Chapter 11
An Introduction to Calculus: Limits, Derivatives, and Integrals

■ Section 11.1 Limits and Motion: The Tangent Problem

Exploration 1

1. $m = \dfrac{4-1}{2-1} = \dfrac{3}{1} = 3$

2. $v_{ave} = \dfrac{\Delta s}{\Delta t} = \dfrac{3 \text{ ft}}{1 \text{ sec}} = 3 \text{ ft/sec}$

3. They are the same.

4. As the slope of the line joining $(a, s(a))$ and $(b, s(b))$

Quick Review 11.1

1. $m = \dfrac{-1-3}{5-(-2)} = \dfrac{-4}{7} = -\dfrac{4}{7}$

2. $m = \dfrac{3-(-1)}{3-(-3)} = \dfrac{4}{6} = \dfrac{2}{3}$

3. $y - 3 = \dfrac{3}{2}(x+2)$ or $y = \dfrac{3}{2}x + 6$

4. $m = \dfrac{-1-6}{4-1} = \dfrac{-7}{3} = -\dfrac{7}{3}, y - 6 = -\dfrac{7}{3}(x-1)$

5. $y - 4 = \dfrac{3}{4}(x-1)$

6. $y - 4 = -\dfrac{4}{3}(x-1)$

7. $\dfrac{4+4h+h^2-4}{h} = \dfrac{4h+h^2}{h} = h + 4$

8. $\dfrac{9+6h+h^2+3+h-12}{h} = \dfrac{h^2+7h}{h} = h + 7$

9. $\dfrac{\dfrac{1}{2+h} - \dfrac{1}{2}}{h} = \dfrac{2-(2+h)}{2(2+h)} \cdot \dfrac{1}{h}$
$= \dfrac{-h}{h} \cdot \dfrac{1}{2(2+h)} = -\dfrac{1}{2(h+2)}$

10. $\dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \dfrac{x-(x+h)}{x(x+h)} \cdot \dfrac{1}{h}$
$= \dfrac{-h}{h} \cdot \dfrac{1}{x(x+h)} = -\dfrac{1}{x(x+h)}$

Section 11.1 Exercises

1. $v_{ave} = \dfrac{\Delta s}{\Delta t} = \dfrac{21 \text{ mi}}{1.75 \text{ hours}} = 12 \text{ mi per hour}$

2. $v_{ave} = \dfrac{\Delta s}{\Delta t} = \dfrac{540 \text{ km}}{4.5 \text{ hours}} = 120 \text{ km per hour}$

3. $s'(4) = \lim\limits_{h \to 0} \dfrac{s(4+h) - s(4)}{h}$
$= \lim\limits_{h \to 0} \dfrac{3(h+4) - 5 - 7}{h}$
$= \lim\limits_{h \to 0} 3 = 3$

4. $s'(2) = \lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$
$= \lim\limits_{h \to 0} \dfrac{\dfrac{2}{h+2+1} - \dfrac{2}{3}}{h}$
$= \lim\limits_{h \to 0} \dfrac{6 - 2(h+3)}{3(h+3)} \cdot \dfrac{1}{h}$
$= \lim\limits_{h \to 0} \dfrac{-2h}{h} \cdot \dfrac{1}{3(h+3)}$
$= \lim\limits_{h \to 0} \dfrac{-2}{3(h+3)} = -\dfrac{2}{9}$

5. $s'(2) = \lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$
$= \lim\limits_{h \to 0} \dfrac{a(h+2)^2 + 5 - (4a+5)}{h}$
$= \lim\limits_{h \to 0} \dfrac{ah^2 + 4ah}{h}$
$= \lim\limits_{h \to 0} (ah + 4a) = 4a$

6. $s'(1) = \lim\limits_{h \to 0} \dfrac{s(1+h) - s(1)}{h}$
$= \lim\limits_{h \to 0} \dfrac{\sqrt{h+2} - \sqrt{2}}{h}$
$= \lim\limits_{h \to 0} \dfrac{\sqrt{h+2} - \sqrt{2}}{h} \cdot \dfrac{\sqrt{h+2} + \sqrt{2}}{\sqrt{h+2} + \sqrt{2}}$
$= \lim\limits_{h \to 0} \dfrac{h+2-2}{h(\sqrt{h+2} + \sqrt{2})}$
$= \lim\limits_{h \to 0} \dfrac{h}{h} \cdot \dfrac{1}{\sqrt{h+2} + \sqrt{2}} = \lim\limits_{h \to 0} \dfrac{1}{\sqrt{h+2} + \sqrt{2}}$
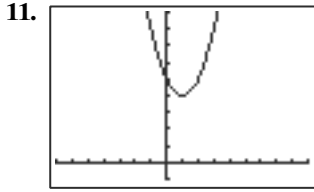$= \dfrac{1}{2\sqrt{2}}$

7. Try $\dfrac{f(1) - f(0)}{1 - 0} = \dfrac{3-2}{1} = 1$

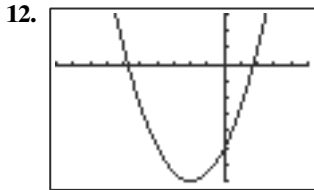**8.** Try $\dfrac{f(2) - f(1)}{2 - 1} = \dfrac{1-2}{1} = -1$

9. No tangent

10. No tangent

11.

$[-7, 9]$ by $[-1, 9]$

$m = 4$

12.

$[-10, 5]$ by $[-7, 3]$

$m = 4$

13.

$[-10, 11]$ by $[-12, 2]$

$m = 12$

14.
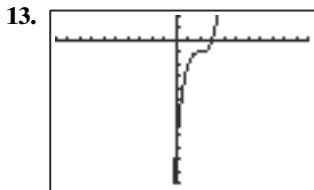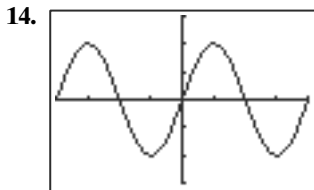
$[-2\pi, 2\pi]$ by $[-3, 3]$

$m = -2$

15. (a) $f'(0) = \lim_{h \to 0} \dfrac{f(0 + h) - f(0)}{h}$

$= \lim_{h \to 0} \dfrac{3 + 48(0 + h) - 16(0 + h)^2 - 3}{h}$

$= \lim_{h \to 0} \dfrac{48h - 16h^2}{h} = 48$

(b) The initial velocity of the rock is $f'(0) = 48$ ft/sec.

16. (a) $f'(0) = \lim_{h \to 0} \dfrac{f(0 + h) - f(0)}{h}$

$= \lim_{h \to 0} \dfrac{170\,(0 + h) - 16(0 + h)^2 - 0}{h}$

$= \lim_{h \to 0}\, (170 - 16h) = 170$

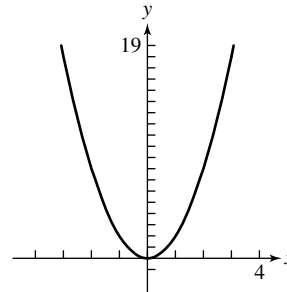**(b)** The initial velocity of the rock is $f'(0) = 170$ ft/sec.

17. (a) $m = \lim_{h \to 0} \dfrac{f(-1 + h) - f(-1)}{h}$

$= \lim_{h \to 0} \dfrac{2(h - 1)^2 - 2}{h} = \lim_{h \to 0} \dfrac{2h^2 - 4h + 2 - 2}{h}$

$= \lim_{h \to 0}\, (2h - 4) = -4$

(b) Since $(-1, f(-1)) = (-1, 2)$ the equation of the tangent line is $y - 2 = -4(x + 1)$.

(c)

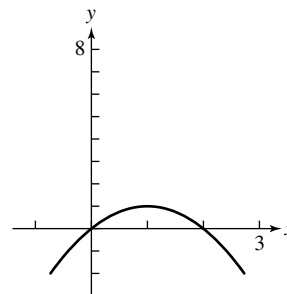**18. (a)** $m = \lim_{h \to 0} \dfrac{f(2 + h) - f(2)}{h}$

$= \lim_{h \to 0} \dfrac{2(h + 2) - (h + 2)^2 - 0}{h}$

$= \lim_{h \to 0} \dfrac{2h + 4 - h^2 - 4h - 4}{h} = \lim_{h \to 0}\, (-h - 2)$

$= -2$

(b) Since $(2, f(2)) = (2, 0)$ the equation of the tangent line is $y = -2(x - 2)$.

(c)

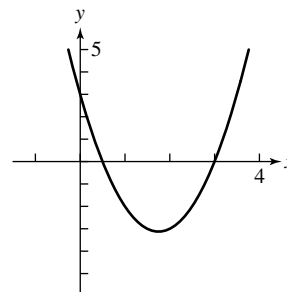**19. (a)** $m = \lim_{h \to 0} \dfrac{f(2 + h) - f(2)}{h}$

$= \lim_{h \to 0} \dfrac{2(h + 2)^2 - 7(h + 2) + 3 - (-3)}{h}$

$= \lim_{h \to 0} \dfrac{2h^2 + 8h + 8 - 7h - 14 + 6}{h}$

$= \lim_{h \to 0}\, (2h + 1) = 1$

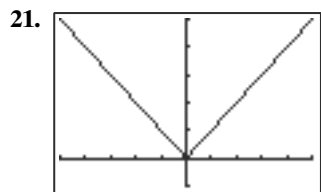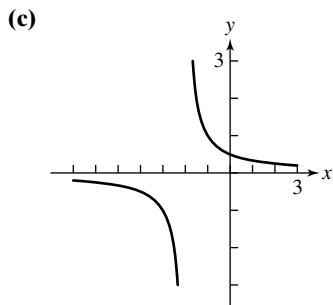**(b)** Since $(2, f(2)) = (2, -3)$ the equation of the tangent line is $y + 3 = 1(x - 2)$, or $y = x - 5$.
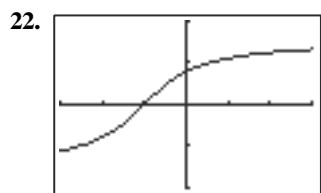
(c)

20. (a) $m = \lim\limits_{h \to 0} \dfrac{f(1 + h) - f(1)}{h}$

$= \lim\limits_{h \to 0} \dfrac{\dfrac{1}{h + 1 + 2} - \dfrac{1}{3}}{h} = \lim\limits_{h \to 0} \dfrac{3 - (h + 3)}{3(h + 3)} \cdot \dfrac{1}{h}$

$= \lim\limits_{h \to 0} \dfrac{-h}{h} \cdot \dfrac{1}{3(h + 3)} = \lim\limits_{h \to 0} \dfrac{-1}{3(h + 3)} = -\dfrac{1}{9}$

(b) Since $(1, f(1)) = \left(1, \dfrac{1}{3}\right)$ the equation of the tangent

line is $y - \dfrac{1}{3} = -\dfrac{1}{9}(x - 1)$.

(c)

21.

$[-5, 5]$ by $[-1, 5]$

At $x = -2$: $m = -1$, at $x = 2$: $m = 1$, at $x = 0$,
m does not exist.

22.

$[-3, 3]$ by $[-2, 2]$

At $x = -2$: $m = 0.5$, at $x = 2$: $m = 0.1$, at $x = 0$,
$m = 0.5$.

23. $\lim\limits_{h \to 0} \dfrac{f(2 + h) - f(2)}{h} = \lim\limits_{h \to 0} \dfrac{1 - (2 + h)^2 - (1 - 4)}{h}$

$= \lim\limits_{h \to 0} \dfrac{-h^2 - 4h - 4 + 4}{h} = \lim\limits_{h \to 0} (-h - 4) = -4$

24. $\lim\limits_{h \to 0} \dfrac{f(2 + h) - f(2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2(2 + h) + \dfrac{1}{2}(2 + h)^2 - 4 - 2}{h}$

$= \lim\limits_{h \to 0} \dfrac{4 + 2h + \dfrac{1}{2}h^2 + 2h + 2 - 6}{h}$

$= \lim\limits_{h \to 0} \left(\dfrac{1}{2}h + 4\right) = 4$

25. $\lim\limits_{h \to 0} \dfrac{f(-2 + h) - f(-2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{3(h - 2)^2 + 2 - (14)}{h}$

$= \lim\limits_{h \to 0} \dfrac{3h^2 - 12h + 12 - 12}{h} = \lim\limits_{h \to 0} (3h - 12) = -12$

26. $\lim\limits_{h \to 0} \dfrac{f(1 + h) - f(1)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(h + 1)^2 - 3(h + 1) + 1 - (-1)}{h}$

$= \lim\limits_{h \to 0} \dfrac{h^2 + 2h + 1 - 3h - 3 + 2}{h} = \lim\limits_{h \to 0} (h - 1)$

$= -1$

27. $\lim\limits_{h \to 0} \dfrac{f(-2 + h) - f(-2)}{h} = \lim\limits_{h \to 0} \dfrac{|h - 2 + 2| - 0}{h}$

$= \lim\limits_{h \to 0} \dfrac{|h|}{h} \cdot$ When $h > 0, \dfrac{|h|}{h} = 1$ while when $h < 0,$

$\dfrac{|h|}{h} = -1.$ The limit does not exist. The derivative does not

exist.

28. $\lim\limits_{h \to 0} \dfrac{f(-1 + h) - f(-1)}{h} = \lim\limits_{h \to 0} \dfrac{\dfrac{1}{h - 1 + 2} - \dfrac{1}{1}}{h}$

$= \lim\limits_{h \to 0} \dfrac{1 - (h + 1)}{h + 1} \cdot \dfrac{1}{h} = \lim\limits_{h \to 0} \dfrac{-h}{h} \cdot \dfrac{1}{h + 1}$

$= \lim\limits_{h \to 0} - \dfrac{1}{h + 1} = -1$

29. $f'(x) = \lim\limits_{h \to 0} \dfrac{2 - 3(x + h) - (2 - 3x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2 - 3x - 3h - 2 + 3x}{h} = \lim\limits_{h \to 0} \dfrac{-3h}{h} = -3$

30. $f'(x) = \lim\limits_{h \to 0} \dfrac{(2 - 3(x + h)^2) - (2 - 3x^2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2 - 3x^2 - 6xh - 3h^2 - 2 + 3x^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{-6xh - 3h^2}{h} = \lim\limits_{h \to 0} (-6x - 3h) = -6x$

31. $f'(x)$

$= \lim\limits_{h \to 0} \dfrac{3(x + h)^2 + 2(x + h) - 1 - (3x^2 + 2x - 1)}{h}$

$= \lim\limits_{h \to 0} \dfrac{3x^2 + 6xh + 3h^2 + 2x + 2h - 1 - 3x^2 - 2x + 1}{h}$

$= \lim\limits_{h \to 0} \dfrac{6xh + 3h^2 + 2h}{h} = \lim\limits_{h \to 0} (6x + 3h + 2) = 6x + 2$

32. $f'(x) = \lim\limits_{h \to 0} \dfrac{\dfrac{1}{(x + h) - 2} - \dfrac{1}{x - 2}}{h}$

$= \lim\limits_{h \to 0} \dfrac{(x - 2) - (x + h - 2)}{(x + h - 2)(x - 2)} \cdot \dfrac{1}{h}$

$= \lim\limits_{h \to 0} \dfrac{-h}{h} \cdot \dfrac{1}{(x + h - 2)(x - 2)}$

$= \lim\limits_{h \to 0} \dfrac{1}{(x + h - 2)(x - 2)} = -\dfrac{1}{(x - 2)^2}$

33. (a) Between 0.5 and 0.6 seconds: $\dfrac{3.2 - 2.3}{0.6 - 0.5} = 9$ ft/sec

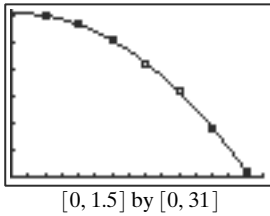Between 0.8 and 0.9 seconds: $\dfrac{7.3 - 5.8}{0.9 - 0.8} = 15$ ft/sec

(b) $f(x) = 8.94x^2 + 0.05x + 0.01$, $x =$ time in seconds

$[-0.1, 1]$ by $[-0.1, 8]$
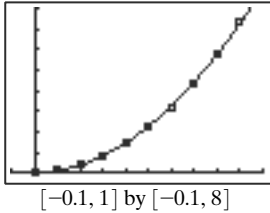
(c) $f(2) \approx 35.9$ ft

34. (a) $\dfrac{15.76 - 21.24}{1.0 - 0.8} = -27.4$ ft/sec

(b) $s(t) = -16.01t^2 + 1.43t + 30.35$

$[0, 1.5]$ by $[0, 31]$

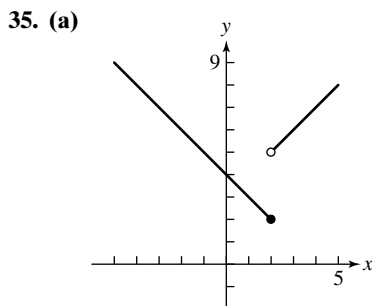**(c)** $s'(t) = \lim\limits_{h \to 0} \dfrac{s(t + h) - s(t)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[-16.015(t + h)^2 + 1.43(t + h) + 30.35] - (-16.015t^2 + 1.43t + 30.35)}{h}$

$= \lim\limits_{h \to 0} \dfrac{-32.03th - 16.015h^2 + 1.43h}{h}$

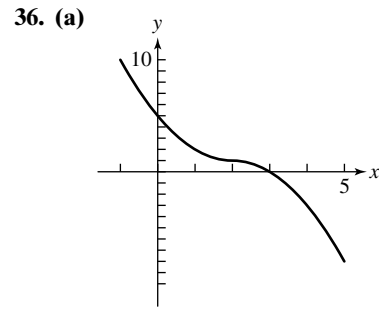$= \lim\limits_{h \to 0} (-32.03t - 16.015h + 16.015)$

$= -32.03t + 1.43;$

$s'(1) = -32.03(1) + 1.43 = -30.6$

At $t = 1$, the velocity is about -30.60 ft/sec.

