ f)(x) = g(f(x)) = g\left(\frac{2x-1}{x+5}\right)$   
 $= \frac{1}{\frac{2x-1}{x+5}} + 1$   
 $= \frac{x+5}{2x-1} + \frac{2x-1}{2x-1}$   
 $= \frac{3x+4}{2x-1}, x \neq -5$

5. Note that  $\sin x^2 = (g \circ f)(x) = g(f(x)) = g(x^2)$ .  
 Therefore:  $g(x) = \sin x, x \geq 0$   
 $(f \circ g)(x) = f(g(x)) = f(\sin x) = (\sin x^2)$  or  $\sin^2 x, x \geq 0$

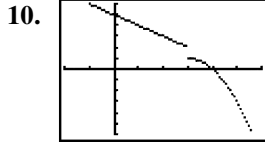
6. Note that  $\frac{1}{x} = (g \circ f)(x) = g(f(x)) = \sqrt{f(x)-1}$ .  
 Therefore,  $\sqrt{f(x)-1} = \frac{1}{x}$  for  $x > 0$ . Squaring both sides gives  $f(x)-1 = \frac{1}{x^2}$ . Therefore,  
 $f(x) = \frac{1}{x^2} + 1, x > 0$ .

$(f \circ g)(x) = f(g(x)) = \frac{1}{(\sqrt{x-1})^2} + 1$   
 $= \frac{1}{x-1} + 1$   
 $= \frac{1+x-1}{x-1}$   
 $= \frac{x}{x-1}, x > 1$

7.  $2x^2 + 9x - 5 = 0$   
 $(2x-1)(x+5) = 0$   
 Solutions:  $x = \frac{1}{2}, x = -5$



9. For  $x \leq 3, f(x) = 4$  when  $5 - x = 4$ , which gives  $x = 1$ . (Note that this value is, in fact,  $\leq 3$ .)  
 For  $x > 3, f(x) = 4$  when  $-x^2 + 6x - 8 = 4$ , which gives  $x^2 - 6x + 12 = 0$ . The discriminant of this equation is  $b^2 - 4ac = (-6)^2 - 4(1)(12) = -12$ . Since the discriminant is negative, the quadratic equation has no solution. The only solution to the original equation is  $x = 1$ .



$[-2.7, 6.7]$  by  $[-6, 6]$

A graph of  $f(x)$  is shown. The range of  $f(x)$  is  $(-\infty, 1) \cup [2, \infty)$ . The values of  $c$  for which  $f(x) = c$  has no solution are the values that are excluded from the range. Therefore,  $c$  can be any value in  $[1, 2)$ .

### Section 2.3 Exercises

- The function  $y = \frac{1}{(x+2)^2}$  is continuous because it is a quotient of polynomials, which are continuous. Its only point of discontinuity occurs where it is undefined. There is an infinite discontinuity at  $x = -2$ .
- The function  $y = \frac{x+1}{x^2-4x+3}$  is continuous because it is a quotient of polynomials, which are continuous. Its only points of discontinuity occur where it is undefined, that is, where the denominator  $x^2 - 4x + 3 = (x-1)(x-3)$  is zero. There are infinite discontinuities at  $x = 1$  and at  $x = 3$ .
- The function  $y = \frac{1}{x^2+1}$  is continuous because it is a quotient of polynomials, which are continuous. Furthermore, the domain is all real numbers because the denominator,  $x^2 + 1$ , is never zero. Since the function is continuous and has domain  $(-\infty, \infty)$ , there are no points of discontinuity.
- The function  $y = |x-1|$  is a composition  $(f \circ g)(x)$  of the continuous functions  $f(x) = |x|$  and  $g(x) = x - 1$ , so it is continuous. Since the function is continuous and has domain  $(-\infty, \infty)$ , there are no points of discontinuity.
- The function  $y = \sqrt{2x+3}$  is a composition  $(f \circ g)(x)$  of the continuous functions  $f(x) = \sqrt{x}$  and  $g(x) = 2x + 3$ , so it is continuous. Its points of discontinuity are the points not in the domain, i.e., all  $x < -\frac{3}{2}$ .
- The function  $y = \sqrt[3]{2x-1}$  is a composition  $(f \circ g)(x)$  of the continuous functions  $f(x) = \sqrt[3]{x}$  and  $g(x) = 2x - 1$ , so it is continuous. Since the function is continuous and has domain  $(-\infty, \infty)$ , there are no points of discontinuity.
- The function  $y = \frac{|x|}{x}$  is equivalent to 
$$y = \begin{cases} -1, & x < 0 \\ 1, & x > 0. \end{cases}$$
 It has a jump discontinuity at  $x = 0$ .
- The function  $y = \cot x$  is equivalent to  $y = \frac{\cos x}{\sin x}$ , a quotient of continuous functions, so it is continuous. Its only points of discontinuity occur where it is undefined. It has infinite discontinuities at  $x = k\pi$  for all integers  $k$ .
- The function  $y = e^{1/x}$  is a composition  $(f \circ g)(x)$  of the continuous functions  $f(x) = e^x$  and  $g(x) = \frac{1}{x}$ , so it is continuous. Its only point of discontinuity occurs at  $x = 0$ , where it is undefined. Since  $\lim_{x \rightarrow 0^+} e^{1/x} = \infty$ , this may be considered an infinite discontinuity.
- The function  $y = \ln(x+1)$  is a composition  $(f \circ g)(x)$  of the continuous functions  $f(x) = \ln x$  and  $g(x) = x + 1$ , so it is continuous. Its points of discontinuity are the points not in the domain, i.e.,  $x \leq -1$ .
- (a) Yes,  $f(-1) = 0$ .  
(b) Yes,  $\lim_{x \rightarrow -1^+} f(x) = 0$ .  
(c) Yes  
(d) Yes, since  $-1$  is a left endpoint of the domain of  $f$  and  $\lim_{x \rightarrow -1^+} f(x) = f(-1)$ ,  $f$  is continuous at  $x = -1$ .
- (a) Yes,  $f(1) = 1$ .  
(b) Yes,  $\lim_{x \rightarrow 1} f(x) = 2$ .

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(c) No

(d) No

13. (a) No

(b) No, since  $x = 2$  is not in the domain.

14. Everywhere in  $[-1, 3)$  except for  $x = 0, 1, 2$ .

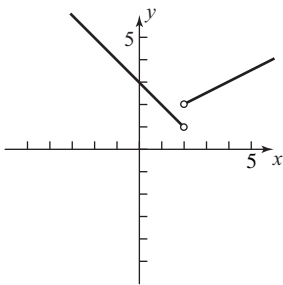
15. Since  $\lim_{x \rightarrow 2} f(x) = 0$ , we should assign  $f(2) = 0$ .

16. Since  $\lim_{x \rightarrow 1} f(x) = 2$ , we should reassign  $f(1) = 2$ .

17. No, because the right-hand and left-hand limits are not the same at zero.

18. Yes, Assign the value 0 to  $f(3)$ . Since 3 is a right endpoint of the extended function and  $\lim_{x \rightarrow 3^-} f(x) = 0$ , the extended function is continuous at  $x = 3$ .

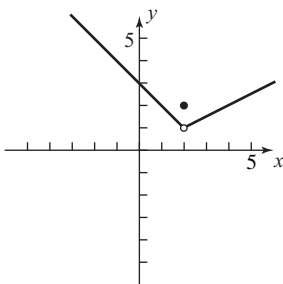
19.



(a)  $x = 2$

(b) Not removable, the one-sided limits are different.

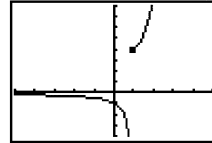
20.



(a)  $x = 2$

(b) Removable, assign the value 1 to  $f(2)$ .

21.

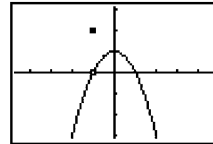


$[-5, 5]$  by  $[-4, 8]$

(a)  $x = 1$

(b) Not removable, it's an infinite discontinuity.

22.



$[-4.7, 4.7]$  by  $[-3.1, 3.1]$

(a)  $x = -1$

(b) Removable, assign the value 0 to  $f(-1)$ .

23. (a) All points not in the domain along with  $x = 0, 1$

(b)  $x = 0$  is a removable discontinuity, assign  $f(0) = 0$ .  
 $x = 1$  is not removable, the one-sided limits are different.

24. (a) All points not in the domain along with  $x = 1, 2$

(b)  $x = 1$  is not removable, the one-sided limits are different.  
 $x = 2$  is a removable discontinuity, assign  $f(2) = 1$ .

25. For  $x \neq -3$ ,

$$f(x) = \frac{x^2 - 9}{x + 3} = \frac{(x + 3)(x - 3)}{x + 3} = x - 3.$$

The extended function is  $y = x - 3$ .

26. For  $x \neq 1$ ,  $f(x) = \frac{x^3 - 1}{x^2 - 1}$

$$= \frac{(x - 1)(x^2 + x + 1)}{(x + 1)(x - 1)}$$

$$= \frac{x^2 + x + 1}{x + 1}.$$

The extended function is  $y = \frac{x^2 + x + 1}{x + 1}$ .

