Chapter 8
Analytic Geometry in Two and Three Dimensions

■ Section 8.1 Conic Sections and Parabolas

Exploration 1

1. From Figure 8.4, we see that the axis of the parabola is $x = 0$. Thus, we want to find the point along $x = 0$ that is equidistant from both $(0, 1)$ and the line $y = -1$. Since the axis is perpendicular to the directrix, the point on the directrix closest to the parabola is $(0, 1)$ and $(0, -1)$, it must be located at $(0, 0)$.

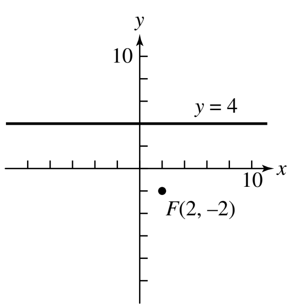
2. Choose any point on the parabola (x, y). From figures 8.3 and 8.4, we see that the distance from (x, y) to the focus is $d_1 = \sqrt{(x-0)^2 + (y-1)^2} = \sqrt{x^2 + (y-1)^2}$ and the distance from (x, y) to the directrix is $d_2 = \sqrt{(x-x)^2 + (y-(-1))^2} = \sqrt{(y+1)^2}$. Since d_1 must equal d_2, we have

$$d_1 = \sqrt{x^2 + (y-1)^2} = \sqrt{(y+1)^2} = d_2$$
$$x^2 + (y-1)^2 = (y+1)^2$$
$$x^2 + y^2 - 2y + 1 = y^2 + 2y + 1$$
$$x^2 = 4y$$
$$\frac{x^2}{4} = y \text{ or } x^2 = 4y$$

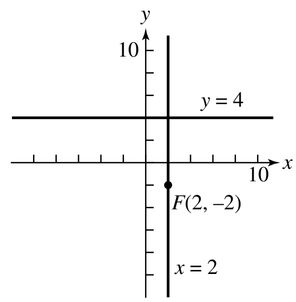
3. From the figure, we see that the first dashed line above $y = 0$ is $y = 1$, and we assume that each subsequent dashed line increases by $y = 1$. Using the equation above, we solve $\left\{1 = \frac{x^2}{4}, 2 = \frac{x^2}{4}, 3 = \frac{x^2}{4}, 4 = \frac{x^2}{4}, 5 = \frac{x^2}{4}, 6 = \frac{x^2}{4}\right\}$ to find:
$\{(-2\sqrt{6}, 6), (-2\sqrt{5}, 5), (-4, 4), (-2\sqrt{3}, 3), (-2\sqrt{2}, 2), (-2, 1), (0, 0), (2, 1), (2\sqrt{2}, 2), (2\sqrt{3}, 3), (4, 4), (2\sqrt{5}, 5), (2\sqrt{6}, 6)\}$

Exploration 2

1.

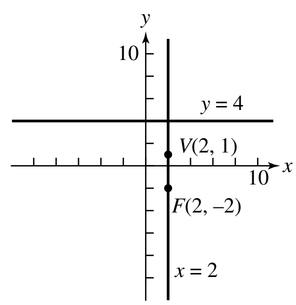


2.



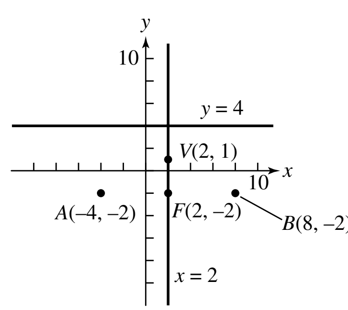
The equation of the axis is $x = 2$.

3.

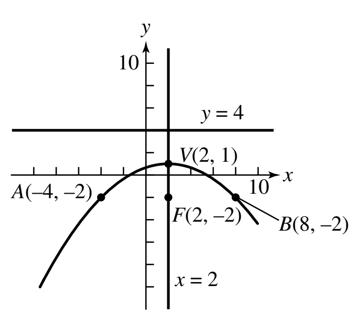


4. Since the focus $(h, k + p) = (2, -2)$ and the directrix $y = k - p = 4$, we have $k + p = -2$ and $k - p = 4$. Thus, $k = 1$, $p = -3$. As a result, the focal length p is -3 and the focal width $|4p|$ is 12.

5. Since the focal width is 12, each endpoint of the chord is 6 units away from the focus $(2, -2)$ along the line $y = -2$. The endpoints of the chord, then, are $(2 - 6, -2)$ and $(2 + 6, -2)$, or $(-4, -2)$ and $(8, -2)$.



6.



7. Downward

8. $h = 2, p = -3, k = 1$, so $(x - 2)^2 = -12(y - 1)$

Quick Review 8.1

1. $\sqrt{(2 - (-1))^2 + (5 - 3)^2} = \sqrt{9 + 4} = \sqrt{13}$

2. $\sqrt{(a - 2)^2 + (b + 3)^2}$

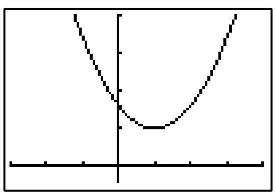
3. $y^2 = 4x, y = \pm 2\sqrt{x}$

4. $y^2 = 5x, y = \pm \sqrt{5x}$

5. $y + 7 = -(x^2 - 2x), y + 7 - 1 = -(x - 1)^2$,
$y + 6 = -(x - 1)^2$

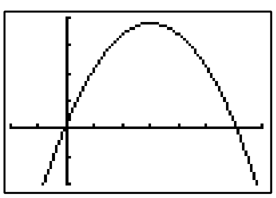
6. $y + 5 = 2(x^2 + 3x), y + 5 + \frac{9}{2} = 2\left(x + \frac{3}{2}\right)^2$
$y + \frac{19}{2} = 2\left(x + \frac{3}{2}\right)^2$

7. Vertex: (1, 5). $f(x)$ can be obtained from $g(x)$ by stretching x^2 by 3, shifting up 5 units, and shifting right 1 unit.



$[-3, 4]$ by $[-2, 20]$

8. Vertex: (3, 19). $f(x) = -2(x - 3)^2 + 19$. $f(x)$ can be obtained from $g(x)$ by stretching x^2 by 2, reflecting across the x-axis, shifting up 19 units and shifting right 3 units.