35. (a)

(b) Since the graph of the function does not have a definable slope at $x = 2$, the derivative of f does not exist at $x = 2$. The function is not continuous at $x = 2$.

(c) Derivatives do not exist at points where functions have discontinuities.

36. (a)

(b) From the graph of the function, it appears that the derivative may exist at $x = 2$. Using the first definition of the derivative and taking secant lines on the left of $x = 2$ (so that $f(x) = 1 + (x - 2)^2$), we have

$\lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2} \dfrac{1 + (x - 2)^2 - 1}{x - 2}$
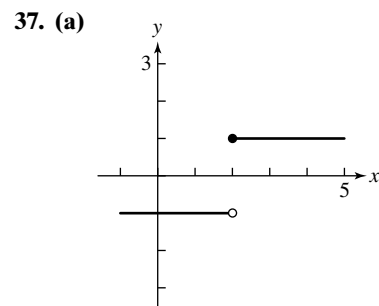
$= \lim\limits_{x \to 2} \dfrac{(x - 2)^2}{x - 2} = 1.$

Now, taking secant lines on the right of $x = 2$, so that $f(x) = 1 - (x - 2)^2$, we have

$\lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2} \dfrac{1 - (x - 2)^2 - 1}{x - 2}$

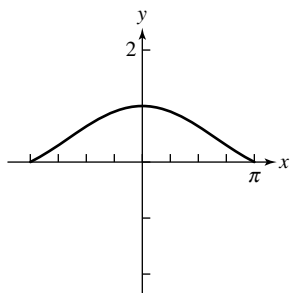$= \lim\limits_{x \to 2} \dfrac{-(x - 2)^2}{x - 2} = \lim\limits_{x \to 2} (-|x - 2|) = 0.$

Since the limits are the same, $f'(x)$ exists at $x = 2$ and $f'(2) = 0$.

37. (a)

(b) Since the graph of the function does not have a definable slope at $x = 2$, the derivative of f does not exist at $x = 2$. The function is not continuous at $x = 2$.

(c) Derivatives do not exist at points where functions have discontinuities.

38. (a)

(b) From the graph of the function, it appears that the derivative may exist at $x = 0$.

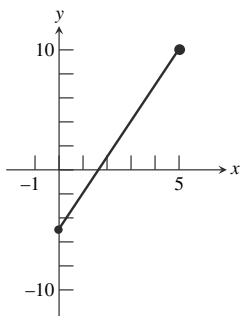$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{\dfrac{\sin h}{h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\sin h - h}{h^2}$$

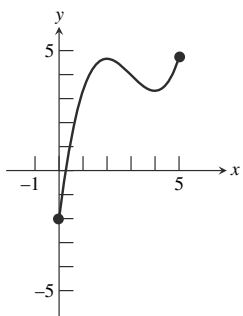This limit cannot be found using algebraic techniques. The table of values below suggests that this limit equals 0.

x	$\dfrac{\sin x - x}{x^2}$
-0.1	0.01666
-0.01	0.00167
-0.001	0.00017
0.001	-0.00017
0.01	-0.00167
0.1	-0.01666

The graph supports this since it appears that there is a horizontal tangent line at $x = 0$. Thus, $f'(0) = 0$.
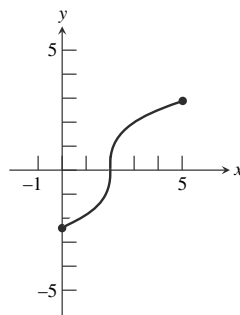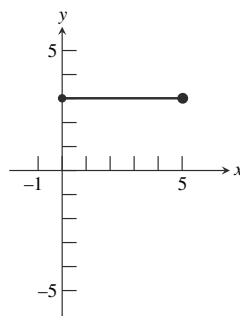
39. Answers will vary. One possibility:

40. Answers will vary. One possibility:

41. Answers will vary. One possibility:

42. Answers will vary. One possibility:

43. Since $f(x) = ax + b$ is a linear function, the rate of change for any x is exactly the slope of the line. No calculations are necessary since it is known that the slope $a = f'(x)$.

44. $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \dfrac{|x| - |0|}{x}$

$= \lim_{x \to 0} \dfrac{|x|}{x}$. Looking at secant lines, we see that this limit

does not exist. If the secant line is to the left of $x = 0$, it will have slope $m = -1$, while if it is to the right of $x = 0$, it will have slope $m = 1$. At $x = 0$, the graph of the function does not have a definable slope.

45. False. The instantaneous velocity is a limit of average velocities. It is nonzero when the ball is moving.

46. True. Both the derivative and the slope equal

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

47. For $Y_1 = x^2 + 3x - 4$, at $x = 0$ the calculator shows $dy/dx = 3$. The answer is D.

48. For $Y_1 = 5x - 3x^2$, at $x = 2$ the calculator shows $dy/dx = -7$. The answer is A.

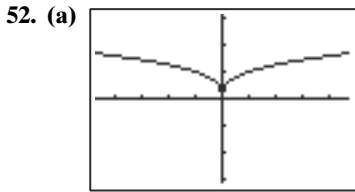**49.** For $Y_1 = x^3$, at $x = 2$ the calculator shows $dy/dx = 12$. The answer is C.

50. For $Y_1 = \dfrac{1}{x - 3}$, at $x = 1$ the calculator shows

$dy/dx = -0.25$. The answer is A.

51. (a)
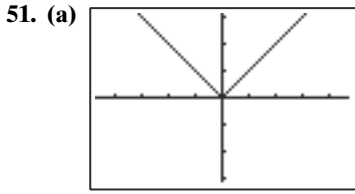
[−4.7, 4.7] by [−3.1, 3.1]

No, there is no derivative because the graph has a corner at $x = 0$.

(b) No

52. (a)
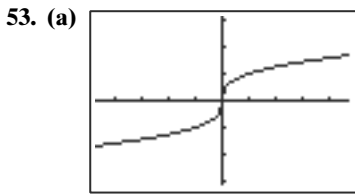

[−4.7, 4.7] by [−3.1, 3.1]

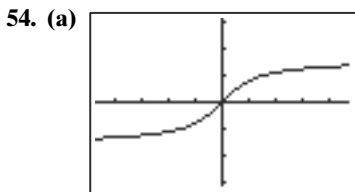No, there is no derivative because the graph has a cusp ("spike") at $x = 0$.

(b) Yes, the tangent line is $x = 0$.

53. (a)

[−4.7, 4.7] by [−3.1, 3.1]

No, there is no derivative because the graph has a vertical tangent (no slope) at $x = 0$.

(b) Yes, the tangent line is $x = 0$.

54. (a)

[−4.7, 4.7] by [−3.1, 3.1]

Yes, there is a derivative because the graph has a nonvertical tangent line at $x = 0$.

(b) Yes, the tangent line is $y = x$.

55. (a) The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{16(3)^2 - 16(0)^2}{3 - 0} = 48 \text{ ft/sec.}$$

(b) The instantaneous velocity is

$$\lim_{h \to 0} \frac{16(3 + h)^2 - 144}{h} = \lim_{h \to 0} \frac{96h + h^2}{h}$$

$$= \lim_{h \to 0} (96 + h) = 96 \text{ ft/sec.}$$

56. (a) $g = \dfrac{y}{t^2} = \dfrac{125}{25} = 5 \text{ m/sec}^2$
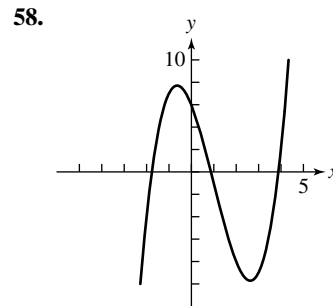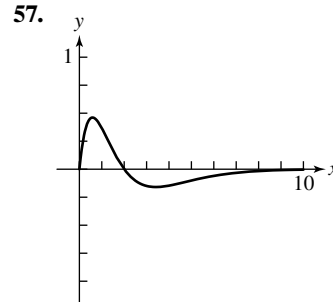
(b) Average speed: $\dfrac{\Delta x}{\Delta t} = \dfrac{125}{5} = 25 \text{ m/sec}$

(c) Since $y = 5t^2$, the instantaneous speed at $t = 5$ is

$$\lim_{h \to 0} \frac{5(5 + h)^2 - 5(5)^2}{h} = \lim_{h \to 0} \frac{50h + 5h^2}{h}$$

$$= \lim_{h \to 0} (50 + 5h)$$

$$= 50 \text{ m/sec.}$$

57.

58.

■ Section 11.2 Limits and Motion: The Area Problem

Exploration 1

1. The total amount of water remains 1 gallon. Each of the 10 teacups holds $\dfrac{1 \text{ gal}}{10} = 0.1$ gallon of water.

2. The total amount of water remains 1 gallon. Each of the 100 teacups holds $\dfrac{1 \text{ gal}}{100} = 0.01$ gallon of water.

3. The total amount of water remains 1 gallon. Each of the 1,000,000,000 teacups holds $\dfrac{1 \text{ gal}}{1,000,000,000} = 0.000\,000\,001$ gallon of water.

4. The total amount of water remains 1 gallon. Each of the teacups holds an amount of water that is less than what was in each of the 1 billion teacups in step 3. Thus each teacup holds about 0 gallons of water.

Quick Review 11.2

1. $\dfrac{1}{8}, \dfrac{1}{2}, \dfrac{9}{8}, 2, \dfrac{25}{8}, \dfrac{9}{2}, \dfrac{49}{8}, 8, \dfrac{81}{8}, \dfrac{25}{2}$

2. $\dfrac{81}{64}, \dfrac{25}{16}, \dfrac{121}{64}, \dfrac{9}{4}, \dfrac{169}{64}, \dfrac{49}{16}, \dfrac{225}{64}, 4, \dfrac{289}{64}, \dfrac{81}{16}$

3. $\frac{1}{2}[2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11] = \frac{65}{2}$

4.
$$
\begin{array}{ccccccc}
2 & + & 3 & + & 4 + \ldots + & n & + (n + 1) \\
+ (n + 1) & + & (n) & + & (n - 1) + \ldots + & 3 & + \ 2 \\
\hline
(n + 3) & + & (n + 3) & + & (n + 3) + \ldots + & (n + 3) & + (n + 3)
\end{array}
$$

Thus $2\sum_{k=1}^{n}(k + 1) = n(n + 3)$, and

$$\sum_{k=1}^{n}(k + 1) = \frac{1}{2}n(n + 3).$$

5. $\frac{1}{2}[4 + 9 + \ldots + 121] = \frac{505}{2}$

6. $\frac{1}{2}[1 + 4 + 9 + \ldots + (n - 1)^2 + n^2]$

$= \frac{1}{2}\left[\frac{n(n + 1)(2n + 1)}{6}\right] = \frac{n(n + 1)(2n + 1)}{12}$

7. $(57 \text{ mph})(4 \text{ hours}) = 228 \text{ miles}$

8. $\left(\frac{5 \text{ gal}}{\text{min}}\right)(120 \text{ min}) = 600 \text{ gallons}$

9. $\left(\frac{200 \text{ ft}^3}{\text{sec}}\right)(6 \text{ hours})\left(\frac{60 \text{ minutes}}{\text{hour}}\right)\left(\frac{60 \text{ seconds}}{\text{minute}}\right)$
$= 4{,}320{,}000 \text{ ft}^3$

10. $\left(\frac{560 \text{ people}}{\text{mi}^2}\right)(35{,}000 \text{ mi}^2) = 19{,}600{,}000 \text{ people}$

Section 11.2 Exercises

1. Let the line $y = 65$ represent the situation. The area under the line is the distance traveled, a rectangle, $(65)(3) = 195$ miles.

2. Let the line $y = 15$ represent the situation. The area under the line is the number of gallons pumped, a rectangle, $(15)(30) = 450$ gallons.

3. Let the line $y = 150$ represent the situation. The area under the line is the total number of cubic feet of water pumped, a rectangle, $(150)(3600) = 540{,}000 \text{ ft}^3$.

4. Let the line $y = 650$ represent the situation. The area under the line is the total population, a rectangle, $(650)(20) = 13{,}000$ people.