27. Since  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ , the extended function is

$$y = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

28. Since  $\lim_{x \rightarrow 0} \frac{\sin 4x}{x} = 4 \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} = 4(1) = 4$ , the

$$\text{extended function is } y = \begin{cases} \frac{\sin 4x}{x}, & x \neq 0 \\ 4, & x = 0 \end{cases}.$$

29. For  $x \neq 4$  (and  $x > 0$ ),

$$f(x) = \frac{x-4}{\sqrt{x}-2} = \frac{(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2} = \sqrt{x}+2.$$

The extended function is  $y = \sqrt{x} + 2$ .

30. For  $x \neq 2$  (and  $x \neq -2$ ),

$$\begin{aligned} f(x) &= \frac{x^3 - 4x^2 - 11x + 30}{x^2 - 4} \\ &= \frac{(x-2)(x-5)(x+3)}{(x-2)(x+2)} \\ &= \frac{(x-5)(x+3)}{x+2} \\ &= \frac{x^2 - 2x - 15}{x+2}. \end{aligned}$$

The extended function is  $y = \frac{x^2 - 2x - 15}{x+2}$ .

31. The domain of  $f$  is all real numbers  $x \neq 3$ .  $f$  is continuous at all those points so  $f$  is a continuous function.
32. The domain of  $g$  is all real numbers  $x > 1$ .  $f$  is continuous at all those points so  $g$  is a continuous function.
33.  $f$  is the composite of two continuous functions  $g \circ h$  where  $g(x) = \sqrt{x}$  and
- $$h(x) = \frac{x}{x+1}.$$
34.  $f$  is the composite of two continuous functions  $g \circ h$  where  $g(x) = \sin x$  and
- $$h(x) = x^2 + 1.$$
35.  $f$  is the composite of three continuous functions  $g \circ h \circ k$  where  $g(x) = \cos x$ ,
- $$h(x) = \sqrt[3]{x}, \text{ and } k(x) = 1 - x.$$

36.  $f$  is the composite of two continuous functions

$$g \circ h \text{ where } g(x) = \tan x \text{ and } h(x) = \frac{x^2}{x^2 + 4}.$$

37. One possible answer:

Assume  $y = x$ , constant functions, and the square root function are continuous.

By the sum theorem,  $y = x + 2$  is continuous.

By the composite theorem,  $y = \sqrt{x+2}$  is continuous.

By the quotient theorem,  $y = \frac{1}{\sqrt{x+2}}$  is

continuous.

Domain:  $(-2, \infty)$

38. One possible answer:

Assume  $y = x$ , constant functions, and the cube root function are continuous.

By the difference theorem,  $y = 4 - x$  is continuous.

By the composite theorem,  $y = \sqrt[3]{4-x}$  is continuous.

By the product theorem,  $y = x^2 = x \cdot x$  is continuous.

By the sum theorem,  $y = x^2 + \sqrt[3]{4-x}$  is continuous.

Domain:  $(-\infty, \infty)$

39. Possible answer:

Assume  $y = x$  and  $y = |x|$  are continuous.

By the product theorem,  $y = x^2 = x \cdot x$  is continuous.

By the constant multiple theorem,  $y = 4x$  is continuous.

By the difference theorem,  $y = x^2 - 4x$  is continuous.

By the composite theorem,  $y = |x^2 - 4x|$  is continuous.

Domain:  $(-\infty, \infty)$

40. One possible answer:

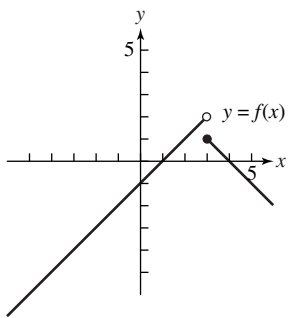
Assume  $y = x$  and  $y = 1$  are continuous.

Use the product, difference, and quotient theorems. One also needs to verify that the limit of this function as  $x$  approaches 1 is 2. Alternately, observe that the function is equivalent to  $y = x + 1$  (for all  $x$ ), which is continuous by the sum theorem.

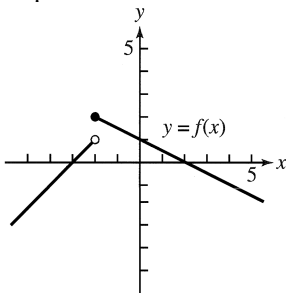
Domain:  $(-\infty, \infty)$



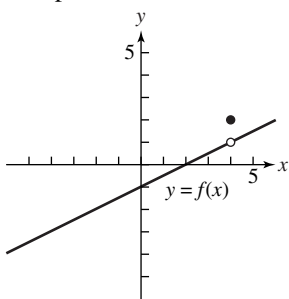
41. One possible answer:



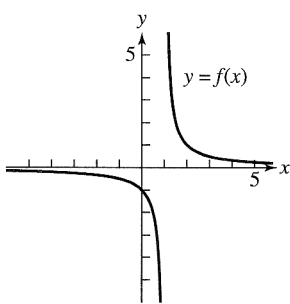
42. One possible answer:



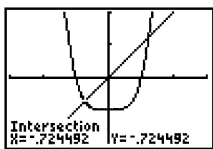
43. One possible answer:



44. One possible answer:



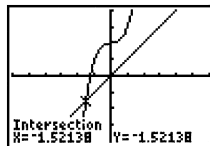
45.



$[-3, 3]$  by  $[-2, 2]$

Solving  $x = x^4 - 1$ , we obtain the solutions  $x = -0.724$  and  $x = 1.221$ .

46.



$[-6, 6]$  by  $[-4, 4]$

Solving  $x = x^3 + 2$ , we obtain the solution  $x = -1.521$ .

47. Since  $f(3) = \lim_{x \rightarrow 3^+} f(x) = 2a(3) = 6a$  and

$\lim_{x \rightarrow 3^-} f(x) = 3^2 - 1 = 8$ , the function will be

continuous if  $6a = 8$ . Thus  $a = \frac{4}{3}$ .

48. Since  $f(2) = \lim_{x \rightarrow 2^-} f(x) = 2(2) + 3 = 7$  and

$\lim_{x \rightarrow 2^+} f(x) = 2a + 1$ , the function will be

continuous if  $2a + 1 = 7$ . Thus  $a = 3$ .

49. Since  $f(-1) = \lim_{x \rightarrow -1^+} f(x) = a(-1)^2 - 1 = a - 1$

and  $\lim_{x \rightarrow -1^-} f(x) = 4 - (-1)^2 = 3$ , the function

will be continuous if  $a - 1 = 3$ . Thus  $a = 4$ .

50. Since  $f(1) = \lim_{x \rightarrow 1^+} f(x) = 1^3 = 1$  and

$\lim_{x \rightarrow 1^-} f(x) = (1)^2 + 1 + a = 2 + a$ , the function

will be continuous if  $2 + a = 1$ . Thus  $a = -1$ .

51. Consider  $f(x) = x - e^{-x}$ .  $f$  is continuous,

$f(0) = -1$ , and  $f(1) = 1 - \frac{1}{e} > 0.5$ . By the

Intermediate Value Theorem, for some  $c$  in  $(0, 1)$ ,  $f(c) = 0$  and  $e^{-c} = c$ .

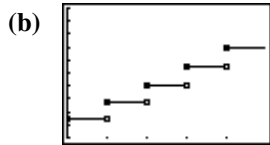
52. (a) Luisa's salary is

$\$36,500 = \$36,500(1.035)^0$  for the first year ( $0 \leq t < 1$ ),  $\$36,500(1.035)$  for the second year ( $1 \leq t < 2$ ),

$\$36,500(1.035)^2$  for the third year

( $2 \leq t < 3$ ), and so on. This corresponds to

$y = 36,500(1.035)^{\text{int}t}$ .



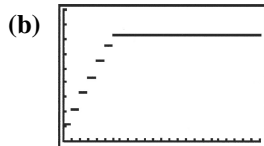
[0, 4.8] by [35000, 45000]  
The function is continuous at all points in the domain [0, 5) except at  $t = 1, 2, 3, 4$ .

53. (a) We require:

$$f(x) = \begin{cases} 0 & x = 0 \\ 1.10, & 0 < x \leq 1 \\ 2.20, & 1 < x \leq 2 \\ 3.30, & 2 < x \leq 3 \\ 4.40, & 3 < x \leq 4 \\ 5.50, & 4 < x \leq 5 \\ 6.60, & 5 < x \leq 6 \\ 7.25, & 6 < x \leq 24. \end{cases}$$

This may be written more compactly as

$$f(x) = \begin{cases} -1.10 \operatorname{int}(-x), & 0 \leq x \leq 6 \\ 7.25, & 6 < x \leq 24 \end{cases}$$



[0, 24] by [0, 9]  
This is continuous for all values of  $x$  in the domain [0, 24] except for  $x = 0, 1, 2, 3, 4, 5, 6$ .

54. False. Consider  $f(x) = \frac{1}{x}$  which is continuous and has a point of discontinuity at  $x = 0$ .

55. False; if  $f$  has a jump discontinuity at  $x = a$ , then  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$ , so  $f$  is not continuous at  $x = a$ .

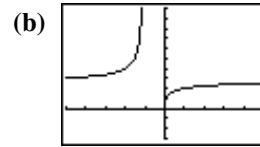
56. B;  $f(x) = \frac{1}{\sqrt{0}}$  is not defined.

57. E;  $f(x) = \sqrt{x-1} = \sqrt{1-1} = \sqrt{0}$  is the only defined option.

58. A;  $f(1) = 1$ .

59. E;  $x = 3$  causes the denominator to be zero even after the rational expression is reduced.

60. (a) The function is defined when  $1 + \frac{1}{x} > 0$ , that is, on  $(-\infty, -1) \cup (0, \infty)$ . (It can be argued that the domain should also include certain values in the interval  $(-1, 0)$ , namely, those rational numbers that have odd denominators when expressed in lowest terms.)