$[-2, 7]$ by $[-10, 20]$

9. $f(x) = a(x + 1)^2 + 3$, so $1 = a + 3, a = -2$,
$f(x) = -2(x + 1)^2 + 3$.

10. $f(x) = a(x - 2)^2 - 5$, so $13 = 9a - 5, a = 2$,
$f(x) = 2(x - 2)^2 - 5$

Section 8.1 Exercises

1. $k = 0, h = 0, p = \frac{6}{4} = \frac{3}{2}$. Vertex: $(0, 0)$, Focus: $\left(0, \frac{3}{2}\right)$,
Directrix: $y = -\frac{3}{2}$, Focal width: $\left|4p\right| = \left|4 \cdot \frac{3}{2}\right| = 6$

2. $k = 0, h = 0, p = \frac{-8}{4} = -2$. Vertex: $(0, 0)$,
Focus: $(-2, 0)$, Directrix: $x = 2$,
Focal width: $|4p| = |4(-2)| = 8$

3. $k = 2, h = -3, p = \frac{4}{4} = 1$. Vertex: $(-3, 2)$,
Focus: $(-2, 2)$, Directrix: $x = -3 - 1 = -4$,
Focal width: $|4p| = |4(1)| = 4$.

4. $k = -1, h = -4, p = \frac{-6}{4} = \frac{-3}{2}$. Vertex: $(-4, -1)$,
Focus: $\left(-4, \frac{-5}{2}\right)$, Directrix: $y = -1 - \left(\frac{-3}{2}\right) = \frac{1}{2}$,
Focal width: $\left|4p\right| = \left|4\left(\frac{-3}{2}\right)\right| = 6$

5. $k = 0, h = 0, 4p = \frac{-4}{3}$, so $p = -\frac{1}{3}$. Vertex: $(0, 0)$,
Focus: $\left(0, -\frac{1}{3}\right)$, Directrix: $y = \frac{1}{3}$, Focal width:
$\left|4p\right| = \left|\left(\frac{-4}{3}\right)\right| = \frac{4}{3}$

6. $k = 0, h = 0, 4p = \frac{16}{5}$, so $p = \frac{4}{5}$. Vertex: $(0, 0)$,
Focus: $\left(\frac{4}{5}, 0\right)$, Directrix: $x = -\frac{4}{5}$,
Focal width: $\left|4p\right| = \left|4\left(\frac{4}{5}\right)\right| = \frac{16}{5}$

7. (c)

8. (b)

9. (a)

10. (d)

For #11–30, recall that the standard form of the parabola is dependent on the vertex (h, k), the focal length p, the focal width $|4p|$, and the direction that the parabola opens.

11. $p = -3$ and the parabola opens to the left, so $y^2 = -12x$.

12. $p = 2$ and the parabola opens upward, so $x^2 = 8y$.

13. $-p = 4$ (so $p = -4$) and the parabola opens downward, so $x^2 = -16y$.

14. $-p = -2$ (so $p = 2$) and the parabola opens to the right, so $y^2 = 8x$.

15. $p = 5$ and the parabola opens upward, so $x^2 = 20y$.

16. $p = -4$ and the parabola opens to the left, so $y^2 = -16x$.

17. $h = 0, k = 0, |4p| = 8 \Rightarrow p = 2$ (since it opens to the right): $(y - 0)^2 = 8(x - 0)$; $y^2 = 8x$.

18. $h = 0, k = 0, |4p| = 12 \Rightarrow p = -3$ (since it opens to the left): $(y - 0)^2 = -12(x - 0)$; $y^2 = -12x$

19. $h = 0, k = 0, |4p| = 6 \Rightarrow p = -\frac{3}{2}$ (since it opens downward): $(x - 0)^2 = -6(y - 0)$; $x^2 = -6y$

20. $h = 0, k = 0, |4p| = 3 \Rightarrow p = \frac{3}{4}$ (since it opens upward): $(x - 0)^2 = 3(y - 0)$; $x^2 = 3y$

21. $h = -4, k = -4, -2 = -4 + p$, so $p = 2$ and the parabola opens to the right; $(y + 4)^2 = 8(x + 4)$

22. $h = -5, k = 6, 6 + p = 3$, so $p = -3$ and the parabola opens downward; $(x + 5)^2 = -12(y - 6)$

23. Parabola opens upward and vertex is halfway between focus and directrix on $x = h$ axis, so $h = 3$ and $k = \frac{4+1}{2} = \frac{5}{2}$; $1 = \frac{5}{2} - p$, so $p = \frac{3}{2}$.
$(x-3)^2 = 6\left(y - \frac{5}{2}\right)$

24. Parabola opens to the left and vertex is halfway between focus and directrix on $y = k$ axis, so $k = -3$ and $h = \frac{2+5}{2} = \frac{7}{2}$; $5 = \frac{7}{2} - p$, so $p = -\frac{3}{2}$.
$(y+3)^2 = -6\left(x - \frac{7}{2}\right)$

25. $h = 4, k = 3$; $6 = 4 - p$, so $p = -2$ and parabola opens to the left. $(y-3)^2 = -8(x-4)$

26. $h = 3, k = 5$; $7 = 5 - p$, so $p = -2$ and the parabola opens downward. $(x-3)^2 = -8(y-5)$

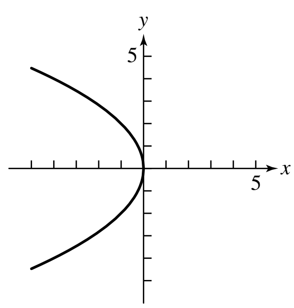
27. $h = 2, k = -1$; $|4p| = 16 \Rightarrow p = 4$ (since it opens upward): $(x-2)^2 = 16(y+1)$

28. $h = -3, k = 3$; $|4p| = 20 \Rightarrow p = -5$ (since it opens downward): $(x+3)^2 = -20(y-3)$

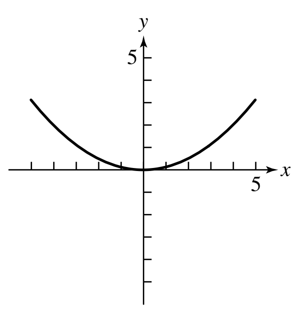
29. $h = -1, k = -4$; $|4p| = 10 \Rightarrow p = -\frac{5}{2}$ (since it opens to the left): $(y+4)^2 = -10(x+1)$

30. $h = 2, k = 3$; $|4p| = 5 \Rightarrow p = \frac{5}{4}$ (since it opens to the right): $(y-3)^2 = 5(x-2)$

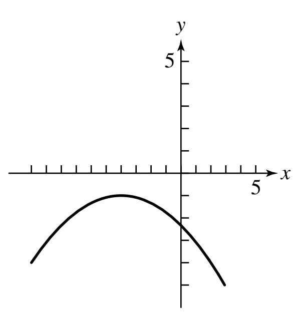
31.