5. $\Delta s = \frac{\Delta s}{\Delta t} \cdot \Delta t = (640 \text{ km/h})(3.4 \text{ h}) = 2176 \text{ km}$

6. $\Delta s = \frac{\Delta s}{\Delta t} \cdot \Delta t = (24 \text{ mi/h})\left(4\frac{5}{6}\text{ h}\right) = 116 \text{ mi}$

7. $\sum_{k=1}^{5}1 \cdot f(k) = f(1) + f(2) + f(3) + f(4) + f(5)$

$= 3\frac{1}{2} + 4\frac{1}{4} + 3\frac{1}{2} + 1\frac{3}{4} + 0 = 13$ (answers will vary)

8. $\sum_{k=1}^{5}1 \cdot f(k) = f(1) + f(2) + f(3) + f(4) + f(5)$

$= 1 + 3 + 4\frac{1}{2} + 4 + 0 = 12\frac{1}{2}$ (answers will vary)

9. $\sum_{k=1}^{5}1 \cdot f(k) = f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5)$

$= 3.5 + 5.25 + 2.75 + 0.25 + 1.25 = 13$ (answers will vary)

10. $\sum_{k=1}^{5}1 \cdot f(k) = f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5)$

$= 3 + 1.5 + 1.75 + 3.25 + 5 = 14.5$ (answers will vary)

11. $\sum_{i=1}^{8}(10 - x_i^2)\Delta x_i$

$= (9 + 9.75 + 10 + 9.75 + 9 + 7.75 + 6 + 3.75)(0.5)$

$= 32.5$ square units

12. $\sum_{i=1}^{8}(10 - x_i^2)\Delta x_i$

$= (9.75 + 10 + 9.75 + 9 + 7.75 + 6 + 3.75 + 1)(0.5)$
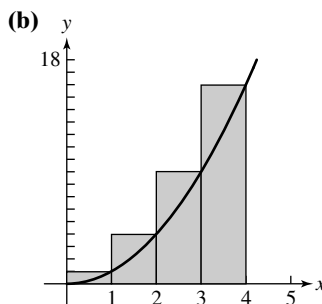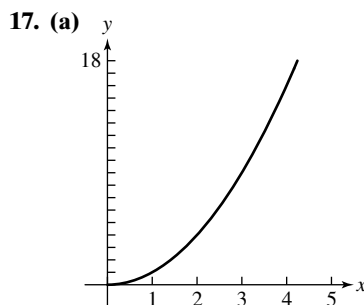
$= 28.5$ square units

13. $\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right]$

14. $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right], \left[1, \frac{5}{4}\right], \left[\frac{5}{4}, \frac{3}{2}\right], \left[\frac{3}{2}, \frac{7}{4}\right], \left[\frac{7}{4}, 2\right]$
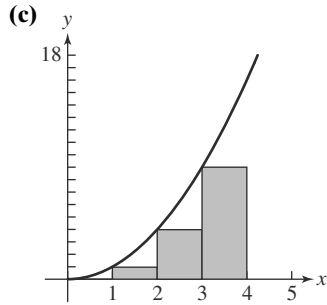
15. $\left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right], \left[2, \frac{5}{2}\right], \left[\frac{5}{2}, 3\right], \left[3, \frac{7}{2}\right], \left[\frac{7}{2}, 4\right]$

16. $\left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right], \left[2, \frac{5}{2}\right], \left[\frac{5}{2}, 3\right], \left[3, \frac{7}{2}\right], \left[\frac{7}{2}, 4\right], \left[4, \frac{9}{2}\right], \left[\frac{9}{2}, 5\right]$

For #17–20, the intervals are of width 1, so the area of each rectangle is $1 \cdot f(k) = f(k)$.
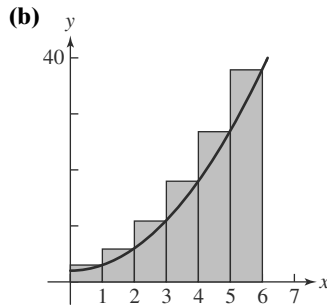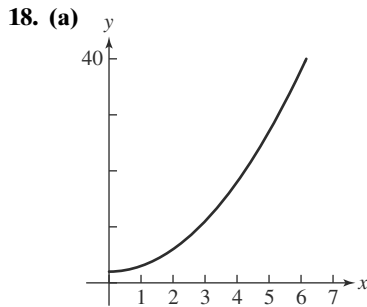
17. (a)

(b)

RRAM: $f(1) + f(2) + f(3) + f(4)$
$= 1 + 4 + 9 + 16 = 30$

(c)

LRAM: $f(0) + f(1) + f(2) + f(3)$
$= 0 + 1 + 4 + 9 = 14$

(d) Average: $\dfrac{14 + 30}{2} = 22$

18. (a)

(b)

RRAM: $f(1) + f(2) + \ldots + f(6)$
$= 3 + 6 + 11 + 18 + 27 + 38 = 103$

(c)
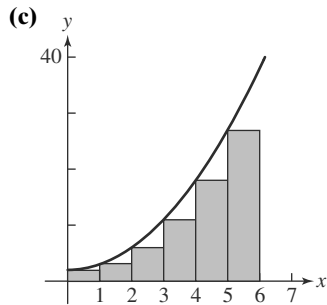
LRAM: $f(0) + f(1) + \ldots + f(5)$
$= 2 + 3 + 6 + 11 + 18 + 27 = 67$

(d) Average: $\dfrac{67 + 103}{2} = 85$

19. (a)

(b)

RRAM: $f(1) + f(2) + f(3) + f(4)$
$= 3 + 4 + 3 + 0 = 10$

(c)
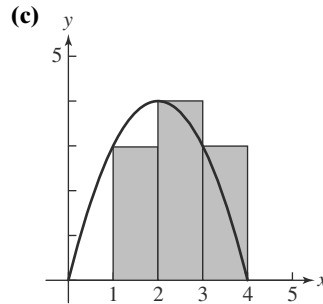
LRAM: $f(0) + f(1) + f(2) + f(3)$
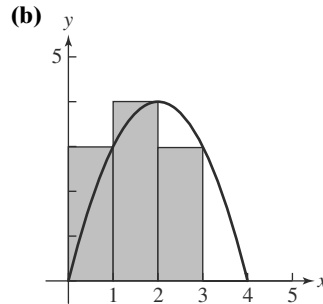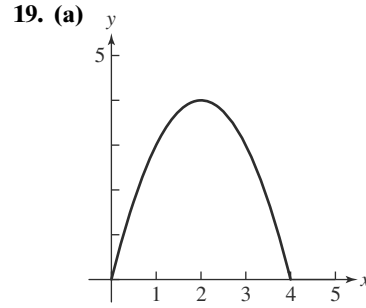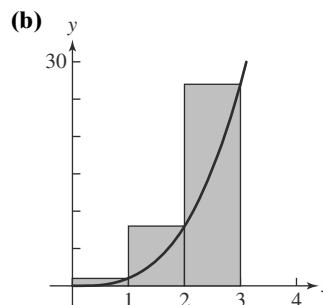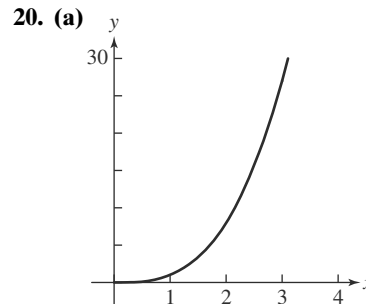$= 0 + 3 + 4 + 3 = 10$

(d) Average: $\dfrac{10 + 10}{2} = 10$

20. (a)

(b)

RRAM: $f(1) + f(2) + f(3) = 1 + 8 + 27 = 36$

(c)
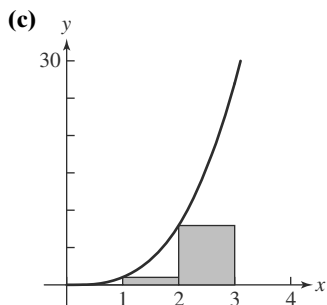
LRAM: $f(0) + f(1) + f(2) = 0 + 1 + 8 = 9$

(d) Average: $\dfrac{9 + 36}{2} = \dfrac{45}{2}$

21. $\displaystyle\int_3^7 5\,dx = 20$ (Rectangle with base 4 and height 5)
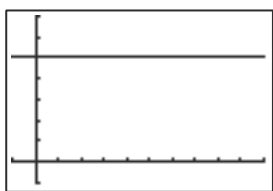
$[-1, 10]$ by $[-1, 7]$

22. $\displaystyle\int_{-1}^4 6\,dx = 30$ (Rectangle with base 5 and height 6)
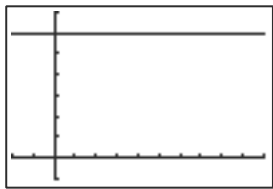
$[-2, 10]$ by $[-1, 7]$

23. $\displaystyle\int_0^5 3x\,dx = 37.5$ (Triangle with base 5 and altitude 15)
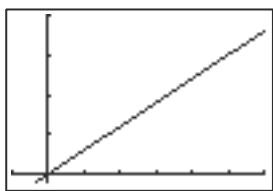
$[-1, 6]$ by $[-1, 20]$

24. $\displaystyle\int_1^7 0.5x\,dx = 12$ (Trapezoid with bases of 0.5 and 3.5 and height 6)
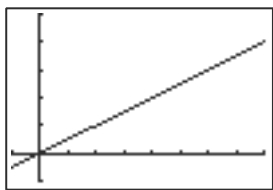
$[-1, 8]$ by $[-1, 5]$

25. $\displaystyle\int_1^4 (x + 3)\,dx = 16.5$ (Trapezoid with bases 4 and 7 and altitude 3)
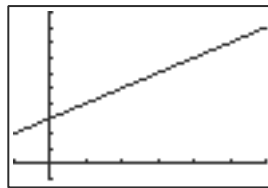
$[-1, 6]$ by $[-1, 12]$

26. $\displaystyle\int_1^4 (3x - 2)\,dx = 16.5$ (Trapezoid with bases 1 and 10 and altitude 3)
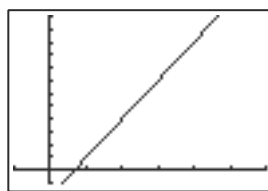
$[-1, 6]$ by $[-1, 12]$

27. $\displaystyle\int_{-2}^2 \sqrt{4 - x^2}\,dx = 2\pi$ (Semicircle with radius 2)
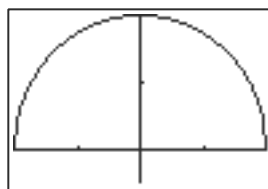
$[-2, 2]$ by $[-0.5, 2]$

28. $\displaystyle\int_0^6 \sqrt{36 - x^2}\,dx = 9\pi$ (Quarter circle with radius 6)
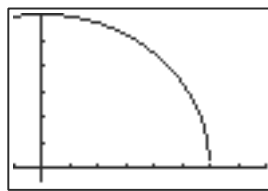
$[-1, 8]$ by $[-0.5, 6]$

29. $\displaystyle\int_0^\pi \sin x\,dx = 2$ (One arch of sine curve)
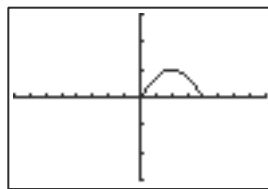
$[-2\pi, 2\pi]$ by $[-3, 3]$

30. $\int_0^\pi (\sin x + 2)\, dx = 2 + 2\pi$ (Arch of sine curve plus

rectangle with base π and height 2)
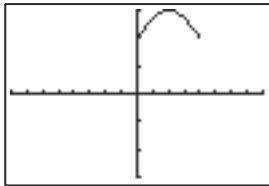
$[-2\pi, 2\pi]$ by $[-3, 3]$

31. $\int_2^\pi \sin(x - 2)\, dx = 2$ (One arch of sine curve translated

2 units right)
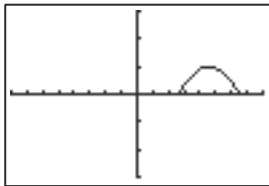
$[-2\pi, 2\pi]$ by $[-3, 3]$

32. $\int_{-\pi/2}^{\pi/2} \cos x\, dx = 2$ (One arch of cosine curve, which is

sine curve translated $\pi/2$ units)
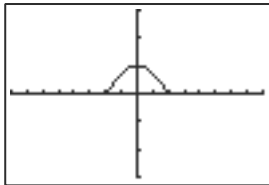
$[-2\pi, 2\pi]$ by $[-3, 3]$

33. $\int_0^{\pi/2} \sin x\, dx = 1$ (Half-arch of sine curve)
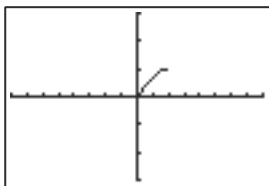
$[-2\pi, 2\pi]$ by $[-3, 3]$

34. $\int_0^{\pi/2} \cos x\, dx = 1$ (Half-arch of cosine curve, congruent

to half-arch of sine curve)
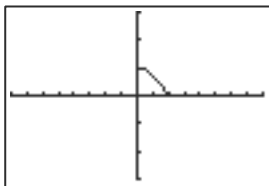
$[-2\pi, 2\pi]$ by $[-3, 3]$

35. $\int_0^\pi 2 \sin x\, dx = 4$ (Rectangles in sum are twice as tall,

yielding twice the sum.)
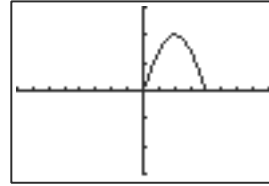
$[-2\pi, 2\pi]$ by $[-3, 3]$

36. $\int_0^{2\pi} \sin\left(\dfrac{x}{2}\right) dx = 4$ (Rectangles in sum are twice as

wide, yielding twice the sum.)
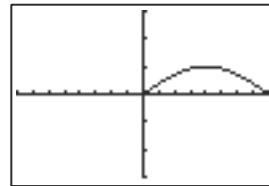
$[-2\pi, 2\pi]$ by $[-3, 3]$

37. $\int_0^{2\pi} |\sin x|\, dx = 4$ (Two arches of the sine curve)
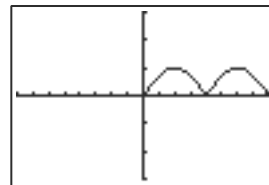
$[-2\pi, 2\pi]$ by $[-3, 3]$

38. $\int_{-\pi}^{3\pi/2} |\cos x|\, dx = 5$ (Two-and-a-half arches of the cosine

curve)
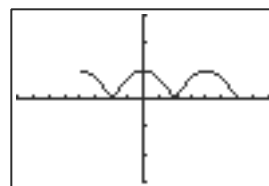
$[-2\pi, 2\pi]$ by $[-3, 3]$

39. The graph of $f(x) = kx + 3$ is a line. If k is a number
between 0 and 4, the integral is the area of a trapezoid
with bases of $0k + 3 = 3$ and $4k + 3$ and height of
$4 - 0 = 4$. The area is $\dfrac{1}{2}(4)(3 + 4k + 3) = 2(4k + 6)$
$= 8k + 12$, so $\int_0^4 (kx + 3)dx = 8k + 12$.

40. The graph of $f(x) = 4x + 3$ is a line. The integral is the
area of a trapezoid with bases of $4 \cdot 0 + 3 = 3$ and
$4k + 3$ and height of $k - 0 = k$. The area is
$\dfrac{1}{2}k(3 + 4k + 3) = \dfrac{1}{2}k(4k + 6) = 2k^2 + 3k$, so
$\int_0^k (4x + 3)dx = 2k^2 + 3k.$

41. The graph of $f(x) = 3x + k$ is a line. The integral is the area of a trapezoid with bases of $3 \cdot 0 + k = k$ and $3 \cdot 4 + k = 12 + k$ and height of $4 - 0 = 4$. The area is
$$\frac{1}{2}(4)(k + 12 + k) = 2(12 + 2k) = 24 + 4k, \text{ so}$$
$$\int_0^4 (3x + k)dx = 24 + 4k.$$

42. The graph of $f(x) = 4x + 3$ is a line. The integral is the area of a trapezoid with bases of $4k + 3$ and $4 \cdot 4 + 3 = 19$ and height of $4 - k$. The area is
$$\frac{1}{2}(4 - k)(4k + 3 + 19) = \frac{1}{2}(4 - k)(22 + 4k)$$
$$= (4 - k)(11 + 2k) = 44 - 2k^2 - 3k.$$

43. Since $g(x) = -f(x)$, we consider g to be symmetric with f about the x-axis. For every value of x in the interval, $|f(x)|$ is the distance to the x-axis and similarly, $|g(x)|$ is the distance to the x-axis; $f(x)$ and $g(x)$ are equidistant from the x-axis. As a result, the area under $f(x)$ must be exactly equal to the area above $g(x)$.

44. The graph of $f(x) = \sqrt{16 - x^2}$ is the top half of a circle of radius 4. The area of the graph from $x = 0$ to $x = 4$ is the area of $\frac{1}{4}$ of the entire circle. Thus the desired area is
$$\frac{1}{4}(\pi \cdot 4^2) = \frac{1}{4}(16\pi) = 4\pi.$$

45. The distance traveled will be the same as the area under the velocity graph, $v(t) = 32t$, over the interval $[0, 2]$. That triangular region has an area of $A = (1/2)(2)(64) = 64$. The ball falls 64 feet during the first 2 seconds.