[-5, 5] by [-3, 10]

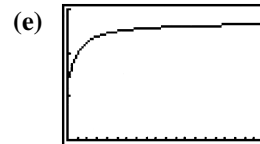
(c) If we attempt to evaluate  $f(x)$  at these values, we obtain

$$f(-1) = \left(1 + \frac{1}{-1}\right)^{-1} = 0^{-1} = \frac{1}{0}$$

$$\text{(undefined) and } f(0) = \left(1 + \frac{1}{0}\right)^0$$

(undefined). Since  $f$  is undefined at these values due to division by zero, both values are points of discontinuity.

(d) The discontinuity at  $x = 0$  is removable because the right-hand limit is 0. The discontinuity at  $x = -1$  is not removable because it is an infinite discontinuity.



[0, 20] by [0, 3]

X	Y1
10	2.5827
100	2.7048
1000	2.7169
10000	2.7181
100000	2.7183
1E6	2.7183
1E7	2.7183

The limit is about 2.718, or  $e$ .

61. This is because  $\lim_{h \rightarrow 0} f(a+h) = \lim_{x \rightarrow a} f(x)$ .

62. Suppose not. Then  $f$  would be negative somewhere in the interval and positive somewhere else in the interval. So, by the Intermediate Value Theorem, it would have to be zero somewhere in the interval, which contradicts the hypothesis.

63. Since the absolute value function is continuous, this follows from the theorem about continuity of composite functions.
64. For any real number  $a$ , the limit of this function as  $x$  approaches  $a$  cannot exist. This is because as  $x$  approaches  $a$ , the values of the function will continually oscillate between 0 and 1.

**Section 2.4** Rates of Change, Tangent Lines, and Sensitivity  
(pp. 87–96)

**Quick Review 2.4**

- $\Delta x = 3 - (-5) = 8$   
 $\Delta y = 5 - 2 = 3$
- $\Delta x = a - 1$   
 $\Delta y = b - 3$
- $m = \frac{-1 - 3}{5 - (-2)} = \frac{-4}{7} = -\frac{4}{7}$
- $m = \frac{3 - (-1)}{3 - (-3)} = \frac{4}{6} = \frac{2}{3}$
- $y = \frac{3}{2}[x - (-2)] + 3$   
 $y = \frac{3}{2}x + 6$
- $m = \frac{-1 - 6}{4 - 1} = \frac{-7}{3} = -\frac{7}{3}$   
 $y = -\frac{7}{3}(x - 1) + 6$   
 $y = -\frac{7}{3}x + \frac{25}{3}$
- $y = -\frac{3}{4}(x - 1) + 4$   
 $y = -\frac{3}{4}x + \frac{19}{4}$
- $m = -\frac{1}{-\frac{3}{4}} = \frac{4}{3}$   
 $y = \frac{4}{3}(x - 1) + 4$   
 $y = \frac{4}{3}x + \frac{8}{3}$

9. Since  $2x + 3y = 5$  is equivalent to

$$y = -\frac{2}{3}x + \frac{5}{3}, \text{ we use}$$

$$m = -\frac{2}{3}.$$

$$y = -\frac{2}{3}[x - (-1)] + 3$$

$$y = -\frac{2}{3}x + \frac{7}{3}$$

10.  $\frac{b-3}{4-2} = \frac{5}{3}$   
 $b-3 = \frac{10}{3}$   
 $b = \frac{19}{3}$

**Section 2.4** Exercises

- (a)  $\frac{\Delta f}{\Delta x} = \frac{f(3) - f(2)}{3 - 2} = \frac{28 - 9}{1} = 19$   
(b)  $\frac{\Delta f}{\Delta x} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1$
- (a)  $\frac{\Delta f}{\Delta x} = \frac{f(2) - f(0)}{2 - 0} = \frac{3 - 1}{2} = 1$   
(b)  $\frac{\Delta f}{\Delta x} = \frac{f(12) - f(10)}{12 - 10} = \frac{7 - \sqrt{41}}{2}$
- (a)  $\frac{\Delta f}{\Delta x} = \frac{f(0) - f(-2)}{0 - (-2)} = \frac{1 - e^{-2}}{2}$   
(b)  $\frac{\Delta f}{\Delta x} = \frac{f(3) - f(1)}{3 - 1} = \frac{e^3 - e}{2}$
- (a)  $\frac{\Delta f}{\Delta x} = \frac{f(4) - f(1)}{4 - 1} = \frac{\ln 4 - 0}{3} = \frac{\ln 4}{3}$   
(b)  $\frac{\Delta f}{\Delta x} = \frac{f(103) - f(100)}{103 - 100} = \frac{\ln 103 - \ln 100}{3} = \frac{1}{3} \ln \frac{103}{100} = \frac{1}{3} \ln 1.03$

5. (a)  $\frac{\Delta f}{\Delta x} = \frac{f\left(\frac{3\pi}{4}\right) - f\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-1-1}{\frac{\pi}{2}} = -\frac{4}{\pi}$

(b)  $\frac{\Delta f}{\Delta x} = \frac{f\left(\frac{\pi}{2}\right) - f\left(\frac{\pi}{6}\right)}{\frac{\pi}{2} - \frac{\pi}{6}} = \frac{0 - \sqrt{3}}{\frac{\pi}{3}} = -\frac{3\sqrt{3}}{\pi}$

6. (a)  $\frac{\Delta f}{\Delta x} = \frac{f(\pi) - f(0)}{\pi - 0} = \frac{1-3}{\pi} = -\frac{2}{\pi}$

(b)  $\frac{\Delta f}{\Delta x} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{1-1}{2\pi} = 0$

7. We use  $Q_1 = (10, 225)$ ,  $Q_2 = (14, 375)$ ,  $Q_3 = (16.5, 475)$ ,  $Q_4 = (18, 550)$ , and  $P = (20, 650)$ .

(a) Slope of  $PQ_1$ :  $\frac{650 - 225}{20 - 10} \approx 43$   
 Slope of  $PQ_2$ :  $\frac{650 - 375}{20 - 14} \approx 46$   
 Slope of  $PQ_3$ :  $\frac{650 - 475}{20 - 16.5} = 50$   
 Slope of  $PQ_4$ :  $\frac{650 - 550}{20 - 18} = 50$

Secant	Slope
$PQ_1$	43
$PQ_2$	46
$PQ_3$	50
$PQ_4$	50

The appropriate units are meters per second.

(b) Approximately 50 m/sec

8. We use  $Q_1 = (5, 20)$ ,  $Q_2 = (7, 38)$ ,  $Q_3 = (8.5, 56)$ ,  $Q_4 = (9.5, 72)$ , and  $P = (10, 80)$ .

(a) Slope of  $PQ_1$ :  $\frac{80 - 20}{10 - 5} = 12$   
 Slope of  $PQ_2$ :  $\frac{80 - 38}{10 - 7} = 14$   
 Slope of  $PQ_3$ :  $\frac{80 - 56}{10 - 8.5} = 16$   
 Slope of  $PQ_4$ :  $\frac{80 - 72}{10 - 9.5} = 16$

Secant	Slope
$PQ_1$	12
$PQ_2$	14
$PQ_3$	16
$PQ_4$	16

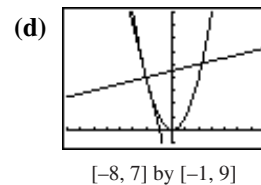
The appropriate units are meters per second.

(b) Approximately 16 m/sec

9. (a)  $\lim_{h \rightarrow 0} \frac{y(-2+h) - y(-2)}{h}$   
 $= \lim_{h \rightarrow 0} \frac{(-2+h)^2 - (-2)^2}{h}$   
 $= \lim_{h \rightarrow 0} \frac{4 - 4h + h^2 - 4}{h}$   
 $= \lim_{h \rightarrow 0} \frac{-4h + h^2}{h}$   
 $= \lim_{h \rightarrow 0} (-4 + h)$   
 $= -4$

(b) The tangent line has slope  $-4$  and passes through  $(-2, y(-2)) = (-2, 4)$   
 $y = -4[x - (-2)] + 4$   
 $y = -4x - 4$

(c) The normal line has slope  $-\frac{1}{-4} = \frac{1}{4}$  and passes through  $(-2, y(-2)) = (-2, 4)$ .  
 $y = \frac{1}{4}[x - (-2)] + 4$   
 $y = \frac{1}{4}x + \frac{9}{2}$



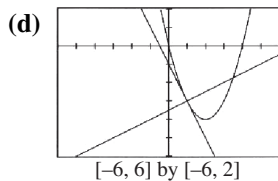
10. (a) 
$$\begin{aligned} \lim_{h \rightarrow 0} \frac{y(1+h) - y(1)}{h} &= \lim_{h \rightarrow 0} \frac{[(1+h)^2 - 4(1+h)] - [1^2 - 4(1)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 + 2h + h^2 - 4 - 4h + 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 - 2h}{h} \\ &= \lim_{h \rightarrow 0} (h - 2) \\ &= -2 \end{aligned}$$

(b) The tangent line has slope  $-2$  and passes through  $(1, y(1)) = (1, -3)$ .  
 $y = -2(x - 1) - 3$   
 $y = -2x - 1$