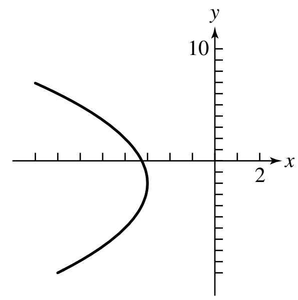
32.



33.



34.



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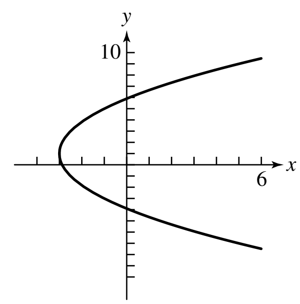
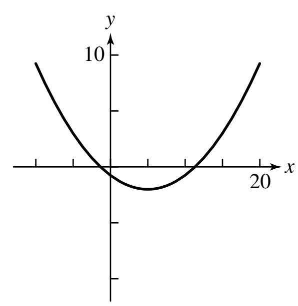


36.



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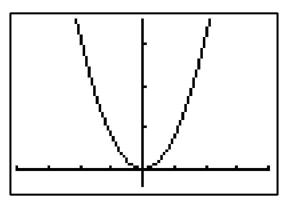
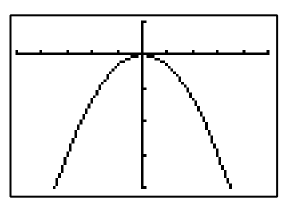


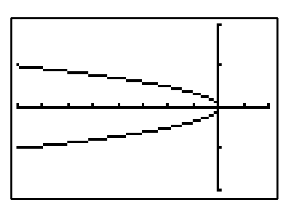
[−4, 4] by [−2, 18]

38.



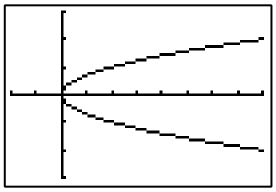
[−10, 10] by [−8, 2]

39.



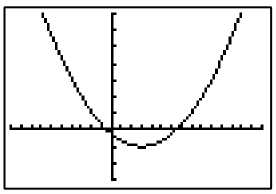
[−8, 2] by [−2, 2]

40.



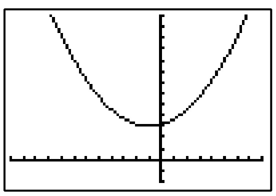
$[-2, 8]$ by $[-3, 3]$

41.



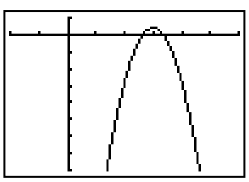
$[-10, 15]$ by $[-3, 7]$

42.



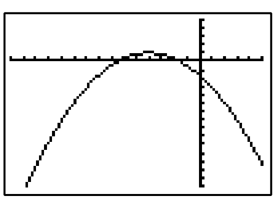
$[-12, 8]$ by $[-2, 13]$

43.



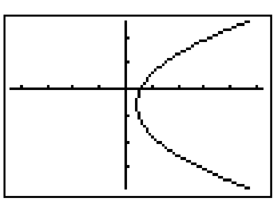
$[-2, 6]$ by $[-40, 5]$

44.



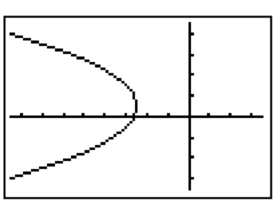
$[-15, 5]$ by $[-15, 5]$

45.



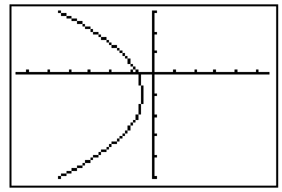
$[-22, 26]$ by $[-19, 13]$

46.



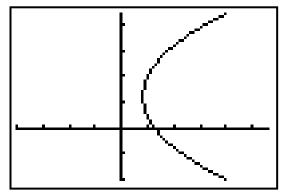
$[-17, 7]$ by $[-7, 9]$

47.



$[-13, 11]$ by $[-10, 6]$

48.



$[-20, 28]$ by $[-10, 22]$

49. Completing the square produces $y - 2 = (x + 1)^2$. The vertex is $(h, k) = (-1, 2)$, so the focus is $(h, k + p) = \left(-1, 2 + \frac{1}{4}\right) = \left(-1, \frac{9}{4}\right)$, and the directrix is $y = k - p = 2 - \frac{1}{4} = \frac{7}{4}$

50. Completing the square produces $2\left(y - \frac{7}{6}\right) = (x - 1)^2$.

The vertex is $(h, k) = \left(1, \frac{7}{6}\right)$, so the focus is $(h, k + p) = \left(1, \frac{7}{6} + \frac{1}{2}\right) = \left(1, \frac{5}{3}\right)$, and the directrix is $y = k - p = \frac{7}{6} - \frac{1}{2} = \frac{2}{3}$.

51. Completing the square produces $8(x - 2) = (y - 2)^2$. The vertex is $(h, k) = (2, 2)$ so the focus is $(h + p, k) = (2 + 2, 2) = (4, 2)$, and the directrix is $x = h - p = 2 - 2 = 0$.