46. The distance traveled will be the same as the area under the velocity graph, $v(t) = 6t$, over the interval $[0, 7]$. That triangular region has an area of $A = (1/2)(7)(42) = 147$. The car travels 147 feet in the first 7 seconds.

47. (a)
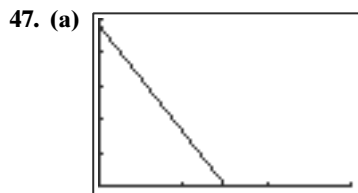
[0, 3] by [0, 50]

(b) The ball reaches its maximum height when the velocity function is zero; this is the point where the ball changes direction and starts its descent. Solving for t when $48 - 32t = 0$, we find $t = 1.5$ sec.

(c) The distance the ball has traveled is the area under the curve, a triangle with base 1.5 and height 48, thus $d = 0.5(1.5)(48) = 36$ units.

48. (a)

[0, 8] by [0, 180]

(b) The rocket reaches its maximum height when the velocity function is zero; this is the point where the rocket changes direction and starts its descent. Solving for t when $170 - 32t = 0$, $t \approx 5.31$ sec.

(c) The distance the rocket has traveled is the area under the curve, a triangle with base 5.3125 and height 170, thus $d = \frac{1}{2}(170)(5.3125) \approx 451.6$ ft.

49. (a)
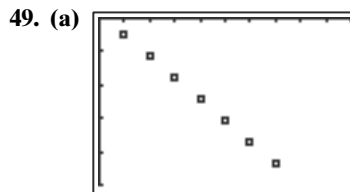
[0, 2] by [−50, 0]

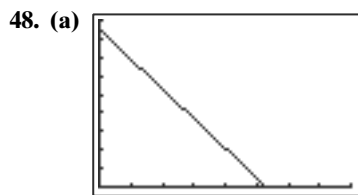**(b)** Each RRAM rectangle will have width 0.2. The heights (using the absolute value of the velocity) are 5.05, 11.43, 17.46, 24.21, 30.62, 37.06, and 43.47. The height of the building is approximately $0.2[5.05 + 11.43 + 17.46 + 24.21 + 30.62 + 37.06 + 43.47] = 33.86$ feet.

50. Work is defined as force times distance. The work done in moving the barrel 35 feet is the area under the curve created by the given data points, assuming the barrel weighs approximately 550 pounds after being moved 35 feet. In this case, the area under the curve is the sum of a rectangle of width 35 and height 550 and a triangle of base 35 and height $(1250 - 550) = 700$. The total work performed is
$$(35)(550) + \frac{1}{2}(35)(700) = 31,500 \text{ ft-pounds.}$$

51. True. The exact area under a curve is given by the limit as n approaches infinity. This is true whether LRAM or RRAM is used.

52. False. The statement $\lim_{x \to \infty} f(x) = L$ means that $f(x)$ gets arbitrarily close to L as x gets arbitrarily large.

53. Since $y = 2\sqrt{x}$ represents a vertical stretch, by a factor of 2, of $y = \sqrt{x}$, the area under the curve between $x = 0$ and $x = 9$ is doubled. The answer is A.

54. Since $y = \sqrt{x} + 5$ represents a vertical shift, by 5 units upward, of $y = \sqrt{x}$, the area is increased by the contribution of a 9-by-5 rectangle — an area of 45 square units. The answer is E.

55. $y = \sqrt{x - 5}$ is shifted right 5 units compared to $y = \sqrt{x}$, but the limits of integration are shifted right 5 units also, so the area is unchanged. The answer is C.

56. $y = \sqrt{3x}$ represents a horizontal compression, by a factor of $1/3$, and the interval of integration is shrunk in the same way. So the new area is $1/3$ of the old area. The answer is D.

57. In the definition of the definite integral, if $f(x)$ is negative, then $\sum_{i=1}^{n} f(x_i)\Delta x$ is negative, so the definite integral is negative. For $f(x) = \sin x$ on $[0, 2\pi]$, the "positive area" (from 0 to π) cancels the "negative area" (from π to 1), so the definite integral is 0.

Since $g(x) = x - 1$ forms a triangle with area $\frac{1}{2}$ below the x-axis on $[0, 1]$, $\int_0^1 (x - 1)dx = -\frac{1}{2}$.

58. (a)
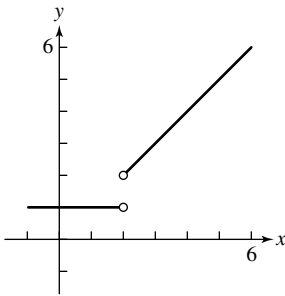
Domain: $(-\infty, 2) \cup (2, \infty)$
Range: $\{1\} \cup (2, \infty)$

(b) The area under f from $x = 0$ to $x = 4$ is a rectangle of width 4 and height 1 and a trapezoid with bases of 1 and 3 and height 2. It does not really make any difference that the function has no value at $x = 2$.

59. True

$$\int_a^b f(x)dx + \int_a^b g(x)dx$$

$$= \left[\lim_{n \to \infty} \sum_{i=1}^n f(x_i)\Delta x \right] + \left[\lim_{n \to \infty} \sum_{i=1}^n g(x_i)\Delta x \right]$$

$$= \lim_{n \to \infty} \left[\sum_{i=1}^n f(x_i)\Delta x + \sum_{i=1}^n g(x_i)\Delta x \right]$$

$$= \lim_{n \to \infty} \sum_{i=1}^n [f(x_i) + g(x_i)]\Delta x$$

$$= \int_a^b (f(x) + g(x))dx$$

Note: There are some subtleties here, because the x_i that are chosen for $f(x)$ may be different from the x_i that are chosen for $g(x)$; however, the result is true, provided the limits exist.

60. True, because multiplying the function by 8 will multiply the area by 8.

61. False. Counterexample: Let $f(x) = 1$, $g(x) = 1$. Then $\int_0^2 f(x)g(x)dx = 2$ but $\int_0^2 f(x)dx \cdot \int_0^2 g(x)dx = 4$.

62. True, because (area from a to c) + (area from c to b) = (area from a to b).

63. False. Interchanging a and b reverses the sign of $\Delta x = \dfrac{b - a}{n}$, which reverses the sign of the integral.

64. True. For any value of n, $\Delta x = \dfrac{a - a}{n} = 0$.

$$\int_a^a f(x)dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n \to \infty} \sum_{i=1}^n f(a) \cdot 0$$

$$= \lim_{n \to \infty} 0 = 0.$$

■ Section 11.3   More on Limits

Exploration 1

1. Answers will vary. Possible answers include:
Solving graphically or algebraically shows that $7x = 14$ when $x = 2$, so we know that 14 is the limit.
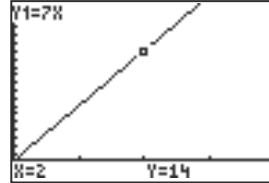
[0, 4] by [−2, 20]

A table of values also shows that the value of the function approaches 14 as x approaches 2 from either direction.

2. Answers will vary. Possible answers include:
The graphs suggest that the limit exists and is 2. Because the graph is a line with a discontinuity at $x = 0$, there is no asymptote at $x = 0$.
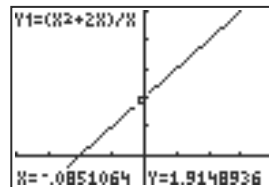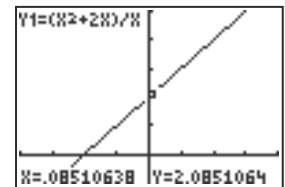
[−4, 4] by [−1, 5]   [−4, 4] by [−1, 5]

A table of values also suggests that the limit is 2.

To show that 2 is the limit and $1.999\overline{9}$ is not, we can solve algebraically.

$$\lim_{x \to 0} \frac{x^2 + 2x}{x} = \lim_{x \to 0} \frac{x(x + 2)}{x}$$

$$= \lim_{x \to 0} (x + 2)$$

$$= 2$$

Exploration 2

1.
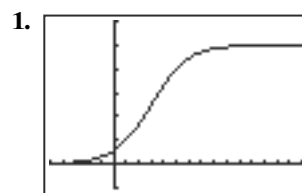
[−5, 15] by [−10, 60]

$$\lim_{x \to \infty} f(x) = 50, \quad \lim_{x \to -\infty} f(x) = 0$$

2. The two horizontal asymptotes are $y = 50$ and $y = 0$.

3. As $x \to \infty$, $2^{3-x} \to 0$ and $1 + 2^{3-x} \to 1$.
As $x \to -\infty$, $2^{3-x} \to \infty$ and $1 + 2^{3-x} \to \infty$.

Quick Review 11.3

1. (a) $f(-2) = \dfrac{-4 + 1}{(-4 - 4)^2} = -\dfrac{3}{64}$

(b) $f(0) = \dfrac{0 + 1}{(0 - 4)^2} = \dfrac{1}{16}$

(c) $f(2) = \dfrac{4 + 1}{(4 - 4)^2}$ is undefined.

2. (a) $f(-2) = \dfrac{\sin(-2)}{-2} = \dfrac{\sin 2}{2} \approx 0.45$

(b) $f(0) = \dfrac{\sin 0}{0}$ is undefined.

(c) $f(2) = \dfrac{\sin 2}{2} \approx 0.45$

3. (a) Since $x^2 - 4 = 0$ and $x = \pm 2$, the graph of f has vertical asymptotes at $x = -2$ and $x = 2$.

(b) Since $\lim\limits_{x \to -\infty} f(x) = 2$ and $\lim\limits_{x \to \infty} f(x) = 2$, the graph of f has a horizontal asymptote of $y = 2$.

4. (a) Since $x^2 + x - 2 = 0$ when $x = -2$ and $x = 1$, the graph of f has vertical asymptotes at $x = -2$ and $x = 1$.

(b) Since $\lim\limits_{x \to -\infty} f(x) = \infty$ and $\lim\limits_{x \to \infty} f(x) = -\infty$, the graph of f has no horizontal asymptotes.

5. Since $\dfrac{2x^3}{-x} = -2x^2$, the end behavior asymptote is

(b) $y = -2x^2$.

6. Since $\dfrac{x^4}{x} = x^3$, the end behavior asymptote is **(c)** $y = x^3$.

7. (a) $[-2, \infty)$

(b) None

8. (a) $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

(b) $x = -2$, $x = 2$

9.

10. Continuous on $(-\infty, 1) \cup (1, \infty)$; discontinuous at $x = 1$

Section 11.3 Exercises

1. $(-1)(-2)^2 = -4$

2. $(2)^{12} = 4096$

3. $8 - 4 + 3 = 7$

4. $-8 + 2 + 5 = -1$

5. $\sqrt{7}$

6. $(-6)^{2/3} \approx 3.30$

7. $\lim\limits_{x \to 0} (e^x \sin(x)) = \lim\limits_{x \to 0} e^x \lim\limits_{x \to 0} \sin(x) = 1 \cdot 0 = 0$

8. $\lim\limits_{x \to \pi} \ln\left(\sin\dfrac{x}{2}\right) = \ln(1) = 0 = 0$

9. $a^2 - 2$

10. $\dfrac{a^2 - 1}{a^2 + 1}$ (Since $a^2 + 1 > 0$ for all a, we don't have to worry about division by zero.)

11. (a) division by zero

(b) $\lim\limits_{x \to -3} \dfrac{x^2 + 7x + 12}{x^2 - 9} = \lim\limits_{x \to -3} \dfrac{(x + 4)(x + 3)}{(x + 3)(x - 3)}$
$= \lim\limits_{x \to -3} \dfrac{x + 4}{x - 3} = -\dfrac{1}{6}$

12. (a) division by zero

(b) $\lim\limits_{x \to 3} \dfrac{(x + 3)(x - 3)}{(x + 5)(x - 3)} = \lim\limits_{x \to 3} \dfrac{x + 3}{x + 5} = \dfrac{6}{8} = \dfrac{3}{4}$

13. (a) division by zero

(b) $\lim\limits_{x \to -1} \dfrac{(x + 1)(x^2 - x + 1)}{x + 1} = \lim\limits_{x \to -1}(x^2 - x + 1) = 3$

14. (a) division by zero

(b) $\lim\limits_{x \to 2} \dfrac{(x - 2)(x^2 + 1)}{x - 2} = \lim\limits_{x \to 2}(x^2 + 1) = 5$
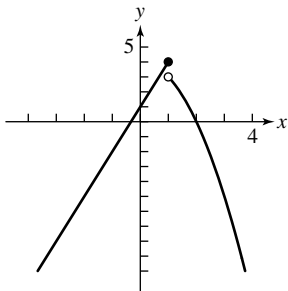
15. (a) division by zero

(b) $\lim\limits_{x \to -2} \dfrac{(x + 2)(x - 2)}{x + 2} = \lim\limits_{x \to -2}(x - 2) = -4$

16. (a) division by zero

(b) $\lim\limits_{x \to -2} \dfrac{|(x + 2)(x - 2)|}{x + 2}$. Check left- and right-hand limits.

Right: $\lim\limits_{x \to -2^+} \dfrac{(-1)(x + 2)(x - 2)}{x + 2} = \lim\limits_{x \to -2^+}(-x + 2) = 4$

Left: $\lim\limits_{x \to -2^-} \dfrac{(x + 2)(x - 2)}{x + 2} = \lim\limits_{x \to -2^-}(x - 2) = -4$

Since $4 \neq -4$, the limit does not exist.

17. (a) The square root of negative numbers is not defined in the real plane.

(b) The limit does not exist.

18. (a) division by zero

(b) The limit does not exist.

19. $\lim\limits_{x \to 0} \dfrac{\sin x}{2x^2 - x} = \lim\limits_{x \to 0} \dfrac{\sin x}{x(2x - 1)} = \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x \to 0} \dfrac{1}{2x - 1}$
$= 1 \cdot -1 = -1$ (Recall Example 11 and the product rule.)
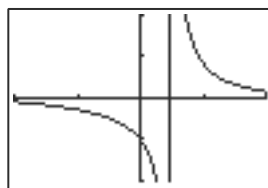
$[-2, 2]$ by $[-2, 2]$

20. $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x} = \lim\limits_{x \to 0} \dfrac{3 \sin (3x)}{(3x)} = \lim\limits_{x \to 0} 3 \cdot \lim\limits_{x \to 0} \dfrac{\sin (3x)}{(3x)}$

$= 3 \cdot 1 = 3$

(Recall Example 11 and the product rule.)
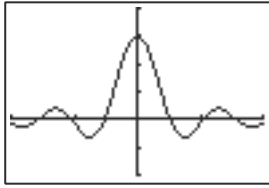
$[-4, 4]$ by $[-2, 4]$

21. $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{x} = \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \sin x = \lim\limits_{x \to 0} \sin x \cdot \lim\limits_{x \to 0} \dfrac{\sin x}{x}$

$= 0 \cdot 1 = 0$
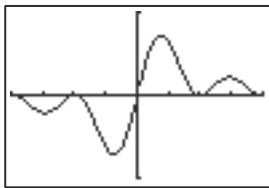
$[-2\pi, 2\pi]$ by $[-1, 1]$

22. $\lim\limits_{x \to 0} \dfrac{x + \sin x}{2x} = \dfrac{1}{2} \lim\limits_{x \to 0} \dfrac{x}{x} + \dfrac{1}{2} \lim\limits_{x \to 0} \dfrac{\sin x}{x}$

$= \dfrac{1}{2} \cdot 1 + \dfrac{1}{2} \cdot 1 = 1$
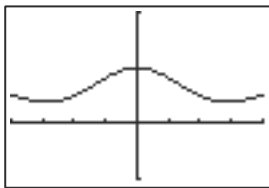
$[-2\pi, 2\pi]$ by $[-1, 2]$

In Exercises #23–26, the function is defined and continuous at the value approached by x, and so the limit is simply the function evaluated at that value.