(c) The normal line has slope  $-\frac{1}{-2} = \frac{1}{2}$  and passes through  $(1, y(1)) = (1, -3)$ .

$$y = \frac{1}{2}(x - 1) - 3$$

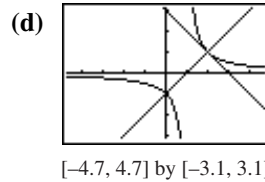
$$y = \frac{1}{2}x - \frac{7}{2}$$



11. (a) 
$$\begin{aligned} \lim_{h \rightarrow 0} \frac{y(2+h) - y(2)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{(2+h)-1} - \frac{1}{2-1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{h+1} - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - (h+1)}{h(h+1)} \\ &= \lim_{h \rightarrow 0} \left( -\frac{1}{h+1} \right) \\ &= -1 \end{aligned}$$

(b) The tangent line has slope  $-1$  and passes through  $(2, y(2)) = (2, 1)$ .  
 $y = -(x - 2) + 1$   
 $y = -x + 3$

(c) The normal line has slope  $-\frac{1}{-1} = 1$  and passes through  $(2, y(2)) = (2, 1)$ .  
 $y = 1(x - 2) + 1$   
 $y = x - 1$



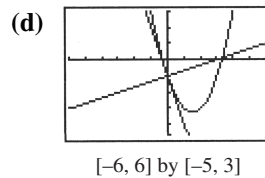
12. (a) 
$$\begin{aligned} \lim_{h \rightarrow 0} \frac{y(0+h) - y(0)}{h} &= \lim_{h \rightarrow 0} \frac{(h^2 - 3h - 1) - (-1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 - 3h}{h} \\ &= \lim_{h \rightarrow 0} (h - 3) \\ &= -3 \end{aligned}$$

(b) The tangent line has slope  $-3$  and passes through  $(0, y(0)) = (0, -1)$ .  
 $y = -3(x - 0) - 1$   
 $y = -3x - 1$

(c) The normal line has slope  $-\frac{1}{-3} = \frac{1}{3}$  and passes through  $(0, y(0)) = (0, -1)$ .

$$y = \frac{1}{3}(x - 0) - 1$$

$$y = \frac{1}{3}x - 1$$



13. (a) Near  $x = 2, f(x) = |x| = x$ .  

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0} \frac{(2+h) - 2}{h} \\ &= \lim_{h \rightarrow 0} 1 \\ &= 1 \end{aligned}$$

(b) Near  $x = -3, f(x) = |x| = -x$ .  

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(-3+h) - f(-3)}{h} &= \lim_{h \rightarrow 0} \frac{(3-h) - 3}{h} \\ &= \lim_{h \rightarrow 0} -1 \\ &= -1 \end{aligned}$$

14. Near  $x = 1$ ,  $f(x) = |x - 2| = -(x - 2) = 2 - x$ .

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0} \frac{[2 - (1+h)] - (2-1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1-h-1}{h} \\ &= \lim_{h \rightarrow 0} -1 \\ &= -1\end{aligned}$$

15. First, note that  $f(0) = 2$ .

$$\begin{aligned}\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^-} \frac{(2 - 2h - h^2) - 2}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{-2h - h^2}{h} \\ &= \lim_{h \rightarrow 0^-} (-2 - h) \\ &= -2\end{aligned}$$

$$\begin{aligned}\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^+} \frac{(2h+2) - 2}{h} \\ &= \lim_{h \rightarrow 0^+} 2 \\ &= 2\end{aligned}$$

No, the slope from the left is  $-2$  and the slope from the right is  $2$ . The two-sided limit of the difference quotient does not exist.

16. First, note that  $f(0) = 0$ .

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{-h-0}{h} = -1$$

$$\begin{aligned}\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^+} \frac{(h^2 - h) - 0}{h} \\ &= \lim_{h \rightarrow 0^+} (h-1) \\ &= -1\end{aligned}$$

Yes. The slope is  $-1$ .

17. First, note that  $f(2) = \frac{1}{2}$

$$\begin{aligned}\lim_{h \rightarrow 0^-} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0^-} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{2 - (2+h)}{2h(2+h)} \\ &= \lim_{h \rightarrow 0^-} \frac{-h}{2h(2+h)} \\ &= \lim_{h \rightarrow 0^-} -\frac{1}{2(2+h)} \\ &= -\frac{1}{4}\end{aligned}$$

$$\begin{aligned}\lim_{h \rightarrow 0^+} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0^+} \frac{\frac{4-(2+h)}{4} - \frac{1}{2}}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{[4 - (2+h)] - 2}{4h} \\ &= \lim_{h \rightarrow 0^+} \frac{-h}{4h} \\ &= -\frac{1}{4}\end{aligned}$$

Yes. The slope is  $-\frac{1}{4}$ .

18. No; the function is discontinuous at  $x = \frac{3\pi}{4}$

because

$$\lim_{x \rightarrow (3\pi/4)^-} f(x) = \lim_{x \rightarrow (3\pi/4)^-} \sin x = \sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\text{but } f\left(\frac{3\pi}{4}\right) = \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}.$$

19. (a)  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

$$\begin{aligned}&= \lim_{h \rightarrow 0} \frac{[(a+h)^2 + 2] - (a^2 + 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 + 2 - a^2 - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2ah + h^2}{h} \\ &= \lim_{h \rightarrow 0} (2a + h) \\ &= 2a\end{aligned}$$

- (b) The slope of the tangent steadily increases as  $a$  increases.

20. (a)  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{2}{a+6} - \frac{2}{a}}{h}$

$$\begin{aligned}&= \lim_{h \rightarrow 0} \frac{2a - 2(a+h)}{ah(a+h)} \\ &= \lim_{h \rightarrow 0} \frac{-2}{a(a+h)} \\ &= -\frac{2}{a^2}\end{aligned}$$

- (b) The slope of the tangent is always negative. The tangents are very steep near  $x = 0$  and nearly horizontal as  $a$  moves away from the origin.

$$\begin{aligned}
 21. \quad (a) \quad & \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{a+h-1} - \frac{1}{a-1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(a-1) - (a+h-1)}{h(a-1)(a+h-1)} \\
 &= \lim_{h \rightarrow 0} -\frac{1}{(a-1)(a+h-1)} \\
 &= -\frac{1}{(a-1)^2}
 \end{aligned}$$

(b) The slope of the tangent is always negative. The tangents are very steep near  $x = 1$  and nearly horizontal as  $a$  moves away from the origin.

$$\begin{aligned}
 22. \quad (a) \quad & \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[9 - (a+h)^2] - (9 - a^2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{9 - a^2 - 2ah - h^2 - 9 + a^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2ah - h^2}{h} \\
 &= \lim_{h \rightarrow 0} (-2a - h) \\
 &= -2a
 \end{aligned}$$

(b) The slope of the tangent steadily decreases as  $a$  increases.

$$\begin{aligned}
 23. \quad & \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3(1+h) - 7 - (3 \cdot 1 - 7)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3 + 3h - 7 - (-4)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3h}{h} \\
 &= \lim_{h \rightarrow 0} 3 \\
 &= 3
 \end{aligned}$$

The instantaneous rate of change is 3 ft/sec.

$$\begin{aligned}
 24. \quad & \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3(3+h)^2 + 2(3+h) - (3 \cdot 3^2 + 2 \cdot 3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3(9 + 6h + h^2) + 6 + 2h - 33}{h} \\
 &= \lim_{h \rightarrow 0} \frac{20h + 3h^2}{h} \\
 &= \lim_{h \rightarrow 0} (20 + 3h) \\
 &= (20 + 3 \cdot 0) \\
 &= 20
 \end{aligned}$$

The instantaneous rate of change is 20 ft/sec.

$$\begin{aligned}
 25. \quad & \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \rightarrow 0} \frac{\frac{(2+h)+1}{2+h} - \frac{2+1}{2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{2(3+h) - 3(2+h)}{2(2+h)}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-h}{2(2+h)h} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{2(2+h)} \\
 &= \frac{-1}{2(2+0)} \\
 &= -\frac{1}{4}
 \end{aligned}$$

The instantaneous rate of change is  $-\frac{1}{4}$  ft/sec.

$$\begin{aligned}
 26. \quad & \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(2+h)^3 - 1 - (2^3 - 1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{8 + 12h + 6h^2 + h^3 - 1 - 7}{h} \\
 &= \lim_{h \rightarrow 0} \frac{12h + 6h^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} (12 + 6h + h^2) \\
 &= 12 + 6 \cdot 0 + 0^2 \\
 &= 12
 \end{aligned}$$

The instantaneous rate of change is 12 ft/sec.

27. Let
- $f(t) = 100 - 4.9t^2$
- .

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[100 - 4.9(2+h)^2] - [100 - 4.9(2)^2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{100 - 19.6 - 19.6h - 4.9h^2 - 100 + 19.6}{h} \\ &= \lim_{h \rightarrow 0} (-19.6 - 4.9h) \\ &= -19.6 \end{aligned}$$

The object is falling at a speed of 19.6 m/sec.

28. Let
- $f(t) = 3t^2$
- .

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(10+h) - f(10)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3(10+h)^2 - 300}{h} \\ &= \lim_{h \rightarrow 0} \frac{300 + 60h + 3h^2 - 300}{h} \\ &= \lim_{h \rightarrow 0} (60 + 3h) \\ &= 60 \end{aligned}$$

The rocket's speed is 60 ft/sec.