52. Completing the square produces $-4\left(x - \frac{13}{4}\right) = (y - 1)^2$. The vertex is $(h, k) = \left(\frac{13}{4}, 1\right)$ so the focus is $(h + p, k) = \left(\frac{13}{4} - 1, 1\right) = \left(\frac{9}{4}, 1\right)$, and the directrix is $x = h - p = \frac{13}{4} + 1 = \frac{17}{4}$.

53. $h = 0$, $k = 2$, and the parabola opens to the left, so $(y - 2)^2 = 4p(x)$. Using $(-6, -4)$, we find $(-4 - 2)^2 = 4p(-6) \Rightarrow 4p = -\frac{36}{6} = -6$. The equation for the parabola is: $(y - 2)^2 = -6x$

54. $h = 1$, $k = -3$, and the parabola opens to the right, so $(y + 3)^2 = 4p(x - 1)$. Using $\left(\frac{11}{2}, 0\right)$, we find $(0 - 3)^2 = 4p\left(\frac{11}{2} - 1\right) \Rightarrow 4p = 9 \cdot \frac{2}{9} = 2$. The equation for the parabola is: $(y + 3)^2 = 2(x - 1)$.

55. $h = 2$, $k = -1$ and the parabola opens down so $(x - 2)^2 = 4p(y + 1)$. Using $(0, -2)$, we find that $(0 - 2)^2 = 4p(-2 + 1)$, so $4 = -4p$ and $p = -1$. The equation for the parabola is: $(x - 2)^2 = -4(y + 1)$.

56. $h = -1$, $k = 3$ and the parabola opens up so $(x + 1)^2 = 4p(y - 3)$. Using $(3, 5)$, we find that $(3 + 1)^2 = 4p(5 - 3)$, so $16 = 8p$ and $p = 2$. The equation for the parabola is $(x + 1)^2 = 8(y - 3)$

57. One possible answer:
If p is replaced by $-p$ in the proof, then the result is $x^2 = -4py$, which is the correct result.

58. One possible answer:
Let $P(x, y)$ be a point on the parabola with focus $(p, 0)$ and directrix $x = -p$. Then $\sqrt{(x - p)^2 + (y - 0)^2} =$ distance from (x, y) to $(p, 0)$ and $\sqrt{(x - (-p))^2 + (y - y)^2} =$ distance from (x, y) to line $x = -p$. Because a point on a parabola is equidistant from the focus and the directrix, we can equate these distances. After squaring both sides, we obtain

$$(x - p)^2 + (y - 0)^2 = (x - (-p))^2 + (y - y)^2$$
$$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2$$
$$y^2 = 4px.$$

59. For the beam to run parallel to the axis of the mirror, the filament should be placed at the focus. As with Example 6, we must find p by using the fact that the points $(\pm 3, 2)$ must lie on the parabola. Then,

$$(\pm 3)^2 = 4p(2)$$
$$9 = 8p$$
$$p = \frac{9}{8} = 1.125 \text{ cm}$$

Because $p = 1.125$ cm, the filament should be placed 1.125 cm from the vertex along the axis of the mirror.

60. For maximum efficiency, the receiving antenna should be placed at the focus of the reflector. As with Example 6, we know that the points $(\pm 2.5, 2)$ lie on the parabola. Solving for p, we find

$$(\pm 2.5)^2 = 4p(2)$$
$$8p = 6.25$$
$$p = 0.78125 \text{ ft}$$

The receiving antenna should be placed 0.78125 ft, or 9.375 inches, from the vertex along the axis of the reflector.

61. $4p = 10$, so $p = \frac{5}{2}$ and the focus is at $(0, p) = (0, 2.5)$.
The electronic receiver is located 2.5 units from the vertex along the axis of the parabolic microphone.

62. $4p = 12$, so $p = 3$ and the focus is at $(0, p) = (0, 3)$. The light bulb should be placed 3 units from the vertex along the axis of the headlight.

63. Consider the roadway to be the axis. Then, the vertex of the parabola is $(300, 10)$ and the points $(0, 110)$ and $(600, 110)$ both lie on it. Using the standard formula, $(x - 300)^2 = 4p(y - 10)$. Solving for $4p$, we have $(600 - 300)^2 = 4p(110 - 10)$, or $4p = 900$, so the formula for the parabola is $(x - 300)^2 = 900(y - 10)$. The length of each cable is the distance from the parabola to the line $y = 0$. After solving the equation of the parabola for y ($y = \frac{1}{900}x^2 - \frac{2}{3}x + 110$), we determine that the length of each cable is
$\sqrt{(x - x)^2 + \left(\frac{1}{900}x^2 - \frac{2}{3}x + 110 - 0\right)^2} =$
$\frac{1}{900}x^2 - \frac{2}{3}x + 110$. Starting at the leftmost tower, the lengths of the cables are: $\approx \{79.44, 54.44, 35, 21.11, 12.78, 10, 12.78, 21.11, 35, 54.44, 79.44\}$.

64. Consider the x-axis as a line along the width of the road and the y-axis as the line from the middle stripe of the road to the middle of the bridge — the vertex of the parabola. Since we want a minimum clearance of 16 feet at each side of the road, we know that the points $(\pm 15, 16)$ lie on the parabola. We also know that the points $(\pm 30, 0)$ lie on the parabola and that the vertex occurs at some height k along the line $x = 0$, or $(0, k)$. From the standard formula, $(x - 0)^2 = 4p(y - k)$, or $x^2 = 4p(y - k)$. Using the points $(15, 16)$, and $(30, 0)$, we have:

$$30^2 = 4p(0 - k)$$
$$15^2 = 4p(16 - k)$$

Solving these two equations gives $4p = -42.1875$ and $k \approx 21.33$. The maximum clearance must be at least 21.33 feet.

65. False. Every point on a parabola is the same distance from its focus and its directrix.

66. False. The directrix of a parabola is perpendicular to the parabola's axis.

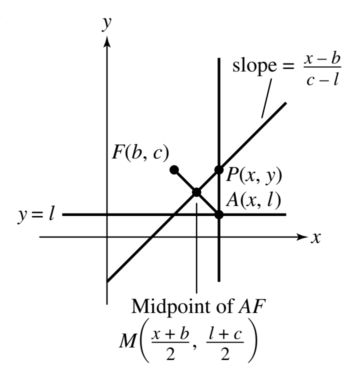
67. The word "oval" does not denote a mathematically precise concept. The answer is D.