23. $\lim\limits_{x \to 0} \dfrac{e^x - \sqrt{x}}{\log_4(x + 2)} = \dfrac{e^0 - \sqrt{0}}{\log_4(0 + 2)} = \dfrac{1}{1/2} = 2$

24. $\lim\limits_{x \to 0} \dfrac{3 \sin x - 4 \cos x}{5 \sin x + \cos x} = \dfrac{3 \sin 0 - 4 \cos 0}{5 \sin 0 + \cos 0} = \dfrac{-4}{1} = -4$

25. $\lim\limits_{x \to \pi/2} \dfrac{\ln(2x)}{\sin^2 x} = \dfrac{\ln \pi}{\sin^2 (\pi/2)} = \dfrac{\ln \pi}{1} = \ln \pi$

26. $\lim\limits_{x \to 27} \dfrac{\sqrt{x + 9}}{\log_3 x} = \dfrac{\sqrt{36}}{\log_3 27} = \dfrac{6}{3} = 2$

27. (a) $\lim\limits_{x \to 2^-} f(x) = 3$

(b) $\lim\limits_{x \to 2^+} f(x) = 1$

(c) $3 \neq 1$, so the limit is not defined.

28. (a) $\lim\limits_{x \to 3^-} f(x) = 2$

(b) $\lim\limits_{x \to 3^+} f(x) = 4$

(c) $2 \neq 4$, so the limit is not defined.

29. (a) $\lim\limits_{x \to 3^-} f(x) = 4$

(b) $\lim\limits_{x \to 3^+} f(x) = 4$

(c) $\lim\limits_{x \to 3} f(x) = 4$

30. (a) $\lim\limits_{x \to 1^-} f(x) = 1$

(b) $\lim\limits_{x \to 1^+} f(x) = 3$

(c) $1 \neq 3$, so the limit is not defined.

31. (a) True

(b) True

(c) False

(d) False

(e) False

(f) False

(g) False

(h) True

(i) False

(j) True

32. (a) True

(b) False

(c) False

(d) True

(e) False

(f) False

(g) False

(h) False (The limit does not exist at $x = 0$.)

(i) True

33.
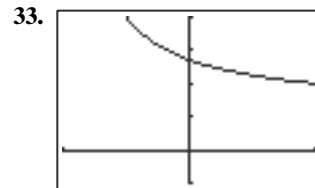
$[-1, 1]$ by $[-1, 4]$

(a) $\lim\limits_{x \to 0^-} f(x) \approx 2.72$

(b) $\lim\limits_{x \to 0^+} f(x) \approx 2.72$

(c) $\lim\limits_{x \to 0} f(x) \approx 2.72$

34.
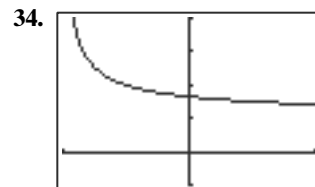
$[-1, 1]$ by $[-1, 4]$

(a) $\lim\limits_{x \to 0^-} f(x) \approx 1.65$

(b) $\lim\limits_{x \to 0^+} f(x) \approx 1.65$

(c) $\lim\limits_{x \to 0} f(x) \approx 1.65$

35. (a) $\lim\limits_{x\to4}(g(x)+2)=4+2=6$ (sum rule)

(b) $\lim\limits_{x\to4}x\cdot f(x)=\lim\limits_{x\to4}x\cdot\lim\limits_{x\to4}f(x)=4(-1)=-4$
(product rule)

(c) $\lim\limits_{x\to4}g^2(x)=\lim\limits_{x\to4}g(x)\cdot\lim\limits_{x\to4}g(x)=4\cdot4=16$
(product rule)

(d) $\lim\limits_{x\to4}\dfrac{g(x)}{f(x)-1}=\dfrac{\lim\limits_{x\to4}g(x)}{\lim\limits_{x\to4}f(x)-\lim\limits_{x\to4}1}=\dfrac{4}{-1-1}=-2$
(quotient rule)

36. (a) $\lim\limits_{x\to a}(f(x)+g(x))=\lim\limits_{x\to a}f(x)+\lim\limits_{x\to a}g(x)$
$=2-3=-1$ (sum rule)

(b) $\lim\limits_{x\to a}(f(x)\cdot g(x))=\lim\limits_{x\to a}f(x)\cdot\lim\limits_{x\to a}g(x)$
$(2)(-3)=-6$ (product rule)

(c) $\lim\limits_{x\to a}(3g(x)+1)=3\lim\limits_{x\to a}g(x)+\lim\limits_{x\to a}1$
$=3(-3)+1=-8$
(constant multiple and sum rules)

(d) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}=\dfrac{2}{-3}=-\dfrac{2}{3}$ (quotient rule)

37. (a)
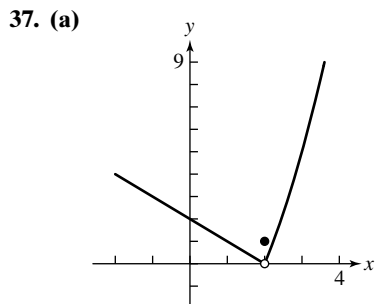
(b) $\lim\limits_{x\to2^+}f(x)=0,\ \lim\limits_{x\to2^-}f(x)=0$

(c) $\lim\limits_{x\to2}f(x)=0$

38. (a)
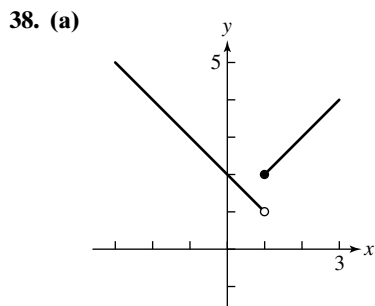
(b) $\lim\limits_{x\to1^+}f(x)=2,\ \lim\limits_{x\to1^-}f(x)=1$

(c) Limit does not exist because $\lim\limits_{x\to1^+}f(x)\neq\lim\limits_{x\to1^-}f(x)$.

39. (a)
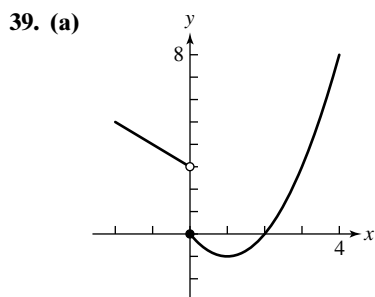
(b) $\lim\limits_{x\to0^+}f(x)=0,\ \lim\limits_{x\to0^-}f(x)=3$

(c) Limit does not exist because $\lim\limits_{x\to0^+}f(x)\neq\lim\limits_{x\to0^-}f(x)$.

40. (a)
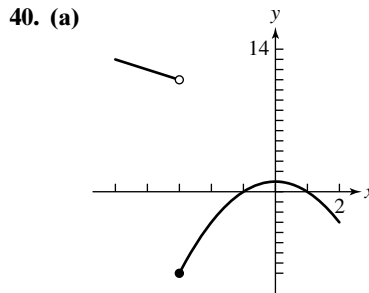
(b) $\lim\limits_{x\to-3^+}f(x)=-8,\ \lim\limits_{x\to-3^-}f(x)=11$

(c) Limit does not exist because $\lim\limits_{x\to-3^+}f(x)\neq\lim\limits_{x\to-3^-}f(x)$.

For Exercises #41–43, use Figure 11.14.

41. $\lim\limits_{x\to2^+}\text{int }x=2$

42. $\lim\limits_{x\to2^-}\text{int }x=1$

43. $\lim\limits_{x\to0.0001}\text{int }x=0$

44. $\lim\limits_{x\to5/2^-}\text{int}(2x)=4$

45. $\lim\limits_{x\to-3^+}\dfrac{x+3}{|x+3|}=\lim\limits_{x\to-3^+}\dfrac{x+3}{x+3}=1$

46. $\lim\limits_{x\to0^-}\dfrac{5x}{|2x|}=\lim\limits_{x\to0^-}\dfrac{5x}{-2x}=-\dfrac{5}{2}$

47. (a) $\lim\limits_{x\to\infty}\dfrac{\cos x}{1+x}=0$

(b) $\lim\limits_{x\to-\infty}\dfrac{\cos x}{x}=0$

48. (a) $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x}=\lim\limits_{x\to\infty}\dfrac{x}{x}+\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=1+0=1$

(b) $\lim\limits_{x\to-\infty}\dfrac{x+\sin x}{x}=\lim\limits_{x\to-\infty}\dfrac{x}{x}+\lim\limits_{x\to-\infty}\dfrac{\sin x}{x}$
$=1+0=1$

49. (a) $\lim\limits_{x\to\infty}(1+2^x)=\infty$

(b) $\lim\limits_{x\to-\infty}(1+2^x)=1$

50. (a) $\lim\limits_{x\to\infty}\dfrac{x}{1+2^x}=0$

(b) $\lim\limits_{x\to-\infty}\dfrac{x}{1+2^x}=-\infty$

51. (a) $\lim\limits_{x\to\infty}(x+\sin x)=\infty$

(b) $\lim\limits_{x\to-\infty}(x+\sin x)=-\infty$

52. (a) $\lim\limits_{x\to\infty}(e^{-x}+\sin x)$ is undefined, because e^{-x} goes to zero but $\sin x$ oscillates.

(b) $\lim\limits_{x\to-\infty}(e^{-x}+\sin x)=\infty$

53. (a) $\lim\limits_{x\to\infty}(-e^x\sin x)$ is undefined, because $\sin x$ oscillates between positive and negative values.

(b) $\lim\limits_{x\to-\infty}(-e^x\sin x)=0$

54. (a) $\lim\limits_{x \to \infty} e^{-x} \cos x = 0$

(b) $\lim\limits_{x \to -\infty} e^{-x} \cos x$ is undefined, because $\cos x$ oscillates between positive and negative values.

55. $\lim\limits_{x \to 3^-} \dfrac{1}{x - 3} = -\infty$; $x = 3$
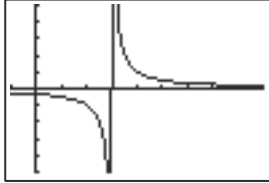
$[-1, 9]$ by $[-5, 5]$

56. $\lim\limits_{x \to 3^+} \dfrac{1}{x - 3} = \infty$; $x = 3$
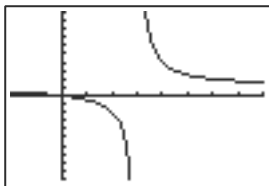
$[-2, 8]$ by $[-10, 10]$

57. $\lim\limits_{x \to -2^+} \dfrac{1}{x + 2} = \infty$; $x = -2$
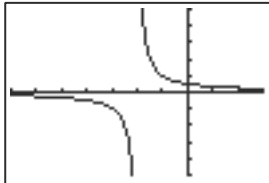
$[-7, 3]$ by $[-5, 5]$

58. $\lim\limits_{x \to -2^-} \dfrac{x}{x - 2} = \infty$; $x = -2$
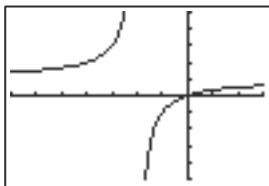
$[-7, 3]$ by $[-5, 5]$

59. $\lim\limits_{x \to 2} \dfrac{1}{(x - 5)^5} = \infty$; $x = 5$
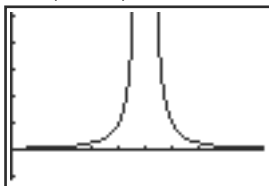
$[0, 9.4]$ by $[-1.1, 5.1]$

60. $\lim\limits_{x \to 2} \dfrac{1}{x^2 - 4}$ is undefined; $x = 2$ and $x = -2$
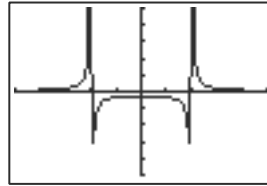
$[-5, 5]$ by $[-5, 5]$

61. $\lim\limits_{x \to 0} \dfrac{(1 + x)^3 - 1}{x} = \lim\limits_{x \to 0} \dfrac{x^3 + 3x^2 + 3x + 1 - 1}{x}$

$= \lim\limits_{x \to 0} \dfrac{x(x^2 + 3x + 3)}{x}$

$= \lim\limits_{x \to 0} (x^2 + 3x + 3) = 3$
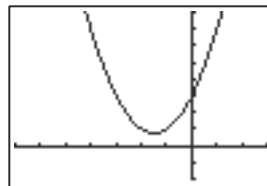
$[-7, 3]$ by $[-2, 8]$

62. $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{3 + x} - \dfrac{1}{3}}{x}$

$= \lim\limits_{x \to 0} \dfrac{3 - (3 + x)}{3(3 + x)} \cdot \dfrac{1}{x} = \dfrac{1}{3} \lim\limits_{x \to 0} -\dfrac{x}{x} \cdot \dfrac{1}{x + 3}$

$\dfrac{1}{3} \lim\limits_{x \to 0} \left(-\dfrac{1}{x + 3}\right) = -\dfrac{1}{9}$
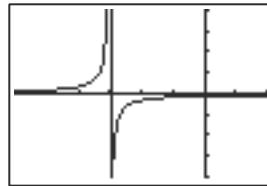
$[-6, 2]$ by $[-4, 4]$

63. $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = \lim\limits_{x \to 0} \dfrac{\sin x}{x \cos x} = \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x \to 0} \dfrac{1}{\cos x}$

$= 1 \cdot 1 = 1$
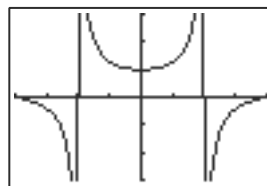
$[-\pi, \pi]$ by $[-3, 3]$

64. $\lim\limits_{x \to 2} \dfrac{x - 4}{x^2 - 4}$ is undefined.
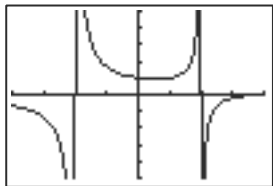
$[-4, 4]$ by $[-5, 5]$

65. $\lim\limits_{x \to 0} \dfrac{|x|}{x^2} = \lim\limits_{x \to 0} \left|\dfrac{x}{x^2}\right| = \lim\limits_{x \to 0} \left|\dfrac{1}{x}\right| = \infty$

66. $\lim\limits_{x \to 0} \dfrac{x^2}{|x|} = \lim\limits_{x \to 0} \left|\dfrac{x^2}{x}\right| = \lim\limits_{x \to 0} |x| = 0$

67. $\lim\limits_{x \to 0} \left[x \sin\left(\dfrac{1}{x}\right)\right] = 0$ because $x \to 0$

and $-1 \leq \sin\left(\dfrac{1}{x}\right) \leq 1$.

68. $\lim\limits_{x \to 27} \cos\left(\dfrac{1}{x}\right) = \cos\left(\dfrac{1}{27}\right) \approx 0.9993$

69. $\lim\limits_{x \to 1} \dfrac{x^2 + 1}{x - 1}$ is undefined, since $\lim\limits_{x \to 1^-} \dfrac{x^2 + 1}{x - 1} = -\infty$.

and $\lim\limits_{x \to 1^+} \dfrac{x^2 + 1}{x - 1} = \infty$

70. $\lim\limits_{x \to \infty} \dfrac{\ln x^2}{\ln x} = \lim\limits_{x \to \infty} \dfrac{2 \ln x}{\ln x} = 2$

71. $\lim\limits_{x \to \infty} \dfrac{\ln x}{\ln x^2} = \lim\limits_{x \to \infty} \dfrac{\ln x}{2 \ln x} = \dfrac{1}{2}$

72. $\lim\limits_{x \to \infty} 3^{-x} = \lim\limits_{x \to \infty} \dfrac{1}{3^x} = 0$

73. False. $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = 5$

74. False. For example, if $f(x) = \sin(1/x)$ and $g(x) = x$, then $\lim\limits_{x \to 0} f(x)$ does not exist but $\lim\limits_{x \to 0} [f(x) \cdot g(x)] = 0$.