29. Let
- $f(r) = \pi r^2$
- , the area of a circle of radius
- $r$
- .

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\pi(3+h)^2 - \pi(3)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{9\pi + 6\pi h + \pi h^2 - 9\pi}{h} \\ &= \lim_{h \rightarrow 0} (6\pi + \pi h) \\ &= 6\pi \end{aligned}$$

The area is changing at a rate of

$6\pi$  in<sup>2</sup>/in., that is,  $6\pi$  square inches of area per inch of radius.

30. Let
- $f(r) = \frac{4}{3}\pi r^3$
- .

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{4}{3}\pi(2+h)^3 - \frac{4}{3}\pi(2)^3}{h} \\ &= \frac{4}{3}\pi \lim_{h \rightarrow 0} \frac{(2+h)^3 - 2^3}{h} \\ &= \frac{4}{3}\pi \lim_{h \rightarrow 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\ &= \frac{4}{3}\pi \lim_{h \rightarrow 0} (12 + 6h + h^2) \\ &= \frac{4}{3}\pi \cdot 12 \\ &= 16\pi \end{aligned}$$

The volume is changing at a rate of

$16\pi$  in<sup>3</sup>/in., that is,  $16\pi$  cubic inches of volume per inch of radius.

- 31.
- $\lim_{h \rightarrow 0} \frac{s(1+h) - s(1)}{h}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{1.86(1+h)^2 - 1.86(1)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{1.86 + 3.72h + 1.86h^2 - 1.86}{h} \\ &= \lim_{h \rightarrow 0} (3.72 + 1.86h) \\ &= 3.72 \end{aligned}$$

The speed of the rock is 3.72 m/sec.

- 32.
- $\lim_{h \rightarrow 0} \frac{s(2+h) - s(2)}{h}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{11.44(2+h)^2 - 11.44(2)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{45.76 + 45.76h + 11.44h^2 - 45.76}{h} \\ &= \lim_{h \rightarrow 0} (45.76 + 11.44h) \\ &= 45.76 \end{aligned}$$

The speed of the rock is 45.76 m/sec.



33. First, find the slope of the tangent at
- $x = a$
- .

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(a+h)^2 + 4(a+h) - 1] - (a^2 + 4a - 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 + 4a + 4h - 1 - a^2 - 4a + 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{2ah + h^2 + 4h}{h} \\ &= \lim_{h \rightarrow 0} (2a + h + 4) \\ &= 2a + 4 \end{aligned}$$

The tangent at  $x = a$  is horizontal when  $2a + 4 = 0$ , or  $a = -2$ . The tangent line is horizontal at  $(-2, f(-2)) = (-2, -5)$ .

34. First, find the slope of the tangent at
- $x = a$
- .

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3 - 4(a+h) - (a+h)^2] - (3 - 4a - a^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3 - 4a - 4h - a^2 - 2ah - h^2 - 3 + 4a + a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{-4h - 2ah - h^2}{h} \\ &= \lim_{h \rightarrow 0} (-4 - 2a - h) \\ &= -4 - 2a \end{aligned}$$

The tangent at  $x = a$  is horizontal when  $-4 - 2a = 0$ , or  $a = -2$ . The tangent line is horizontal at  $(-2, f(-2)) = (-2, 7)$ .

35. (a) From Exercise 21, the slope of the curve

at  $x = a$ , is  $-\frac{1}{(a-1)^2}$ . The tangent has

slope  $-1$  when  $-\frac{1}{(a-1)^2} = -1$ , which

gives  $(a-1)^2 = 1$ , so  $a = 0$  or  $a = 2$ . Note

that  $y(0) = \frac{1}{0-1} = -1$  and

$y(2) = \frac{1}{2-1} = 1$ , so we need to find the

equations of lines of slope  $-1$  passing through  $(0, -1)$  and  $(2, 1)$ , respectively.

At  $x = 0$ :  $y = -1(x-0) - 1$   
 $y = -x - 1$

At  $x = 2$ :  $y = -1(x-2) + 1$   
 $y = -x + 3$

- (b) The normal has slope
- $-\frac{1}{-1} = 1$
- since the

tangent has slope  $-1$ , so we again need to find lines through  $(0, -1)$  and  $(2, 1)$ , this time using slope  $1$ .

At  $x = 0$ :  $y = 1(x-0) - 1$

$$y = x - 1$$

At  $x = 2$ :  $y = 1(x-2) + 1$

$$y = x - 1$$

There is only one such line. It is normal to the curve at two points and its equation is  $y = x - 1$ .

36. Consider a line that passes through
- $(1, 12)$
- and a point
- $(a, 9 - a^2)$
- on the curve. Using the result of Exercise 22, this line will be tangent to the curve at
- $a$
- if its slope is
- $-2a$
- .

$$\begin{aligned} \frac{(9 - a^2) - 12}{a - 1} &= -2a \\ 9 - a^2 - 12 &= -2a(a - 1) \\ -a^2 - 3 &= -2a^2 + 2a \\ a^2 - 2a - 3 &= 0 \\ (a + 1)(a - 3) &= 0 \end{aligned}$$

$$a = -1 \text{ or } a = 3$$

At  $a = -1$  (or  $x = -1$ ), the slope is  $-2(-1) = 2$ .

$$y = 2(x - 1) + 12$$

$$y = 2x + 10$$

At  $a = 3$  (or  $x = 3$ ), the slope is  $-2(3) = -6$ .

$$y = -6(x - 1) + 12$$

$$y = -6x + 18$$

- 37.
- $$\begin{aligned} \lim_{h \rightarrow 0} \frac{\Delta T}{h} &= \lim_{h \rightarrow 0} \frac{99 + \frac{4}{1+(2+h)} - \left(99 + \frac{4}{1+2}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-4h}{3(3+h)h} \\ &= \frac{-4}{9} \text{ degrees per mg.} \end{aligned}$$

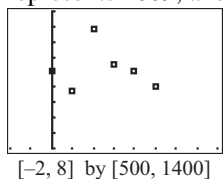
This means that when the dosage is 2 mg, a small additional dosage,  $\Delta D$ , will result in a drop in the patient's temperature of

approximately  $\frac{4}{9} \Delta D$  degrees.

$$\begin{aligned}
 38. \quad \lim_{h \rightarrow 0} \frac{\Delta H}{h} &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{(40+h)^2}{64} - \frac{40^2}{64} \right] \\
 &= \lim_{h \rightarrow 0} \frac{80h+h^2}{64h} \\
 &= \frac{80}{64} \\
 &= 1.25 \text{ ft per ft/s}
 \end{aligned}$$

This means that if the initial speed is increased by a small amount,  $\Delta v$ , the maximum height will increase by approximately  $1.25\Delta v$  feet.

39. (a) Note:  $x = 0$  represents 2008,  $x = 1$  represents 2009, and so on.



(b) slope of  $PQ_1 = \frac{1051-1015}{3-0} = \frac{36}{3} = 12$

The slope of  $PQ_1$  is 12 million bushels per year.

slope of  $PQ_2 = \frac{1007-1015}{4-0} = \frac{-8}{4} = -2$

The slope of  $PQ_2$  is -2 million bushels per year.

slope of  $PQ_3 = \frac{900-1015}{5-0} = \frac{-115}{5} = -23$

The slope of  $PQ_3$  is -23 million bushels per year.

40. (a)  $\frac{\$(633-616) \text{ billion}}{(2013-2008) \text{ years}}$   
 $= \frac{\$17 \text{ billion}}{5 \text{ years}}$   
 $= \$3.4 \frac{\text{billion}}{\text{year}}$

(b)  $\frac{\$(706-616) \text{ billion}}{(2011-2008) \text{ years}}$   
 $= \$30 \frac{\text{billion}}{\text{year}}$

(c)  $\frac{\$(633-706) \text{ billion}}{(2013-2011) \text{ years}} = -\$36.5 \frac{\text{billion}}{\text{year}}$

- (d) One possible reason is that the war in Afghanistan and increased spending to prevent terrorist attacks in the U.S. caused an unusual increase in defense spending.

41. True. The normal line is perpendicular to the tangent line at the point.

42. False. There's no tangent at  $x = 0$  because  $f$  has no slope at  $x = 0$ .

43. D;  $\frac{-3-5}{-1-2} = \frac{8}{3}$

44. E;  $\frac{f(3)-f(1)}{3-1} = \frac{3^2+3-1^2-1}{2} = \frac{10}{2} = 5$

45. C;

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h} &= \lim_{h \rightarrow 0} \left( \frac{2}{x+h} - \frac{2}{x} \right) \frac{1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2x-2x-2h}{x^2+hx} \left( \frac{1}{h} \right) \\
 &= -\frac{2}{x^2}
 \end{aligned}$$

$$y = -\frac{2}{(1)^2} = -2$$

$$y = m(x-x_1) + y_1$$

$$y = -2(x-1) + 2$$

$$y = -2x + 4$$

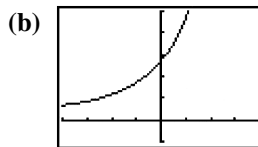
46. A; From 39,  $m_2 = -\frac{1}{m_1}$

$$m_2 = \frac{1}{2}$$

$$y = \frac{1}{2}(x-1) + 2$$

$$y = \frac{1}{2}x + \frac{3}{2}$$

47. (a)  $\frac{f(1+h)-f(1)}{h} = \frac{e^{1+h}-e}{h}$



$[-4, 4]$  by  $[-1, 5]$

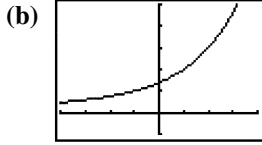
X	Y1
-.01	2.7047
-.5E-4	2.7176
-1E-4	2.7181
0	ERR:00
1E-4	2.7184
.5E-4	2.7189
.01	2.7319

Limit  $\approx 2.718$

- (c) They're about the same.