68. $(0)^2 = 4p(0)$ is true no matter what p is. The answer is D.

69. The focus of $y^2 = 4px$ is $(p, 0)$. Here $p = 3$, so the answer is B.

70. The vertex of a parabola with equation $(y - k)^2 = 4p(x - h)$ is (h, k). Here, $k = 3$ and $h = -2$. The answer is D.

71. (a)–(c)



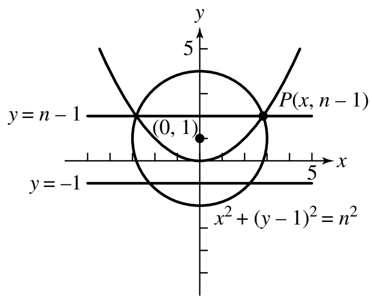
(d) As A moves, P traces out the curve of a parabola.

(e) With labels as shown, we can express the coordinates of P using the point-slope equation of the line PM:

$$y - \frac{\ell + c}{2} = \frac{x - b}{c - \ell}\left(x - \frac{x + b}{2}\right)$$
$$y - \frac{\ell + c}{2} = \frac{(x - b)^2}{2(c - \ell)}$$
$$2(c - \ell)\left(y - \frac{\ell + c}{2}\right) = (x - b)^2$$

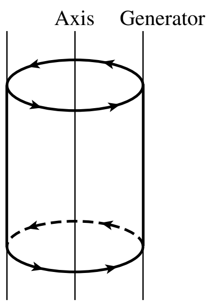
This is the equation of a parabola with vertex at $\left(b, \frac{\ell + c}{2}\right)$ and focus at $\left(b, \frac{\ell + c}{2} + p\right)$ where $p = \frac{c - \ell}{2}$.

72. (a)–(d)



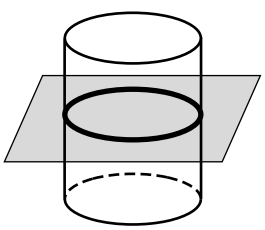
(e) A parabola with directrix $y = -1$ and focus at $(0, 1)$ has equation $x^2 = 4y$. Since P is on the circle $x^2 + (y - 1)^2 = n^2$ and on the line $y = n - 1$, its x-coordinate of P must be
$x = \sqrt{n^2 - ((n - 1) - 1)^2} = \sqrt{n^2 - (n - 2)^2}$.
Substituting $(\sqrt{n^2 - (n - 2)^2}, n - 1)$ into $x^2 = 4y$ shows that $(\sqrt{n^2 - (n - 2)^2})^2 = 4(n - 1)$ so P lies on the parabola $x^2 = 4y$.

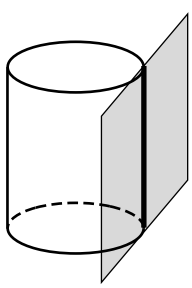
73. (a)

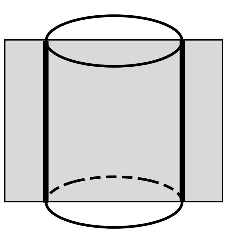


Cylinder

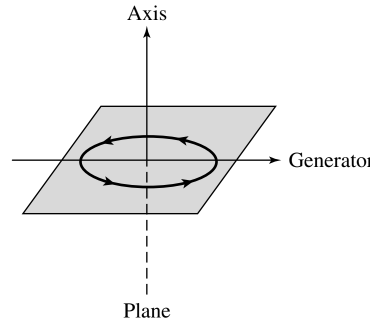
(b)


Circle


Single line

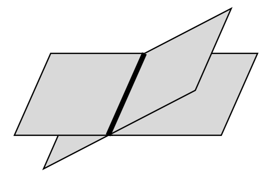

Two parallel lines

(c)



(d)


Plane


Line

74. The point (a, b) is on the parabola $y = \frac{1}{4p}x^2$ if and only if $b = \frac{a^2}{4p}$. The parabola $y = \frac{1}{4p}x^2$ and the line $y = m(x - a) + \frac{a^2}{4p}$ intersect in exactly one point (namely the point $\left(a, \frac{a^2}{4p}\right)$) if and only if the quadratic equation $\frac{1}{4p}x^2 - mx + am - \frac{a^2}{4p} = 0$ has exactly one solution. This happens if and only if the discriminant of the quadratic formula is zero.

$$(-m)^2 - 4\left(\frac{1}{4p}\right)\left(am - \frac{a^2}{4p}\right) = m^2 - \frac{am}{p} + \frac{a^2}{4p^2}$$
$$= \left(m - \frac{a}{2p}\right)^2 = 0 \text{ if and only if } m = \frac{a}{2p}.$$

Substituting $m = \frac{a}{2p}$ and $x = 0$ into the equation of the line gives the y-intercept

$$y = \frac{a}{2p}(0 - a) + \frac{a^2}{4p} = -\frac{a^2}{2p} + \frac{a^2}{4p} = -\frac{a^2}{4p} = -b.$$

75. (a) The focus of the parabola $y = \frac{1}{4p}x^2$ is at $(0, p)$ so any line with slope m that passes through the focus must have equation $y = mx + p$.
The endpoints of a focal chord are the intersection points of the parabola $y = \frac{1}{4p}x^2$ and the line $y = mx + p$.
Solving the equation $\frac{1}{4p}x^2 - mx - p = 0$ using the quadratic formula, we have

$$x = \frac{m \pm \sqrt{m^2 - 4\left(\frac{1}{4p}\right)(-p)}}{2\left(\frac{1}{4p}\right)}$$
$$= \frac{m \pm \sqrt{m^2 + 1}}{\frac{1}{2p}} = 2p(m \pm \sqrt{m^2 + 1}).$$

(b) The y-coordinates of the endpoints of a focal chord are

$$y = \frac{1}{4p}\left(2p\left(m + \sqrt{m^2+1}\right)\right)^2 \text{ and}$$
$$y = \frac{1}{4p}\left(2p\left(m - \sqrt{m^2+1}\right)\right)^2$$
$$\frac{1}{4p}(4p^2)(m^2 + 2m\sqrt{m^2+1} + (m^2+1))$$
$$= \frac{1}{4p}(4p^2)(m^2 - 2m\sqrt{m^2+1} + (m^2+1))$$
$$= p(2m^2 + 2m\sqrt{m^2+1} + 1)$$
$$= p(2m^2 - 2m\sqrt{m^2+1} + 1)$$