75. $\lim\limits_{x \to 3} \dfrac{x^2 - 2x - 3}{x - 3} = \lim\limits_{x \to 3} \dfrac{(x + 1)(x - 3)}{x - 3}$

$= \lim\limits_{x \to 3} (x + 1) = 4$. The answer is B.

76. $\lim\limits_{x \to 3^-} \dfrac{x^2 + 2x + 3}{x - 3} = -\infty, \ \lim\limits_{x \to 3^+} \dfrac{x^2 + 2x + 3}{x - 3} = \infty$.

The answer is A.

77. $\lim\limits_{x \to 3^-} \dfrac{x^2 - 2x - 9}{x - 3} = \infty, \ \lim\limits_{x \to 3^+} \dfrac{x^2 - 2x - 9}{x - 3} = -\infty$.

The answer is C.

78. $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x - 3} = \lim\limits_{x \to 3} \dfrac{(x - 3)(x^2 + 3x + 9)}{x - 3}$

$= \lim\limits_{x \to 3} (x^2 + 3x + 9) = 27$. The answer is D.

79. (a)
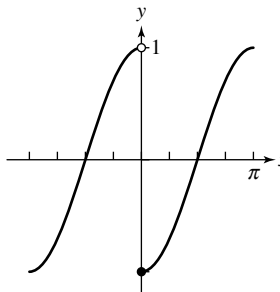
(b) $(-\pi, 0) \cup (0, \pi)$

(c) $x = \pi$

(d) $x = -\pi$

80. (a)
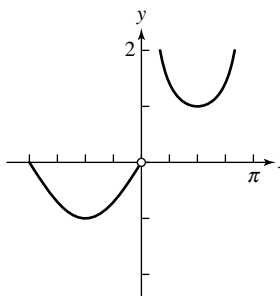
(b) $(-\pi, 0) \cup (0, \pi)$

(c) $x = 0$

(d) $x = -\pi$

81. (a)
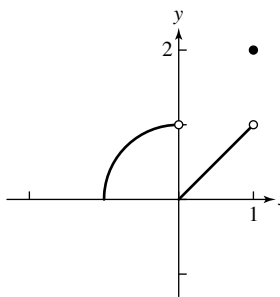
(b) $(-1, 0) \cup (0, 1)$

(c) $x = 1$

(d) $x = -1$

82. (a)

(b) $(-\infty, -2) \cup (-2, \infty)$

(c) None

(d) None

83. (a)

[−2, 25] by [0, 60]
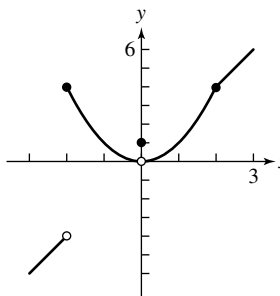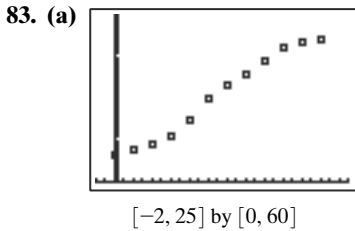
(b) $f(x) \approx \dfrac{57.71}{1 + 6.39e^{-0.19x}}$, where $x =$ the number of
months $\lim\limits_{x \to \infty} f(x) \approx 57.71$.

(c) The rabbit population will stabilize at a little less than 58,000.

(d) One possible answer: As populations burgeon, resources such as food, water, and safe havens from predators become more scarce and the population tends to stabilize based on the resources available to it—this is what is often call a maximum sustainable population.

84.

85.

86.

87.
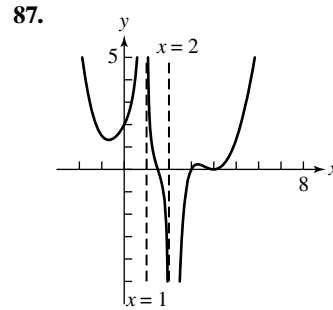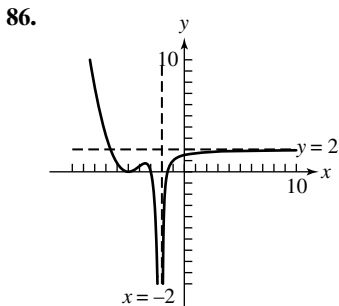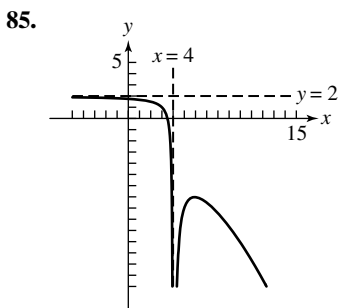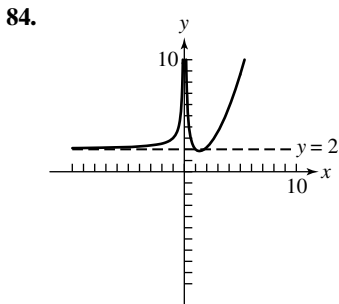
88. (a) $\lim\limits_{x \to 0} \dfrac{2}{x^2} = \infty$; $\lim\limits_{x \to 0} x^2 = 0$; $\lim\limits_{x \to 0} \dfrac{2}{x^2} \cdot x^2 = \lim\limits_{x \to 0} 2 = 2$

(b) $\lim\limits_{x \to 0} \left|\dfrac{1}{x}\right| = \infty$; $\lim\limits_{x \to 0} \sqrt[3]{x} = 0$; $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{x}}{|x|}$ is not defined

(c) $\lim\limits_{x \to 1} \left|\dfrac{3}{x-1}\right| = \infty$; $\lim\limits_{x \to 1} (x-1)^2 = 0$;

$\lim\limits_{x \to 1^+} \dfrac{3(x-1)^2}{|x-1|} = \lim\limits_{x \to 1^+} \dfrac{3(x-1)^2}{x-1}$
$= \lim\limits_{x \to 1^+} 3(x-1) = 0$

$\lim\limits_{x \to 1^-} \dfrac{3(x-1)^2}{|x-1|} = \lim\limits_{x \to 1^-} \dfrac{3(x-1)^2}{-(x-1)}$
$= \lim\limits_{x \to 1^-} -3(x-1) = 0$

Thus, $\lim\limits_{x \to 1} fg = 0$

(d) $\lim\limits_{x \to 1} \dfrac{1}{(x-1)^4} = \infty$; $\lim\limits_{x \to 1} (x-1)^2 = 0$;

$\lim\limits_{x \to 1} \dfrac{(x-1)^2}{(x-1)^4} = \lim\limits_{x \to 1} \dfrac{1}{(x-1)^2} = \infty$

(e) One possible answer: Nothing can really be said—the limit may be undefined, ∞, or a number.

89. (a) For an 8-sided polygon, we have 8 isosceles triangles of area $\dfrac{1}{2}bh$. Thus, $A = 8 \cdot \dfrac{1}{2}bh = 4bh$. Similarly, for an n-sided polygon, we have n triangles of area $\dfrac{1}{2}bh$.
Thus $A = n \cdot \dfrac{1}{2} \cdot bh = \dfrac{1}{2}nhb$.

(b) Consider an n-sided polygon inscribed in a circle of radius r. Since a circle always is 360°, we see that each angle extending from the center of the circle to two consecutive vertices is an angle of $\dfrac{360°}{n}$. Dropping a perpendicular from the center of the circle to the midpoint of the base of the triangle (which is also one of the n sides) results in an angle of $\dfrac{360°}{2n}$. Since
$\tan\left(\dfrac{360°}{2n}\right) = \dfrac{(b/2)}{h}$, we have $\dfrac{b}{2} = h\tan\left(\dfrac{360°}{2n}\right)$
and finally $b = 2h\tan\left(\dfrac{360°}{2n}\right)$.

(c) Since $A = \dfrac{1}{2}nhb$ and $b = 2h\tan\left(\dfrac{360°}{2n}\right)$, we have
$A = \dfrac{1}{2}nh\left(2h\tan\left(\dfrac{360°}{2n}\right)\right) = nh^2\tan\left(\dfrac{360°}{2n}\right)$.

(d)

n	A
4	4
8	3.3137
16	3.1826
100	3.1426
500	3.1416
1,000	3.1416
5,000	3.1416
10,000	3.1416
100,000	3.1416

Yes, $A \to \pi$ as $n \to \infty$.

(e)

n	A
4	36
8	29.823
16	28.643
100	28.284
500	28.275
1,000	28.274
5,000	28.274
10,000	28.274
100,000	28.274

Yes, $n \to \infty$, $A \to 9\pi$.

(f) One possible answer:

$$\lim_{n \to \infty} A = \lim_{n \to \infty} nh^2 \tan\left(\frac{180°}{n}\right)$$

$$= h^2 \lim_{n \to \infty} n \tan\left(\frac{180°}{n}\right)$$

$$= h^2\pi = \pi h^2$$

As the number of sides of the polygon increases, the distance between h and the edge of the circle becomes progressively smaller. As $n \to \infty$, $h \to$ radius of the circle.

90. (a)
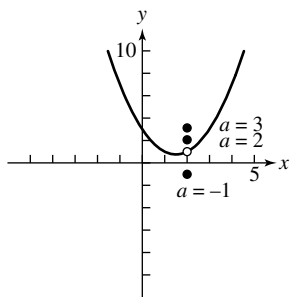
(b) Since $(2)^2 - 3(2) + 3 = 4 - 6 + 3 = 1$, $a = 1$.

91. (a)
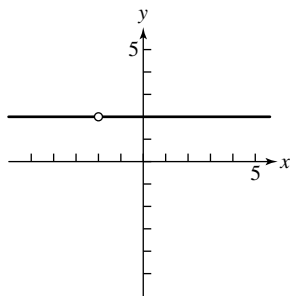
(b) $f(x) = \dfrac{2x + 4}{x + 2} = \dfrac{2(x + 2)}{x + 2} = 2$

(c) $g(x) = 2$

92. (a)
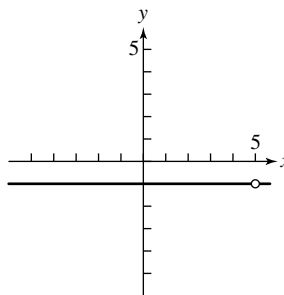
(b) $\dfrac{x - 5}{5 - x} = -\dfrac{(x - 5)}{(x - 5)} = -1$

(c) $g(x) = -1$

93. (a)
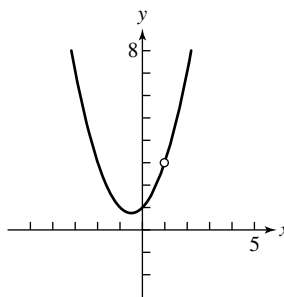
(b) $\dfrac{x^3 - 1}{x - 1} = \dfrac{(x - 1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1$

(c) $g(x) = x^2 + x + 1$

■ Section 11.4 Numerical Derivatives and Integrals

Exploration 1

1. RRAM value ≈ 1.364075504 and the NINT value ≈ 1.386294361.

2. The new command is sum(seq($1/(1 + K * 3/100) * 3/100, K, 1, 100$)). The calculated value ≈ 1.375114667, which is a better approximation than for 50 rectangles.

3. The integral is $\displaystyle\int_0^\pi \sin x \, dx$. The RRAM value is ≈ 1.999342 and the NINT value is 2.

4. The command is sum(seq($\sqrt{(4 + K * 5/50)} * 5/50, K, 1, 50$)). The calculated value ≈ 12.7166 and the NINT value is 12.666667.

Quick Review 11.4

1. $\dfrac{\Delta y}{\Delta x} = \dfrac{4^2 - 1^2}{4 - 1} = \dfrac{15}{3} = 5$

2. $\dfrac{\Delta y}{\Delta x} = \dfrac{\sqrt{4} - \sqrt{1}}{4 - 1} = \dfrac{2 - 1}{3} = \dfrac{1}{3}$

3. $\dfrac{\Delta y}{\Delta x} = \dfrac{\log_2 4 - \log_2 1}{4 - 1} = \dfrac{2 - 0}{3} = \dfrac{2}{3}$

4. $\dfrac{\Delta y}{\Delta x} = \dfrac{3^4 - 3^1}{4 - 1} = \dfrac{81 - 3}{3} = 26$

5. $\dfrac{\Delta y}{\Delta x} = \dfrac{11 - 2}{4 - 1} = \dfrac{9}{3} = 3$

6. $\dfrac{\Delta y}{\Delta x} = \dfrac{10 - (-2)}{4 - 1} = \dfrac{12}{3} = 4$

7. $\dfrac{\sin(1.01) - \sin(0.99)}{2(0.01)} \approx 0.5403$

8. $\dfrac{1.001^4 - 0.999^4}{2(0.001)} \approx 4.000$

9. $\dfrac{\ln 1.001 - \ln 0.999}{2(0.001)} \approx 1.000$

10. $\dfrac{e^{1.0001} - e^{0.9999}}{2(0.0001)} \approx 2.7183$

Exercises 11.4

In #1–10, use NDER on a calculator to find the numerical derivative of the function at the specific point.

1. -4

2. 4

3. -12

4. -1

5. 0

6. ≈ -1.0000

7. ≈ 1.0000

8. ≈ 2.0000

9. ≈ -3.0000

10. ≈ -3.0000

In #11–22, use NINT on a calculator to find the numerical integral of the function over the specified interval.

11. $\dfrac{64}{3}$

12. $\dfrac{64}{3}$

13. 2

14. -2

15. ≈ 0

16. 2

17. 1

18. ≈ 0.69315

19. ≈ 3.1416

20. 10

21. ≈ 106.61 mi

22. ≈ 16.95 mi

23. (a) $v_{ave} = \dfrac{435 - 485}{2 - 1} = \dfrac{-50}{1} = -50$ ft/sec

(b)

$[-1, 6]$ by $[0, 550]$
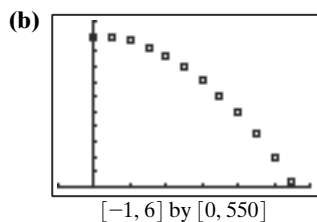
(c) $s(t) \approx -16.08t^2 + 0.36t + 499.77$

(d) $v(1.5 \text{ sec}) \approx -47.88$ ft/sec

(e) Set $s(t)$ equal to zero and solve for t using the quadratic equation.

$$t = \dfrac{-0.36 - \sqrt{0.36^2 - 4(-16.08)(499.77)}}{2(-16.08)}$$

≈ 5.586 sec. (The minus sign was chosen to give $t \geq 0$.) Using NDER at $t = 5.586$ sec gives $v \approx -179.28$ ft/sec.

24. (a) The average rate of change between two data points is found by examining $\dfrac{\Delta y}{\Delta x}$. The average rate of change of the gross domestic product from 2006–2007 is $\dfrac{13{,}362 - 12{,}364}{2007 - 2006} = \$998$ billion per year. The average rate of change of the gross domestic product from 2010–2011 is $\dfrac{14{,}582 - 14{,}044}{2011 - 2010} = \$538$ billion per year.

(b) The quadratic regression model for the data is $y = -53.92x^2 + 929.54x + 10{,}868.08$.
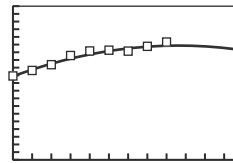
$[0, 12]$ by $[0, 20000]$

(c) To estimate the rate of change of the gross domestic product in 2006, we will use the calculator NDER computation and evaluate that when $x = 2$ to obtain about \$714 billion per year.