(d) Yes, it has a tangent whose slope is about  $e$ .

48. (a) 
$$\frac{f(1+h) - f(1)}{h} = \frac{2^{1+h} - 2}{h}$$



[-4, 4] by [-1, 5]

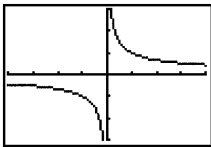
X	Y1
0.000000	1.38115
1E-4	1.3861
1E-4	1.3862
0	ERROR
1E-4	1.3863
5E-4	1.3865
.01	1.3911

Limit  $\approx 1.386$

(c) They're about the same.

(d) Yes, it has a tangent whose slope is about  $\ln 4$ .

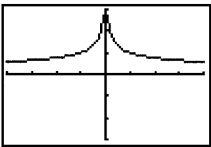
49. Let  $f(x) = x^{2/5}$ . The graph of 
$$y = \frac{f(0+h) - f(0)}{h} = \frac{f(h)}{h}$$
 is shown.



[-4, 4] by [-3, 3]

The left- and right-hand limits are  $-\infty$  and  $\infty$ , respectively. Since they are not the same, the curve does not have a vertical tangent at  $x = 0$ . No.

50. Let  $f(x) = x^{3/5}$ . The graph of 
$$y = \frac{f(0+h) - f(0)}{h} = \frac{f(h)}{h}$$
 is shown.

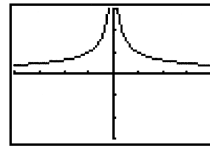


[-4, 4] by [-3, 3]

Yes, the curve has a vertical tangent at  $x = 0$

because  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \infty$ .

51. Let  $f(x) = x^{1/3}$ . The graph of 
$$y = \frac{f(0+h) - f(0)}{h} = \frac{f(h)}{h}$$
 is shown.

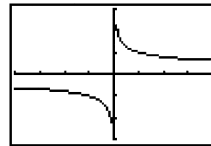


[-4, 4] by [-3, 3]

Yes, the curve has a vertical tangent at  $x = 0$

because  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \infty$ .

52. Let  $f(x) = x^{2/3}$ . The graph of 
$$y = \frac{f(0+h) - f(0)}{h} = \frac{f(h)}{h}$$
 is shown.



[-4, 4] by [-3, 3]

The left- and right-hand limits are  $-\infty$  and  $\infty$ , respectively. Since they are not the same, the curve does not have a vertical tangent at  $x = 0$ . No.

53. This function has a tangent with slope zero at the origin.

It is squeezed between two functions,  $y = x^2$  and  $y = -x^2$ , both of which have slope zero at the origin.

Looking at the difference quotient,

$$-h \leq \frac{f(0+h) - f(0)}{h} \leq h, \text{ so the Squeeze}$$

Theorem tells us the limit is 0.

54. This function does not have a tangent line at the origin. As the function oscillates between  $y = x$  and  $y = -x$  infinitely often near the origin, there are an infinite number of difference quotients (secant line slopes) with a value of 1 and with a value of  $-1$ . Thus the limit of the difference quotient doesn't exist.

The difference quotient is

$$\frac{f(0+h) - f(0)}{h} = \sin \frac{1}{h} \text{ which oscillates}$$

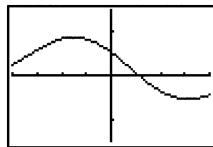
between 1 and  $-1$  infinitely often near zero.

55. Let  $f(x) = \sin x$ . The difference quotient is

$$\frac{f(1+h) - f(1)}{h} = \frac{\sin(1+h) - \sin(1)}{h}.$$

A graph and table for the difference quotient

are shown.



$[-4, 4]$  by  $[-1.5, 1.5]$

x	f(x)
-0.05	.5424
-0.02	.54114
-0.01	.54072
0	ERROR
.001	.53988
.002	.53946
.005	.5382

Since the limit as  $h \rightarrow 0$  is about 0.540, the slope of  $y = \sin x$  at  $x = 1$  is about 0.540.

56. The average rate of change of  $f$  over the interval  $[3, 3+h]$  is
- $$\frac{f(3+h) - f(3)}{(3+h) - 3} = \frac{f(3+h) - f(3)}{h}$$
- which is the difference quotient with  $a = 3$ .

57. (a) If  $x = a + h$ , then  $h = x - a$ . Replace  $a + h$  with  $x$  and  $h$  with  $x - a$ .

$$\frac{f(a+h) - f(a)}{h} = \frac{f(x) - f(a)}{x - a}$$

- (b) Depending on the situation, one form might be more convenient to use than the other.

### Quick Quiz Sections 2.3 and 2.4

1. D;  $\frac{f(3) - f(0)}{3 - 0} = \frac{\sqrt{3+1} - \sqrt{0+1}}{3} = \frac{2-1}{3} = \frac{1}{3}$

2. E;  $f(4-h) \approx \frac{3}{4}(4) \approx 3$  where  $h \rightarrow 0$

$$f(4) = 2$$

$$f(4+h) \approx -4+7 \approx 3$$

3. B;  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

$$\lim_{h \rightarrow 0} \frac{9 - (x+h)^2 - (9 - x^2)}{h}$$

$$\lim_{h \rightarrow 0} \frac{-2xh - h^2}{h} = \lim_{h \rightarrow 0} (-2x - h) = -2x$$

$$y = 9 - x^2 = 9 - (2)^2$$

$$y = 5$$

$$y' = -2(2) = -4$$

$$y = -4(x-2) + 5$$

$$y = -4x + 13$$

4. (a)  $f(3) = 2(3) - (3)^2 = 6 - 9 = -3$

(b)  $f(3+h) = 2(3+h) - (3+h)^2$   
 $= 6 + 2h - (9 + 6h + h^2)$   
 $= -3 - 4h - h^2$

(c)  $\frac{f(3+h) - f(3)}{h} = \frac{-3 - 4h - h^2 - (-3)}{h}$   
 $= -4 - h$

(d)  $\lim_{h \rightarrow 0} (-4 - h) = -4$

### Chapter 2 Review Exercises (pp. 97–99)

1.  $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 1) = (-2)^3 - 2(-2)^2 + 1 = -15$

2.  $\lim_{x \rightarrow -2} \frac{x^2 + 1}{3x^2 - 2x + 5} = \frac{(-2)^2 + 1}{3(-2)^2 - 2(-2) + 5} = \frac{5}{21}$

3. No limit, because the expression  $\sqrt{1-2x}$  is undefined for values of  $x$  near 4.

4. No limit, because the expression  $\sqrt[4]{9-x^2}$  is undefined for values of  $x$  near 5.

5.  $\lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim_{x \rightarrow 0} \frac{2 - (2+x)}{2x(2+x)}$   
 $= \lim_{x \rightarrow 0} \frac{-x}{2x(2+x)}$   
 $= \lim_{x \rightarrow 0} \left( -\frac{1}{2(2+x)} \right)$   
 $= -\frac{1}{2(2+0)}$   
 $= -\frac{1}{4}$

6.  $\lim_{x \rightarrow \pm\infty} \frac{2x^2 + 3}{5x^2 + 7} = \lim_{x \rightarrow \pm\infty} \frac{2x^2}{5x^2} = \frac{2}{5}$

7. An end behavior model for

$$\frac{x^4 + x^3}{12x^3 + 128} \text{ is } \frac{x^4}{12x^3} = \frac{1}{12}x.$$

Therefore

$$\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{12x^3 + 128} = \lim_{x \rightarrow \infty} \frac{1}{12}x = \infty$$

$$\lim_{x \rightarrow -\infty} \frac{x^4 + x^3}{12x^3 + 128} = \lim_{x \rightarrow -\infty} \frac{1}{12}x = -\infty$$

8.  $\lim_{x \rightarrow 0} \frac{\sin 2x}{4x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} = \frac{1}{2}(1) = \frac{1}{2}$

9. Multiply the numerator and denominator by  $\sin x$ .

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x \csc x + 1}{x \csc x} &= \lim_{x \rightarrow 0} \frac{x + \sin x}{x} \\ &= \lim_{x \rightarrow 0} \left( 1 + \frac{\sin x}{x} \right) \\ &= \left( \lim_{x \rightarrow 0} 1 \right) + \left( \lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

10.  $\lim_{x \rightarrow 0} e^x \sin x = e^0 \sin 0 = 1 \cdot 0 = 0$

11. Let  $x = \frac{7}{2} + h$ , where  $h$  is in  $\left(0, \frac{1}{2}\right)$ . Then  
 $\text{int}(2x - 1) = \text{int}\left[2\left(\frac{7}{2}\right) + 2h - 1\right] = \text{int}(6 + 2h) = 6$ ,  
 because  $6 + 2h$  is in  $(6, 7)$ . Therefore,  
 $\lim_{x \rightarrow 7/2^+} \text{int}(2x - 1) = \lim_{x \rightarrow 7/2^+} 6 = 6$ .