Using the distance formula for $(2p(m - \sqrt{m^2+1}), p(2m^2 - 2m\sqrt{m^2+1} + 1))$ and $(2p(m + \sqrt{m^2+1}),$ $p(2m^2 + 2m\sqrt{m^2+1} + 1))$, we know that the length of any focal chord is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(4p\sqrt{m^2+1})^2 + (4mp\sqrt{m^2+1})^2}$$
$$= \sqrt{(16m^2p^2 + 16p^2) + (16m^4p^2 + 16m^2p^2)}$$
$$= \sqrt{16m^4p^2 + 32m^2p^2 + 16p^2}$$

The quantity under the radical sign is smallest when $m = 0$. Thus the smallest focal chord has length $\sqrt{16p^2} = |4p|$.

76. (a) For the parabola $x^2 = 4py$, the axis and directrix intersect at the point $(0, -p)$. since the latus rectum is perpendicular to the axis of symmetry, its slope is 0, and from Exercise 65 we know the endpoints are $(-2p, p)$ and $(2p, p)$. These points are symmetric about the y-axis, so the distance from $(-2p, p)$ to $(0, -p)$ equals the distance from $(2p, p)$ to $(0, -p)$. The slope of the line joining $(0, -p)$ and $(2p, p)$ is $\frac{-p - p}{0 - (-2p)} = -1$ and the slope of the line joining $(0, -p)$ and $(2p, p)$ is $\frac{-p - p}{0 - 2p} = 1$. So the lines are perpendicular, and we know that the three points form a right triangle.

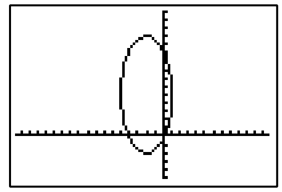
(b) By Exercise 64, the line passing through $(2p, p)$ and $(0, -p)$ must be tangent to the parabola; similarly for $(-2p, p)$ and $(0, -p)$.

■ Section 8.2 Ellipses

Exploration 1

1. The equations $x = -2 + 3\cos t$ and $y = 5 + 7\sin t$ can be rewritten as $\cos t = \frac{x+2}{3}$ and $\sin t = \frac{y-5}{7}$.
Substituting these into the identity $\cos^2 t + \sin^2 t = 1$ yields the equation $\frac{(x+2)^2}{9} + \frac{(y-5)^2}{49} = 1$.

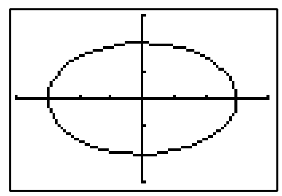
2.



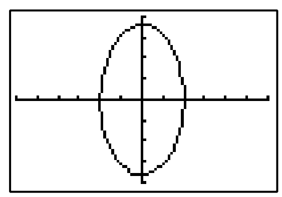
$[-17.5, 12.5]$ by $[-5, 15]$

3. Example 1: Since $\frac{x^2}{9} + \frac{y^2}{4} = 1$, a parametric solution is $x = 3\cos t$ and $y = 2\sin t$.
Example 2: Since $\frac{y^2}{13} + \frac{x^2}{4} = 1$, a parametric solution is $y = \sqrt{13}\sin t$ and $x = 2\cos t$.
Example 3: Since $\frac{(x-3)^2}{25} + \frac{(y+1)^2}{16} = 1$, a parametric solution is $x = 5\cos t + 3$ and $y = 4\sin t - 1$.

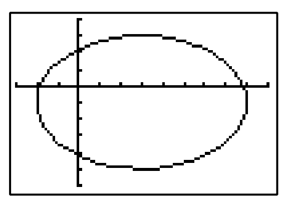
4.



$[-4, 4]$ by $[-3, 3]$



$[-6, 6]$ by $[-4, 4]$



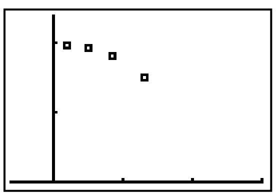
$[-3, 9]$ by $[-6, 4]$

Answers may vary. In general, students should find that the eccentricity is equal to the ratio of the distance between foci over distance between vertices.

5. Example 1: The equations $x = 3\cos t$, $y = 2\sin t$ can be rewritten as $\cos t = \frac{x}{3}$, $\sin t = \frac{y}{2}$, which using $\cos^2 t + \sin^2 t = 1$ yield $\frac{x^2}{9} + \frac{y^2}{4} = 1$ or $4x^2 + 9y^2 = 36$.
Example 2: The equations $x = 2\cos t$, $y = \sqrt{13}\sin t$ can be rewritten as $\cos t = \frac{x}{2}$, $\sin t = \frac{y}{\sqrt{13}}$, which using $\sin^2 t + \cos^2 t = 1$ yield $\frac{y^2}{13} + \frac{x^2}{4} = 1$.
Example 3: By rewriting $x = 3 + 5\cos t$, $y = -1 + 4\sin t$ as $\cos t = \frac{x-3}{5}$, $\sin t = \frac{y+1}{4}$ and using $\cos^2 t + \sin^2 t = 1$, we obtain
$$\frac{(x-3)^2}{25} + \frac{(y+1)^2}{16} = 1.$$

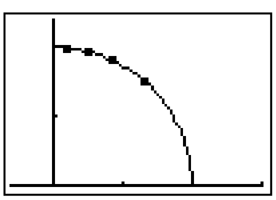
Exploration 2

Answers will vary due to experimental error. The theoretical answers are as follows.