To estimate the rate of change of the gross domestic product in 2010, we will use the calculator NDER computation and evaluate that when $x = 6$ to obtain about \$283 billion per year.

(d) Using the quadratic regression model $y = -53.92x^2 + 929.54x + 10{,}868.08$ to predict the gross domestic product in 2016 where $x = 12$ yields $-53.92(12)^2 + 929.54(12) + 10{,}868.08 = \$14{,}258$ billion or \$14.3 trillion.
This answer is not reasonable. The GDP is likely to continue to grow in the long run.

25. (a) The midpoints of the subintervals will be 0.25, 0.75, 1.25, etc. The average velocities will be the successive height differences divided by 0.5 — that is, times 2.

Midpoint	$\Delta s / \Delta t$
0.25	-10 ft/sec
0.75	-20
1.25	-40
1.75	-60
2.25	-70
2.75	-90
3.25	-100
3.75	-120
4.25	-140
4.75	-150
5.25	-170

(b)
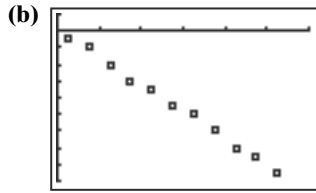
[0, 6] by [−180, 20]

$y \approx -32.18t + 0.32$

(c) Substituting $t = 1.5$ leads to $y \approx -47.95$ ft/sec. This is close to the value of -47.88 ft/sec found in Exercise 23(d).

26. (a) Let 2004 be $x = 0$. The first subinterval has a midpoint of $\dfrac{0 + 1}{2} = 0.5$. On that interval, the average rate of change is $\dfrac{\Delta y}{\Delta x} = \dfrac{11{,}631 - 10{,}908}{1 - 0} = \dfrac{723}{1} = 723$

The rest of the midpoints and values of $\Delta s / \Delta t$ are computed similarly and are shown in the following table.

Midpoint	$\Delta s/\Delta t$	
0.5	723	(2005)
1.5	733	(2006)
2.5	998	
3.5	633	
4.5	302	
5.5	−253	(2010)
6.5	538	
7.5	512	

(b)
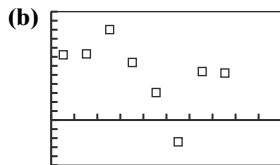
[0, 10] by [−500, 1200]

(c) The linear regression model for the data is $y = -77.809x + 834.488$. Using this regression model to approximate the rate of change in 2006 yields $y = -77.809(3) + 834.488 = 679$. Using the same model to approximate the rate of change in 2010 yields $y = -77.809(6) + 834.488 = 367.7$. These values are not close to the results of NDER from Exercise 24(c).

27. The average velocities, $\Delta s/\Delta t$, for the successive 0.5-sec intervals are 8, 24, 40, 56, and 72 ft/sec. Multiplying each by 0.5 sec and then summing them gives the estimated distance: 100 ft.

28. The average velocities, $\Delta s/\Delta t$, for the successive 0.2-sec intervals are as given:

Interval	$\Delta s/\Delta t$
1	1.09 m/sec
2	0.85
3	0.61
4	0.42
5	0.32
6	0.37
7	0.615
8	1.095

Multiplying each $\Delta s/\Delta t$ by 0.2 sec and then summing them gives the estimated distance: 1.074 m.

29. The program accepts inputs which determine the width and number of approximating rectangles. These rectangles are summed and the result is output to the screen.

30. The program accepts inputs which determine the width and number of approximating rectangles. These rectangles are summed and the result is output to the screen.

For #31–42, verify the function is nonnegative by graphing it over the interval.

31. (b)

N	LRAM	RRAM	Average
10	15.04	19.84	17.44
20	16.16	18.56	17.36
50	16.86	17.82	17.34
100	17.09	17.57	17.33

(c) fnInt gives 17.33; at N_{100}, the average is 17.3344.

32. (b)

N	LRAM	RRAM	Average
10	96.72	132.72	114.72
20	105.18	123.18	114.18
50	110.43	117.63	114.03
100	112.21	115.81	114.01

**(c)** fnInt gives 114; N_{100} is very close at 114.01.

33. (b)

N	LRAM	RRAM	Average
10	7.84	11.04	9.44
20	8.56	10.16	9.36
50	9.02	9.66	9.34
100	9.17	9.49	9.33

(c) fnInt gives 9.33; at N_{100}, the average is 9.3344.

34. (b)

N	LRAM	RRAM	Average
10	107.76	132.96	120.36
20	113.79	126.39	120.09
50	117.49	122.53	120.01
100	118.74	121.26	120

(c) fnInt gives 120; N_{100} has the same result.

35. (b)

N	LRAM	RRAM	Average
10	98.24	112.64	105.44
20	101.76	108.96	105.36
50	103.90	106.78	105.34
100	104.61	106.05	105.33

(c) fnInt gives 105.33; at N_{100}, the average is 105.3344.

36. (b)

N	LRAM	RRAM	Average
10	136.16	185.76	160.96
20	147.84	172.64	160.24
50	155.08	165.00	160.04
100	157.53	162.49	160.01

(c) fnInt gives 160, very close to N_{100} of 160.01.

37. (b)

N	LRAM	RRAM	Average
10	7.70	8.12	7.91
20	7.81	8.02	7.91
50	7.87	7.95	7.91
100	7.89	7.93	7.91

(c) fnInt gives 7.91, the same result as N_{100}.

38. (b)

N	LRAM	RRAM	Average
10	4.51	4.81	4.66
20	4.59	4.74	4.67
50	4.64	4.70	4.67
100	4.65	4.68	4.67

(c) fnInt gives 4.67, the same result as N_{100}.

39. (b)

N	LRAM	RRAM	Average
10	1.08	0.92	1.00
20	1.04	0.96	1.00
50	1.02	0.98	1.00
100	1.01	0.99	1.00

(c) fnInt gives 1, the same result as N_{100}.

40. (b)

N	LRAM	RRAM	Average
10	0.57	0.57	0.57
20	0.57	0.57	0.57
50	0.57	0.57	0.57
100	0.57	0.57	0.57

(c) fnInt $= 0.57$, the same result as N_{100}.

41. (b)

N	LRAM	RRAM	Average
10	0.56	0.62	0.59
20	0.58	0.61	0.59
50	0.59	0.60	0.59
100	0.59	0.60	0.59

(c) fnInt $= 0.594$, the same result as N_{100}.

42. (b)

N	LRAM	RRAM	Average
10	1.17	1.03	1.10
20	1.13	1.07	1.10
50	1.11	1.09	1.10
100	1.11	1.09	1.10

(c) fnInt gives 1.10, which is the same result as N_{100}.

43. True. The notation NDER refers to a symmetric difference quotient using $\Delta x = h = 0.001$.

44. False. NINT will vary the value of Δx until the numerical integral gets close to a limiting value.

45. NINT will use as many rectangles as are needed to obtain an accurate estimate. The answer is B. (Note: NDER estimates the derivative, not the integral.)

46. The most accurate estimate is a symmetric difference quotient with a small h (and of course with $2h$, not h, in the denominator). The answer is E.

47. Instantaneous velocity is the derivative, not an integral, of the position function. The answer is C.

48. Area under a curve that represents $f(x)$ is an integral, not the derivative, of $f(x)$. The answer is D.

49. (a) $f'(x)$

$$= \lim \frac{2(x+h)^2 + 3(x+h) + 1 - 2x^2 - 3x - 1}{h}$$

$$= \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) + 3x + 3h - 2x^2 - 3x}{h}$$

$$= \lim_{h\to 0} \frac{4xh + 2h^2 + 3h}{h} = \lim_{h\to 0} 4x + 2h + 3$$

$$= 4x + 3$$

(b) $g'(x) = \lim_{h\to 0} \dfrac{(x+h)^3 + 1 - x^3 - 1}{h}$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

(c) Standard: $\dfrac{f(2.001) - f(2)}{0.001} = \dfrac{15.011002 - 15}{0.001} = 11.002$

Symmetric: $\dfrac{f(2.001) - f(1.999)}{0.002}$

$$= \frac{15.011002 - 14.989002}{0.002} = 11$$

(d) The symmetric method provides a closer approximation to $f'(2) = 11$.

(e) Standard: $\dfrac{g(2.001) - g(2)}{0.001} \approx \dfrac{9.012006 - 9}{0.001}$

$$= 12.006001$$

Symmetric: $\dfrac{g(2.001) - g(1.999)}{2(0.001)} \approx 12.000001$

The symmetric method provides a closer approximation to $g'(2) = 12$.

50. (a)

$[0, 1]$ by $[0, 4]$
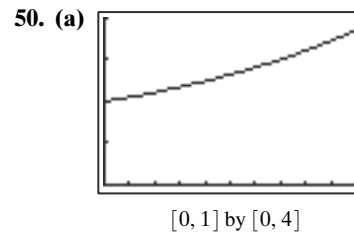
(b) ≈ 2.72

(c) ≈ 2.72

(d) Answers will vary but the true answer is $e \approx 2.72$.

51. $f'(0) = \lim_{h\to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim_{h\to 0} \dfrac{|h| - 0}{h}$

$$= \lim_{h\to 0} \frac{|h|}{h} = 1$$

if 0 is approached from the right, and -1 if 0 is approached from the left. This occurs because calculators tend to take average values for derivatives instead of applying the definition. For example, a calculator may calculate the derivative of $f(0)$ by taking

$$\frac{f(0.0001) - f(-0.0001)}{0.0001 - (-0.0001)} = 0.$$

52.

$[-5, 5]$ by $[-3, 3]$
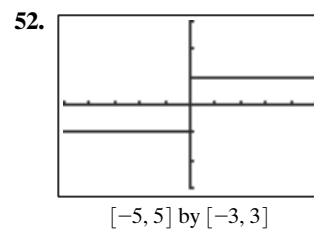
$f'(0)$ does not exist because $f(x)$ is discontinuous at $x = 0$. The calculator gives an incorrrect answer, NDER $f(0) = 1000$, because it divides by $2h = 0.002$ instead of letting $h \to 0$.

53. (a) Let y_1 = abs(sin (x)), which is $|f(x)|$. Then
NINT $(Y_1, X, 0, 2\pi)$ gives 4.

   (b) Let y_1 = abs($x^2 - 2x - 3$), which is $|f(x)|$. Then
NINT $(Y_1, X, 0, 5)$ gives ≈ 19.67.

54. Some functions, such as $\dfrac{1}{x - 2}$, have a singularity; they

   cease to exist at that point and the $\lim\limits_{x \to 2} \dfrac{1}{x - 2} = \infty$. Using

   our rectangular approximations, however, we can find the
area under the curve with successively smaller widths.
Since each of these widths are finite, we simply determine
"how close" our approximation must be to determine the
finite area under the curve. Eventually the rectangle
"next" to $x = 2$ becomes so thin as to render its area
"close enough" to zero to be ignored.

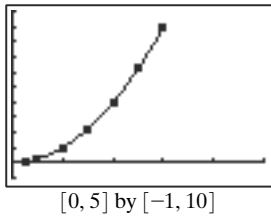**55.** Since $f(x) \geq g(x)$ for all values of x on the interval,

$$A = \lim_{N \to \infty} \sum_{k=1}^{n} \left\{ \left[\left(\frac{b - a}{N} \right) f \left(a + \frac{k(b - a)}{N} \right) \right] \right.$$
$$\left. - \left[\left(\frac{b - a}{N} \right) g \left(a + \frac{k(b - a)}{N} \right) \right] \right\}$$
$$= \lim_{N \to \infty} \sum_{k=1}^{n} \left(\frac{b - a}{N} \right) \left[f \left(a + \frac{k(b - a)}{N} \right) \right.$$
$$\left. - g \left(a + \frac{k(b - a)}{N} \right) \right].$$
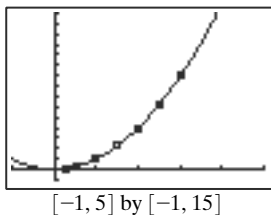
   If the area of both curves is already known and
$f(x) \geq g(x)$ for all values of x, the area between the
curves is simply the area under f minus the area under g.

56. (b)

x	$A(x)$
0.25	0.0625
0.5	0.25
1	1
1.5	2.25
2	4
2.5	6.25
3	9

$[0, 5]$ by $[-1, 10]$

   (c) $y = x^2$

$[-1, 5]$ by $[-1, 15]$

   (d) The data seem to support a curve of $A(x) = x^2$.

(e) $A'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(h)}{h} = \lim\limits_{h \to 0} \dfrac{(x + h)^2 - x^2}{h}$
$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = 2x.$$
The two functions are exactly the same.

57. (b)

x	$A(x)$
0.25	0.0156
0.5	0.125
1	1
1.5	3.375
2	8
2.5	15.625
3	27

$[0, 5]$ by $[-5, 30]$

   (c) $f(x) \approx x^3$

$[-2, 5]$ by $[-5, 30]$
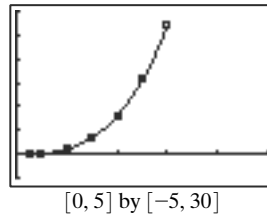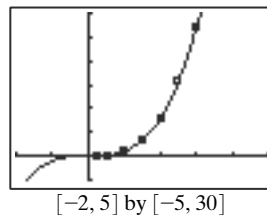
   (d) The exact value of $A(x)$ for any x greater than zero
appears to be x^3.

   (e) $A'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{(x + h)^3 - x^3}{h}$
$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2}{h} = 3x^2.$$
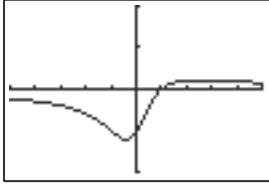The functions are exactly the same.
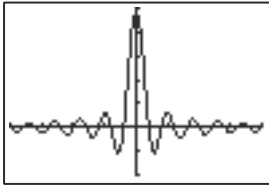
58. Answers may vary.

■ Chapter 11 Review

   1. (a) 2

   (b) Does not exist.

   2. (a) -1

   (b) Does not exist.