12. Let  $x = \frac{7}{2} + h$ , where  $h$  is in  $\left(-\frac{1}{2}, 0\right)$ . Then  
 $\text{int}(2x - 1) = \text{int}\left[2\left(\frac{7}{2}\right) + 2h - 1\right] = \text{int}(6 + 2h) = 5$ ,  
 because  $6 + 2h$  is in  $(5, 6)$ . Therefore,  
 $\lim_{x \rightarrow 7/2^-} \text{int}(2x - 1) = \lim_{x \rightarrow 7/2^-} 5 = 5$

13. Since  $\lim_{x \rightarrow \infty} (-e^{-x}) = \lim_{x \rightarrow \infty} e^{-x} = 0$ , and  
 $-e^{-x} \leq e^{-x} \cos x \leq e^{-x}$  for all  $x$ , the Sandwich  
 Theorem gives  $\lim_{x \rightarrow \infty} e^{-x} \cos x = 0$ .

14. Since the expression  $x$  is an end behavior  
 model for both  
 $x + \sin x$  and  $x + \cos x$ ,  $\lim_{x \rightarrow \infty} \frac{x + \sin x}{x + \cos x} = \lim_{x \rightarrow \infty} \frac{x}{x} = 1$ .

15. Limit exists.

16. Limit exists.

17. Limit exists.

18. Limit does not exist.

19. Limit exists.

20. Limit exists.

21. Yes

22. No

23. No

24. Yes

25. (a)  $\lim_{x \rightarrow 3^-} g(x) = 1$

(b)  $g(3) = 1.5$

(c) No, since  $\lim_{x \rightarrow 3^-} g(x) \neq g(3)$ .

(d)  $g$  is discontinuous at  $x = 3$  (and at points not in the domain).

(e) Yes, the discontinuity at  $x = 3$  can be removed by assigning the value 1 to  $g(3)$ .

26. (a)  $\lim_{x \rightarrow 1^-} k(x) = 1.5$

(b)  $\lim_{x \rightarrow 1^+} k(x) = 0$

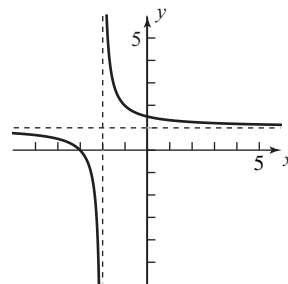
(c)  $k(1) = 0$

(d) No, since  $\lim_{x \rightarrow 1^-} k(x) \neq k(1)$ .

(e)  $k$  is discontinuous at  $x = 1$  (and at points not in the domain).

(f) No, the discontinuity at  $x = 1$  is not removable because the one-sided limits are different.

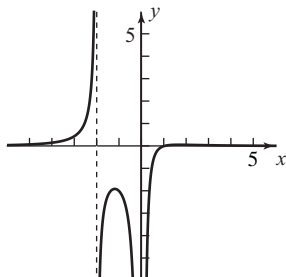
27.



(a) Vertical asymptote:  $x = -2$

$$\begin{aligned} \text{(b) Left-hand limit} &= \lim_{x \rightarrow 2^-} \frac{x+3}{x+2} = -\infty \\ \text{Right-hand limit:} & \lim_{x \rightarrow 2^+} \frac{x+3}{x+2} = \infty \end{aligned}$$

28.



(a) Vertical asymptotes:  $x = 0$ ,  $x = -2$

(b) At  $x = 0$ :

$$\text{Left-hand limit} = \lim_{x \rightarrow 0^-} \frac{x-1}{x^2(x+2)} = -\infty$$

$$\text{Right-hand limit} = \lim_{x \rightarrow 0^+} \frac{x-1}{x^2(x+2)} = -\infty$$

At  $x = -2$ :

$$\text{Left-hand limit} = \lim_{x \rightarrow -2^-} \frac{x-1}{x^2(x+2)} = \infty$$

$$\text{Right-hand limit} = \lim_{x \rightarrow -2^+} \frac{x-1}{x^2(x+2)} = -\infty$$

29. (a) At  $x = -1$ :

$$\begin{aligned} \text{Left-hand limit} &= \lim_{x \rightarrow -1^-} f(x) \\ &= \lim_{x \rightarrow -1^-} (1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{Right-hand limit} &= \lim_{x \rightarrow -1^+} f(x) \\ &= \lim_{x \rightarrow -1^+} (-x) \\ &= 1 \end{aligned}$$

At  $x = 0$ :

$$\begin{aligned} \text{Left-hand limit} &= \lim_{x \rightarrow 0^-} f(x) \\ &= \lim_{x \rightarrow 0^-} (-x) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Right-hand limit} &= \lim_{x \rightarrow 0^+} f(x) \\ &= \lim_{x \rightarrow 0^+} (-x) \\ &= 0 \end{aligned}$$

At  $x = 1$ :

$$\begin{aligned} \text{Left-hand limit} &= \lim_{x \rightarrow 1^-} f(x) \\ &= \lim_{x \rightarrow 1^-} (-x) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{Right-hand limit} &= \lim_{x \rightarrow 1^+} f(x) \\ &= \lim_{x \rightarrow 1^+} (1) \\ &= 1 \end{aligned}$$

(b) At  $x = -1$ : Yes, the limit is 1.  
At  $x = 0$ : Yes, the limit is 0.  
At  $x = 1$ : No, the limit doesn't exist because the two one-sided limits are different.

(c) At  $x = -1$ : Continuous because  $f(-1) =$  the limit.

At  $x = 0$ : Discontinuous because  $f(0) \neq$  the limit.

At  $x = 1$ : Discontinuous because the limit does not exist.

$$\begin{aligned} \text{30. (a) Left-hand limit} &= \lim_{x \rightarrow 1^-} f(x) \\ &= \lim_{x \rightarrow 1^-} |x^3 - 4x| \\ &= |(1)^3 - 4(1)| \\ &= |-3| \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{Right-hand limit} &= \lim_{x \rightarrow 1^+} f(x) \\ &= \lim_{x \rightarrow 1^+} (x^2 - 2x - 2) \\ &= (1)^2 - 2(1) - 2 \\ &= -3 \end{aligned}$$

(b) No, because the two one-sided limits are different.

(c) Every place except for  $x = 1$

(d) At  $x = 1$

31. Since  $f(x)$  is a quotient of polynomials, it is continuous and its points of discontinuity are the points where it is undefined, namely  $x = -2$  and  $x = 2$ .

32. There are no points of discontinuity, since  $g(x)$  is continuous and defined for all real numbers.

33. (a) End behavior model:  $\frac{2x}{x^2}$ , or  $\frac{2}{x}$

(b) Horizontal asymptote:  $y = 0$  (the  $x$ -axis)

34. (a) End behavior model:  $\frac{2x^2}{x^2}$ , or 2

(b) Horizontal asymptote:  $y = 2$

35. (a) End behavior model:  $\frac{x^3}{x}$ , or  $x^2$

(b) Since the end behavior model is quadratic, there are no horizontal asymptotes.

36. (a) End behavior model:  $\frac{x^4}{x^3}$ , or  $x$

(b) Since the end behavior model represents a nonhorizontal line, there are no horizontal asymptotes.

37. (a) Since  $\lim_{x \rightarrow \infty} \frac{x + e^x}{e^x} = \lim_{x \rightarrow \infty} \left( \frac{x}{e^x} + 1 \right) = 1$ , a right end behavior model is  $e^x$ .

(b) Since  $\lim_{x \rightarrow -\infty} \frac{x + e^x}{x} = \lim_{x \rightarrow -\infty} \left( 1 + \frac{e^x}{x} \right) = 1$ , a left end behavior model is  $x$ .

38. (a, b) Note that

$$\lim_{x \rightarrow \pm\infty} \left( -\frac{1}{\ln|x|} \right) = \lim_{x \rightarrow \pm\infty} \left( \frac{1}{\ln|x|} \right) = 0 \text{ and}$$

$$-\frac{1}{\ln|x|} < \frac{\sin x}{\ln|x|} < \frac{1}{\ln|x|} \text{ whenever } |x| > 1.$$

Therefore, the Sandwich Theorem gives

$$\lim_{x \rightarrow \pm\infty} \frac{\sin x}{\ln|x|} = 0. \text{ Hence}$$

$$\begin{aligned} \lim_{x \rightarrow \pm\infty} \frac{\ln|x| + \sin x}{\ln|x|} &= \lim_{x \rightarrow \pm\infty} \left( 1 + \frac{\sin x}{\ln|x|} \right) \\ &= 1 + 0 \\ &= 1, \end{aligned}$$

so  $\ln|x|$  is both a right end behavior model and a left end behavior model.

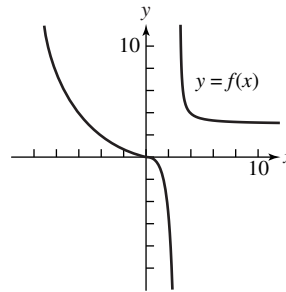
39. 
$$\begin{aligned} \lim_{x \rightarrow 3} f(x) &= \lim_{x \rightarrow 3} \frac{x^2 + 2x - 15}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 5)}{x - 3} \\ &= \lim_{x \rightarrow 3} (x + 5) \\ &= 3 + 5 \\ &= 8 \end{aligned}$$

Assign the value  $k = 8$ .