2. $a = 9$ cm, $b = \sqrt{80} \approx 8.94$ cm, $c = 1$ cm, $e = 1/9 \approx 0.11$, $b/a \approx 0.99$.
3. $a = 8$ cm, $b = \sqrt{60} \approx 7.75$ cm, $c = 2$ cm, $e = 1/4 = 0.25$, $b/a \approx 0.97$;
   $a = 7$ cm, $b = \sqrt{40} \approx 6.32$ cm, $c = 3$ cm, $e = 3/7 \approx 0.43$, $b/a \approx 0.90$;
   $a = 6$ cm, $b = \sqrt{20} \approx 4.47$ cm, $c = 4$ cm, $e = 2/3 \approx 0.67$, $b/a \approx 0.75$.
4. The ratio b/a decreases slowly as $e = c/a$ increases rapidly. The ratio b/a is the height-to-width ratio, which measures the shape of the ellipse—when b/a is close to 1, the ellipse is nearly circular; when b/a is close to 0, the ellipse is elongated. The eccentricity ratio $e = c/a$ measures how off-center the foci are—when e is close to 0, the foci are near the center of the ellipse; when e is close to 1, the foci are far from the center and near the vertices of the ellipse. The foci must be extremely off-center for the ellipse to be significantly elongated.
5. 

[–0.3, 1.5] by [0, 1.2]

$$\frac{b}{a} = \frac{\sqrt{a^2 - c^2}}{a} = \sqrt{1 - \frac{c^2}{a^2}} = \sqrt{1 - e^2}$$



[–0.3, 1.5] by [0, 1.2]

Quick Review 8.2

1. $\sqrt{(2 - (-3))^2 + (4 - (-2))^2} = \sqrt{5^2 + 6^2} = \sqrt{61}$
2. $\sqrt{(a - (-3))^2 + (b - (-4))^2}$
   $= \sqrt{(a + 3)^2 + (b + 4)^2}$
3. $4y^2 + 9x^2 = 36$, $4y^2 = 36 - 9x^2$,
   $y = \pm\sqrt{\dfrac{36 - 9x^2}{4}} = \pm\dfrac{3}{2}\sqrt{4 - x^2}$
4. $25x^2 + 36y^2 = 900$, $36y^2 = 900 - 25x^2$,
   $y = \pm\sqrt{\dfrac{900 - 25x^2}{36}} = \pm\dfrac{5}{6}\sqrt{36 - x^2}$
5. $3x + 12 = (10 - \sqrt{3x - 8})^2$
   $3x + 12 = 100 - 20\sqrt{3x - 8} + 3x - 8$
   $-80 = -20\sqrt{3x - 8}$
   $4 = \sqrt{3x - 8}$
   $16 = 3x - 8$
   $3x = 24$
   $x = 8$
6. $6x + 12 = (1 + \sqrt{4x + 9})^2$
   $6x + 12 = (1 + 2\sqrt{4x + 9} + 4x + 9)$
   $2x + 2 = 2\sqrt{4x + 9}$
   $x + 1 = \sqrt{4x + 9}$
   $x^2 + 2x + 1 = 4x + 9$
   $x^2 - 2x - 8 = 0$
   $(x - 4)(x + 2) = 0$
   $x = 4$
7. $6x^2 + 12 = (11 - \sqrt{6x^2 + 1})^2$
   $6x^2 + 12 = 121 - 22\sqrt{6x^2 + 1} + 6x^2 + 1$
   $-110 = -22\sqrt{6x^2 + 1}$
   $6x^2 + 1 = 25$
   $6x^2 - 24 = 0$
   $x^2 - 4 = 0$
   $x = 2, x = -2$
8. $2x^2 + 8 = (8 - \sqrt{3x^2 + 4})^2$
   $2x^2 + 8 = 64 - 16\sqrt{3x^2 + 4} + 3x^2 + 4$
   $0 = x^2 - 16\sqrt{3x^2 + 4} + 60$
   $x^2 + 60 = (16\sqrt{3x^2 + 4})^2$
   $x^4 + 120x^2 + 3600 = 256\,(3x^2 + 4)$
   $x^4 - 648x^2 + 2576 = 0$
   $x = 2, x = -2$
9. $2\left(x - \dfrac{3}{2}\right)^2 - \dfrac{15}{2} = 0$, so $x = \dfrac{3 \pm \sqrt{15}}{2}$
10. $2(x + 1)^2 - 7 = 0$, so $x = -1 \pm \sqrt{\dfrac{7}{2}}$

Section 8.2 Exercises

1. $h = 0, k = 0, a = 4, b = \sqrt{7}$, so $c = \sqrt{16 - 7} = 3$
   Vertices: $(4, 0), (-4, 0)$; Foci: $(3, 0), (-3, 0)$
2. $h = 0, k = 0, a = 5, b = \sqrt{21}$, so $c = \sqrt{25 - 21} = 2$
   Vertices: $(0, 5), (0, -5)$; Foci: $(0, 2), (0, -2)$
3. $h = 0, k = 0, a = 6, b = 3\sqrt{3}$, so $c = \sqrt{36 - 27} = 3$
   Vertices: $(0, 6), (0, -6)$; Foci: $(0, 3), (0, -3)$
4. $h = 0, k = 0, a = \sqrt{11}, b = \sqrt{7}$, so $c = \sqrt{11 - 7} = 2$
   Vertices: $(\sqrt{11}, 0), (-\sqrt{11}, 0)$; Foci: $(2, 0), (-2, 0)$
5. $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$. $h = 0, k = 0, a = 2, b = \sqrt{3}$, so
   $c = \sqrt{4 - 3} = 1$
   Vertices: $(2, 0), (-2, 0)$; Foci: $(1, 0), (-1, 0)$
6. $\dfrac{y^2}{9} + \dfrac{x^2}{4} = 1$. $h = 0, k = 0, a = 3, b = 2$, so
   $c = \sqrt{9 - 4} = \sqrt{5}$.
   Vertices: $(0, 3), (0, -3)$; Foci: $(0, \sqrt{5}), (0, -\sqrt{5})$

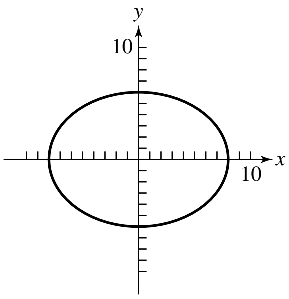
7. (d)

8. (c)

9. (a)

10. (b)

11.