   3. (a) 2

   (b) 2

   4. (a) 2

   (b) 2

5. $\lim\limits_{x \to -1} \dfrac{x - 1}{x^2 + 1} = \dfrac{-1 - 1}{2} = -1$
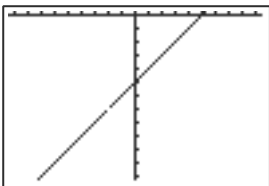

$[-5, 5]$ by $[-2, 2]$

6. $\lim\limits_{x \to 0} \dfrac{\sin 5x}{x} = \lim\limits_{x \to 0} \dfrac{\sin(2x)\cos(3x) + \cos(2x)\sin(3x)}{x}$

$= \lim\limits_{x \to 0} \dfrac{(2\sin x \cos x)(4\cos^3 x - 3\cos x) + (2\cos^2 x - 1)(3\sin x - 4\sin^3 x)}{x}$

$= \lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right)[8\cos^4 x - 6\cos^2 x + (2\cos^2 x - 1)(3 - 4\sin^2 x)]$

$= 1 \cdot [8 - 6 + (2 - 1)(3 - 0)] = 5$

$[-2\pi, 2\pi]$ by $[-2, 5]$

7. $\lim\limits_{x \to -2} \dfrac{(x - 5)(x + 2)}{x + 2} = \lim\limits_{x \to -2} x - 5 = -7$

$[-9.4, 9.4]$ by $[-12.2, 0.2]$

8. $\lim\limits_{x \to 1} |x - 1|$

$= \left\{\begin{array}{lll} \lim\limits_{x \to 1^+} & |x - 1| = \lim\limits_{x \to 1^+} & (x - 1) = 0 \\ \lim\limits_{x \to 1^-} & |x - 1| = \lim\limits_{x \to 1^-} & (-x + 1) = 0 \end{array}\right\}$
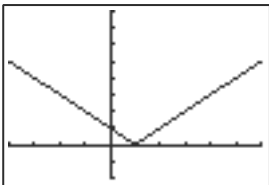
$[-4, 6]$ by $[-2, 8]$

9. $\lim\limits_{x \to 0} 2\tan^{-1}(x) = \lim\limits_{x \to 0} 2\dfrac{\sin^{-1}(x)}{\cos^{-1}(x)} = \dfrac{2 \cdot 0}{1} = 0$
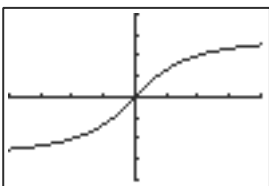
$[-\pi, \pi]$ by $[-4, 4]$

10. $\lim\limits_{x \to 0} \dfrac{2}{1 - 2^x} = \left\{\begin{array}{l} \lim\limits_{x \to 0^+} \quad \dfrac{2}{1 - 2^x} = -\infty \\ \lim\limits_{x \to 0^-} \quad \dfrac{2}{1 - 2^x} = \infty \end{array}\right. \Rightarrow$ undefined
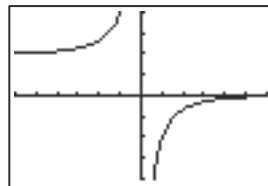
$[-6, 6]$ by $[-4, 4]$

11.

x	$f(x)$
-3	-1
-10	$-\dfrac{1}{64}$
-100	$-\dfrac{1}{9604}$
-1000	$-\dfrac{1}{996{,}004}$

$\Rightarrow \lim\limits_{x \to -\infty} f(x) = 0$

12.

x	$f(x)$
4	9
8	$\dfrac{13}{5}$
16	$\dfrac{21}{13}$
32	$\dfrac{37}{29}$
128	$\dfrac{133}{125}$
1024	$\dfrac{1029}{1021}$

$\Rightarrow \lim\limits_{x \to \infty} f(x) = 1$

13.

x	$f(x)$
1	1
2	-1
5	-4.6
10	-9.8
100	-99.98
1000	-999.998

$\Rightarrow \lim\limits_{x \to \infty} f(x) = -\infty$

14.

x	$f(x)$
0	0
-1	$-\dfrac{1}{3}$
-5	$-\dfrac{25}{7}$
-10	$-\dfrac{25}{3}$
-100	$-\dfrac{5000}{51}$
-1000	$-\dfrac{500,000}{501}$

$\Rightarrow \lim\limits_{x \to -\infty} f(x) = -\infty$

15. $\lim\limits_{x \to 2^+} \dfrac{1}{x - 2} = \infty$

16. $\lim\limits_{x \to 2^-} \dfrac{1}{x^2 - 4} = -\infty$

17. $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x} = \lim\limits_{x \to 0} \dfrac{2 - (2 + x)}{2(2 + x)} \cdot \dfrac{1}{x}$

$= \lim\limits_{x \to 0} \dfrac{-x}{x} \cdot \dfrac{1}{2(2 + x)} = -\dfrac{1}{4}$

18. $\lim\limits_{x \to 0} \dfrac{(x + 2)^3 - 8}{x} = \lim\limits_{x \to 0} \dfrac{x^3 + 6x^2 + 12x}{x}$

$= \lim\limits_{x \to 0} x^2 + 6x + 12 = 12$

19. $f(x) = \dfrac{x - 5}{(x + 5)(x + 1)}$, so f has vertical asymptotes at

$x = -1$ and $x = -5$. Since $\lim\limits_{x \to \infty} f(x) = 0$ and $\lim\limits_{x \to -\infty}$

$f(x) = 0$, f also has a horizontal asymptote at $y = 0$.

20. $f(x) = \dfrac{x^2 + 1}{2(x - 2)}$, so f has a vertical asymptote at $x = 2$.

Since $\lim\limits_{x \to \infty} f(x) = \infty$ and $\lim\limits_{x \to -\infty} f(x) = -\infty$, f has no

horizontal asymptotes.

21. $\lim\limits_{x \to 3} \dfrac{(x + 5)(x - 3)}{3 - x} = \lim\limits_{x \to 3} -x - 5 = -8.$

22. $\lim\limits_{x \to 1} \dfrac{(x - 3)(x - 1)}{x - 1} = \lim\limits_{x \to 1} x - 3 = -2$

23. $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{x - 3} + \dfrac{1}{3}}{x} = \lim\limits_{x \to 0} \dfrac{3 + (x - 3)}{3(x - 3)} \cdot \dfrac{1}{x}$

$= \lim\limits_{x \to 0} \dfrac{x}{x} \cdot \dfrac{1}{3(x - 3)} = -\dfrac{1}{9}$

24. $\lim\limits_{x \to 2} \dfrac{\sin(3x - 6)}{\cos(3x - 6)} \cdot \dfrac{1}{x - 2}$

$= \lim\limits_{x \to 2} \dfrac{\sin(3(x - 2))}{\cos(3x - 6)} \cdot \dfrac{1}{(x - 2)} =$

$= \lim\limits_{x \to 2} \dfrac{(3\sin(x - 2) - 4\sin^3(x - 2))}{(x - 2)} \cdot \dfrac{1}{\cos(3x - 6)}$

$= \lim\limits_{x \to 2} \left(3\dfrac{\sin(x - 2)}{x - 2} - 4\dfrac{\sin^2(x - 2)\sin(x - 2)}{x - 2} \right) \cdot$

$\dfrac{1}{\cos(3x - 6)} = (3 - 0 \cdot 1)\left(\dfrac{1}{1}\right) = 3$

25. $\lim\limits_{x \to 2} \dfrac{(x - 2)(x - 3)}{(x - 2)(x - 1)} = \lim\limits_{x \to 2} \dfrac{x - 3}{x - 1} = -\dfrac{1}{1} = -1$

26. $\lim\limits_{x \to 3} (x - 3) = 0$

27. $\lim\limits_{x \to 1} \dfrac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim\limits_{x \to 1} (x^2 + x + 1) = 3$

$F(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1} & x \neq 1 \\ 3 & x = 1 \end{cases}$

28. $\lim\limits_{x \to 5} \dfrac{(x - 5)(x - 1)}{(x - 5)} = \lim\limits_{x \to 5} x - 1 = 4$

$F(x) = \begin{cases} \dfrac{x^2 - 6x + 5}{x - 5} & x \neq 5 \\ 4 & x = 5 \end{cases}$

29. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(2 + h) - f(2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{1 - (h + 2) - 2(h + 2)^2 - (-9)}{h}$

$= \lim\limits_{h \to 0} \dfrac{-2h^2 - 8h - 8 - h + 8}{h} = \lim\limits_{h \to 0} \dfrac{-2h^2 - 9h}{h}$

$= \lim\limits_{h \to 0} (-2h - 9) = -9$

30. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(2 + h) - f(2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(2 + h + 3)^2 - 25}{h} = \lim\limits_{h \to 0} \dfrac{h^2 + 10h}{h}$

$= \lim\limits_{h \to 0} h + 10 = 10$

31. (a) $\dfrac{f(3.01) - f(3)}{3.01 - 3} = \dfrac{12.0801 - 12}{0.01} = 8.01$

(b) $\displaystyle\lim_{h \to 0} \dfrac{f(3 + h) - f(3)}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{(h + 3)^2 + 2(h + 3) - 3 - 12}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{h^2 + 6h + 9 + 2h + 6 - 15}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{h^2 + 8h}{h} = 8$

32. (a) $\dfrac{f(3.01) - f(3)}{3.01 - 3} = \dfrac{\dfrac{3}{5.01} - 0.6}{0.01} = -\dfrac{20}{167} \approx -0.12$

(b) $\displaystyle\lim_{h \to 0} \dfrac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \dfrac{\dfrac{3}{h + 3 + 2} - \dfrac{3}{5}}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{15 - 3(h + 5)}{5(h + 5)} \cdot \dfrac{1}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{-3h}{h} \cdot \dfrac{1}{5(h + 5)} = -\dfrac{3}{25}$

33. (a) $m = \displaystyle\lim_{h \to 0} \dfrac{f(h + 1) - f(1)}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{(h + 1)^3 - 2(h + 1) + 1 - 0}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{h^3 + 3h^2 + h}{h} = \lim_{h \to 0} h^2 + 3h + 1 = 1$

(b) $(1, f(1)) = (1, 0)$ so the equation of the tangent line at $x = 1$ is $y = x - 1$.

34. (a) $m = \displaystyle\lim_{h \to 0} \dfrac{f(h + 7) - h(7)}{h} = \lim_{h \to 0} \dfrac{\sqrt{h + 3} - \sqrt{3}}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{h + 3 - 3}{h(\sqrt{h + 3} + \sqrt{3})}$

$= \displaystyle\lim_{h \to 0} \dfrac{1}{\sqrt{h + 3} + \sqrt{3}} = \dfrac{\sqrt{3}}{6}$

(b) $(7, f(7)) = (7, \sqrt{3})$ so the equation of the tangent line at $x = 7$ is $y = \dfrac{\sqrt{3}}{6} x - \dfrac{\sqrt{3}}{6}$.

35. $\displaystyle\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{5(x + h)^2 + 7(x + h) - 1 - 5x^2 - 7x + 1}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{5x^2 + 10xh + 5h^2 + 7x + 7h - 5x^2 - 7x}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{10xh + 5h^2 + 7h}{h} = 10x + 7$

36. $\displaystyle\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{3(x + h)^2 - 8(x + h) + 2 - 3x^2 + 8x - 2}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{6xh + 3h^2 - 8h}{h} = \lim_{h \to 0} 6x + 3h - 8$

$= 6x - 8$

For #37 and 38, verify the function is nonnegative through graphical or numerical analysis.

37. (b) LRAM: 42.2976

RRAM: 40.3776

Average: $\dfrac{42.2976 + 40.3776}{2} = 41.3376$

38. (b) LRAM: 49.2352

RRAM: 52.1152

Average: $\dfrac{49.2352 + 52.1152}{2} = 50.6752$

39. (a) Using $x = 0$ for 1995, the scatter plot of the data is:
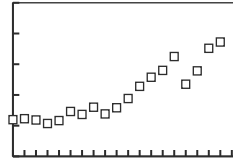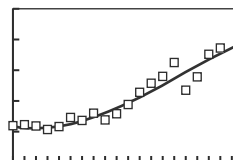

[0, 20] by [0, 5]

(b) The average rate of change for 2000 to 2001 is found by examining $\dfrac{\Delta x}{\Delta y}$.

$\dfrac{\Delta y}{\Delta x} = \dfrac{146 - 151}{1 - 0} = -5$ cents per year.

The average rate of change for 2010 to 2011 is found by examining $\dfrac{\Delta x}{\Delta y}$.

$\dfrac{\Delta y}{\Delta x} = \dfrac{353 - 279}{1 - 0} = 74$ cents per year.

(c) The average rate exhibits the greatest increase from one year to the next consecutive year in the interval from 2010 to 2011.

(d) The average rate exhibits the greatest decrease from one year to the next consecutive year in the interval from 2008 to 2009.

(e) The linear regression model for the data is:

$y = 0.154x + 0.746$.
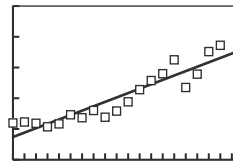
[0, 20] by [0, 5]

(f) The cubic regression model for the data is:

$y = -0.00069717x^3 + 0.0249x^2 - 0.0846x + 1.2110$.


[0, 20] by [0, 5]

Most of the data points touch the regression curve, which would suggest that this is a fairly good model. One possible answer: The cubic regression model is the best. It shows a larger rate of increase in fuel prices, which is what is happening in today's market.

(g) The cubic regression model for the data is:
$y = -0.00069717x^3 + 0.0249x^2 - 0.0846x + 1.2110$.
Using NDER with this regression model yields the following instantaneous rates of change for:

2000	112 cents per year
2004	19.4 cents per year
2007	21.2 cents per year
2011	17.7 cents per year

The cubic regression model is showing a rate of increase. The true rate of change, however, may be higher than what the model indicates.

(h) If we use the given cubic regression model, the average price of a gallon of regular gas in 2011 will be $3.96 per gallon. Based on what is happening with the current fuel prices, this prediction is likely too high, but as time progresses, the model will eventually predict prices that are too low.
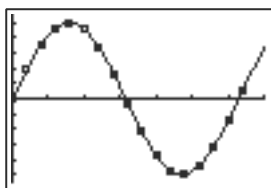
40. (a)

x	$A(x)$
0	0
0.4	0.38942
0.8	0.71736
1.2	0.93204
1.6	0.99957
2.0	0.90930
2.4	0.67546
2.8	0.33499
3.2	−0.05837
3.6	−0.44252
4.0	−0.75680
4.4	−0.95160
4.8	−0.99616
5.2	−0.88345
5.6	−0.63127
6.0	−0.27942
6.4	0.11655
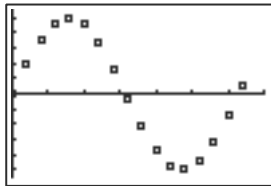
[0, 7] by [−1.1, 1.1]

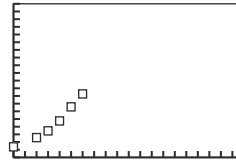**(b)** $f(x) = \sin x$

[0, 7] by [−1.1, 1.1]

(c) $f'(x) = \cos x$, the function being integrated.

(d) The derivative of $\text{NINT}(f(t), t, 0, x)$ gives $f(t)$.

Chapter 11 Project

1. The scatter plot of the population data for Austin, TX, is as follows. The year 1950 is represented by $t = 1950$.

[1950, 2150] by [0, 2000000]

2. The average population growth rate from 1950–2011 is

$$\frac{820,611 - 132,459}{2011 - 1950} = \frac{688,152}{61} = 11,281 \text{ persons/year.}$$

The average population growth rate from 1970–2011 is

$$\frac{820,611 - 253,539}{2011 - 1970} = \frac{567,072}{41} = 13,831 \text{ persons/year.}$$

The average population growth rate from 1980–2011 is

$$\frac{820,611 - 345,890}{2011 - 1980} = \frac{474,721}{31} = 15,314 \text{ persons/year.}$$

The average population growth rate from 2000–2011 is

$$\frac{820,611 - 656,562}{2011 - 2000} = \frac{164,049}{11} = 14,914 \text{ persons/year.}$$

3. The logistic regression model for the population data is

$$y = \frac{1,872,804}{1 + (4.9170441 \times 10^{34})e^{-0.039605897t}}.$$ The logistic

growth model is more realistic than the exponential growth model since the population of Austin will not increase without bound, but have an upper population limit.
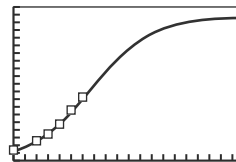
[1950, 2150] by [0, 2000000]

4. If we use NDER with the given exponential logistic model, the instantaneous population growth rate in 2011 is 18,294 persons/year.

5. Using the exponential regression model

$$y = \frac{1,872,804}{1 + e^{-4.9170441 \times 10^{34} \cdot t}}.$$ to predict the population of

Austin for the years 2013, 2020, and 2030 yields:

$$y = \frac{1,872,804}{1 + e^{-4.9170441 \times 10^{34}(2013)}} = 864,474 \text{ persons}$$

$$y = \frac{1,872,804}{1 + e^{-4.9170441 \times 10^{34}(2020)}} = 994,063 \text{ persons}$$

$$y = \frac{1,872,804}{1 + e^{-4.9170441 \times 10^{34}(2040)}} = 1,337,387 \text{ persons}$$

The predictions from the logistic model and the newspaper are relatively close to each other and both seem reasonable.