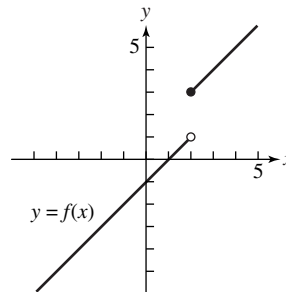
40. 
$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{1}{2}(1) = \frac{1}{2}$$

Assign the value  $k = \frac{1}{2}$ .

41. One possible answer:



42. One possible answer:



43. 
$$\frac{f\left(\frac{\pi}{2}\right) - f(0)}{\left(\frac{\pi}{2}\right) - 0} = \frac{2 - 1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

44. 
$$\begin{aligned} \lim_{h \rightarrow 0} \frac{V(a+h) - V(a)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{3}\pi(a+h)^2 H - \frac{1}{3}\pi a^2 H}{h} \\ &= \frac{1}{3}\pi H \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - a^2}{h} \\ &= \frac{1}{3}\pi H \lim_{h \rightarrow 0} (2a + h) \\ &= \frac{1}{3}\pi H(2a) \\ &= \frac{2}{3}\pi aH \end{aligned}$$
45. 
$$\begin{aligned} \lim_{h \rightarrow 0} \frac{S(a+h) - S(a)}{h} &= \lim_{h \rightarrow 0} \frac{6(a+h)^2 - 6a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{6a^2 + 12ah + 6h^2 - 6a^2}{h} \\ &= \lim_{h \rightarrow 0} (12a + 6h) \\ &= 12a \end{aligned}$$
46. 
$$\begin{aligned} \lim_{h \rightarrow 0} \frac{y(a+h) - y(a)}{h} &= \lim_{h \rightarrow 0} \frac{[(a+h)^2 - (a+h) - 2] - (a^2 - a - 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - a - h - 2 - a^2 + a + 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2ah + h^2 - h}{h} \\ &= \lim_{h \rightarrow 0} (2a + h - 1) \\ &= 2a - 1 \end{aligned}$$
47. (a) 
$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0} \frac{[(1+h)^2 - 3(1+h)] - (-2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 + 2h + h^2 - 3 - 3h + 2}{h} \\ &= \lim_{h \rightarrow 0} (-1 + h) \\ &= -1 \end{aligned}$$
- (b) The tangent at  $P$  has slope  $-1$  and passes through  $(1, -2)$ .  
 $y = -1(x-1) - 2$   
 $y = -x - 1$

- (c) The normal at  $P$  has slope  $1$  and passes through  $(1, -2)$ .  
 $y = 1(x-1) - 2$   
 $y = x - 3$

48. At  $x = a$ , the slope of the curve is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \rightarrow 0} \frac{[(a+h)^2 - 3(a+h)] - (a^2 - 3a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - 3a - 3h - a^2 + 3a}{h} \\ &= \lim_{h \rightarrow 0} \frac{2ah - 3h + h^2}{h} \\ &= \lim_{h \rightarrow 0} (2a - 3 + h) \\ &= 2a - 3 \end{aligned}$$

The tangent is horizontal when  $2a - 3 = 0$ , at

$$a = \frac{3}{2} \left( \text{or } x = \frac{3}{2} \right). \text{ Since } f\left(\frac{3}{2}\right) = -\frac{9}{4}, \text{ the}$$

point where this occurs is  $\left(\frac{3}{2}, -\frac{9}{4}\right)$ .

49. 
$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\Delta H}{h} &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{(30+h)^2}{64} - \frac{30^2}{64} \right] \\ &= \lim_{h \rightarrow 0} \frac{60h + h^2}{64h} \\ &= \frac{60}{64} \\ &= 0.9375 \text{ ft per ft/s} \end{aligned}$$

This means that if the initial speed is increased by a small amount,  $\Delta v$ , the maximum height will increase by approximately  $0.9375\Delta v$  feet.

50. 
$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\Delta A}{h} &= \lim_{h \rightarrow 0} \frac{\pi(2+h)^2 - \pi 2^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{4\pi h + \pi h^2}{h} \\ &= 4\pi \text{ m}^2 \text{ per m} \end{aligned}$$

This means that if the initial measurement is increased by a small amount,  $\Delta r$ , the area will increase by approximately  $4\pi \Delta r$  m<sup>2</sup>.

51. (a)  $p(0) = \frac{200}{1 + 7e^{-0.1(0)}} = \frac{200}{8} = 25$

Perhaps this is the number of bears placed in the reserve when it was established.

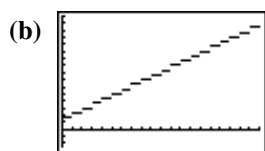


(b)  $\lim_{t \rightarrow \infty} p(t) = \lim_{t \rightarrow \infty} \frac{200}{1 + 7e^{-0.1t}} = \frac{200}{1} = 200$

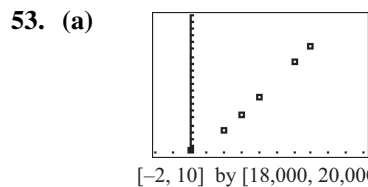
(c) Perhaps this is the maximum number of bears which the reserve can support due to limitations of food, space, or other resources. Or, perhaps the number is capped at 200 and excess bears are moved to other locations.

52. (a)  $f(x) = \begin{cases} 3.20 - 1.35 \text{int}(-x + 1), & 0 < x \leq 20 \\ 0, & x = 0 \end{cases}$

(Note that we cannot use the formula  $f(x) = 3.20 + 1.35 \text{int } x$ , because it gives incorrect results when  $x$  is an integer.)



$[0, 20]$  by  $[-5, 32]$   
 $f$  is discontinuous at integer values of  $x$ : 0, 1, 2, ..., 19.



(b) Let  $x = 0$  correspond to 2006.

$$PQ_1 = \frac{18,538 - 18,019}{3 - 0} = \frac{519}{3} = 173$$

The slope of  $PQ_1$  is 173 thousand people per year.

$$PQ_2 = \frac{19,321 - 18,019}{6 - 0} = \frac{1302}{6} = 217$$

The slope of  $PQ_2$  is 217 thousand people per year.

$$PQ_3 = \frac{19,553 - 18,019}{7 - 0} = \frac{1534}{7} \approx 219.1$$

The slope of  $PQ_3$  is approximately 219.1 thousand people per year.

(c) The slopes in part (b) can be interpreted as the average rates of change. The average rate of change in the population from  $Q_1$  to  $P$ , or from 2006 to 2009, is 173 thousand people per year. The average rate of change in the population from  $Q_2$  to  $P$ , or from 2006 to

2012, is 217 thousand people per year. The average rate of change in the population from  $Q_3$  to  $P$ , or from 2006 to 2013, is 219.1 thousand people per year.

(d) Answers will vary. We could argue that the average rate of change in the population from 2012 to 2013, 219.1 thousand people per year, is a good estimate for the instantaneous rate of change on July 1, 2013, since the data were collected on July 1 each year.

(e) Answers will vary. A linear regression equation for the data is  $y = 227x + 17,930$ , where  $x = 0$  represents 2006. Using this model, the population of Florida in 2020 will be  $17,930 + 227(14) = 21,108$  thousand people.

54. Let  $A = \lim_{x \rightarrow c} f(x)$  and  $B = \lim_{x \rightarrow c} g(x)$ . Then

$A + B = 2$  and  $A - B = 1$ . Adding, we have

$$2A = 3, \text{ so } A = \frac{3}{2}, \text{ whence } \frac{3}{2} + B = 2, \text{ which}$$

gives  $B = \frac{1}{2}$ . Therefore,  $\lim_{x \rightarrow c} f(x) = \frac{3}{2}$  and

$$\lim_{x \rightarrow c} g(x) = \frac{1}{2}.$$

55. (a)  $x^2 - 9 \neq 0$

All  $x$  not equal to  $-3$  or  $3$ .

(b)  $x = -3, x = 3$

(c)  $\lim_{x \rightarrow \infty} \frac{x}{|x^2 - 9|} = 0$   
 $y = 0$

(d) The function is odd:  $f(-x) = -f(x)$

(e)  $f$  is discontinuous at  $-3$  and  $3$ . These are nonremovable discontinuities.

56. (a)  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^2 - a^2x)$   
 $= (2)^2 - a^2(2)$   
 $= 4 - 2a^2$

$$\begin{aligned}
 \text{(b)} \quad \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (4 - 2x^2) \\
 &= 4 - 2(2)^2 \\
 &= 4 - 8 \\
 &= -4
 \end{aligned}$$

(c) For  $x \neq 2$ ,  $f$  is continuous. For  $x = 2$ , we have

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2) = -4$$

as long as  $a = \pm 2$ .

57. (a) By inspection,  $x^3 - 2x^2 + 1 = 0$  when  $x = 1$ . Use synthetic division to write  $x^3 - 2x^2 + 1 = (x - 1)(x^2 - x - 1)$ ; then use the quadratic formula to find the zeros of  $x^2 - x - 1$  to be  $\frac{1 \pm \sqrt{5}}{2}$ . The zeros of  $f$  are 1,  $\frac{1 + \sqrt{5}}{2}$ , and  $\frac{1 - \sqrt{5}}{2}$ .

(b) A right end-behavior model for  $f$  is

$$y = \frac{x^3}{x^2} = x.$$

(c)  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^3 - 2x^2 + 1}{x^2 + 3} = +\infty$  and

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x^3 - 2x^2 + 1}{x^3 + 3x} = 1.$$