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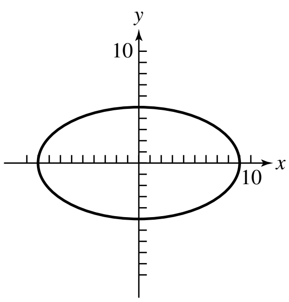

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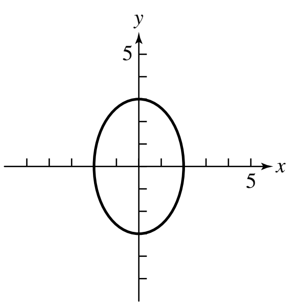

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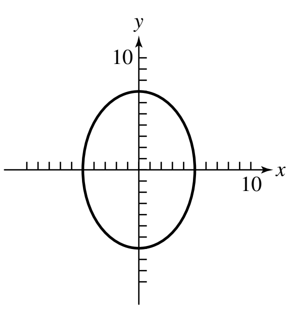

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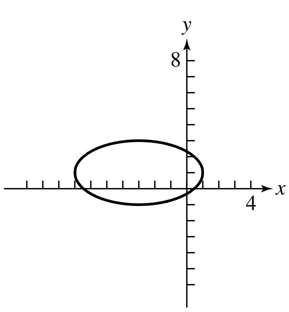

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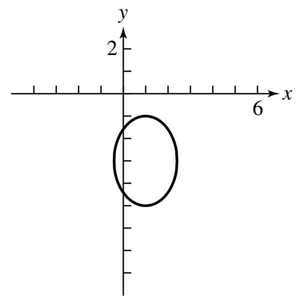

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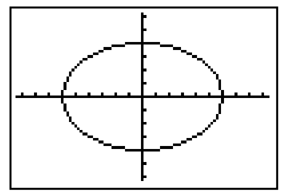
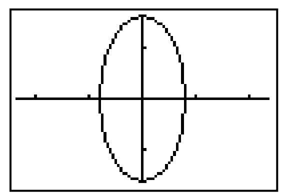


[–9.4, 9.4] by [–6.2, 6.2]

$y = \pm\frac{2}{3}\sqrt{-x^2 + 36}$

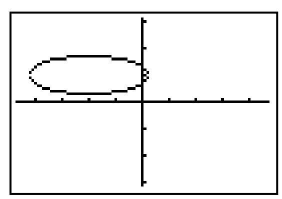
18.



[–11.75, 11.75] by [–8.1, 8.1]

$y = \pm 2\sqrt{-x^2 + 16}$

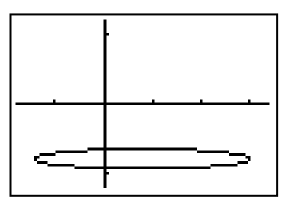
19.



[–4.7, 4.7] by [–3.1, 3.1]

$y = 1 \pm \sqrt{-\frac{(x + 2)^2}{10} + \frac{1}{2}}$

20.



[–9, 17] by [–6, 6]

$y = -4 \pm \frac{1}{16}\sqrt{-x^2 + 8x + 112}$

21. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

22. $\frac{x^2}{49} + \frac{y^2}{25} = 1$

23. $c = 2$ and $a = \frac{10}{2} = 5$, so $b = \sqrt{a^2 - c^2} = \sqrt{21}$:

$\frac{x^2}{25} + \frac{y^2}{21} = 1$

24. $c = 3$ and $b = \dfrac{10}{2} = 5$, so $a = \sqrt{b^2 - c^2} = \sqrt{16} = 4$:
$\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

25. $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

26. $\dfrac{x^2}{49} + \dfrac{y^2}{16} = 1$

27. $b = 4; \dfrac{x^2}{16} + \dfrac{y^2}{36} = 1$

28. $b = 2; \dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$

29. $a = 5; \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

30. $a = 13; \dfrac{x^2}{144} + \dfrac{y^2}{169} = 1$

31. The center (h, k) is $(1, 2)$ (the midpoint of the axes); a and b are half the lengths of the axes (4 and 6, respectively): $\dfrac{(x-1)^2}{16} + \dfrac{(y-2)^2}{36} = 1$

32. The center (h, k) is $(-2, 2)$ (the midpoint of the axes); a and b are half the lengths of the axes (2 and 5, respectively): $\dfrac{(x+2)^2}{4} + \dfrac{(y-2)^2}{25} = 1$

33. The center (h, k) is $(3, -4)$ (the midpoint of the major axis); $a = 3$, half the lengths of the major axis. Since $c = 2$ (half the distance between the foci), $b = \sqrt{a^2 - c^2} = \sqrt{5}$: $\dfrac{(x-3)^2}{9} + \dfrac{(y+4)^2}{5} = 1$

34. The center (h, k) is $(-2, 3)$ (the midpoint of the major axis); $b = 4$, half the lengths of the major axis. Since $c = 2$ (half the distance between the foci), $a = \sqrt{b^2 - c^2} = \sqrt{12}$: $\dfrac{(x+2)^2}{12} + \dfrac{(y-3)^2}{16} = 1$

35. The center (h, k) is $(3, -2)$ (the midpoint of the major axis); a and b are half the lengths of the axes (3 and 5, respectively):
$\dfrac{(x-3)^2}{9} + \dfrac{(y+2)^2}{25} = 1$

36. The center (h, k) is $(-1, 2)$ (the midpoint of the major axis); a and b are half the lengths of the axes (4 and 3, respectively): $\dfrac{(x+1)^2}{16} + \dfrac{(y-2)^2}{9} = 1$

For #37–40, an ellipse with equation $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ has center (h, k), vertices $(h \pm a, k)$, and foci $(h \pm c, k)$ where $c = \sqrt{a^2 - b^2}$.

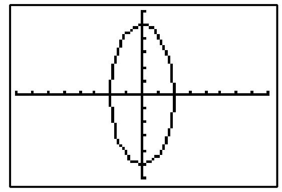
37. Center $(-1, 2)$; Vertices $(-1 \pm 5, 2) = (-6, 2), (4, 2)$; Foci $(-1 \pm 3, 2) = (-4, 2), (2, 2)$

38. Center $(3, 5)$; Vertices: $(3 \pm \sqrt{11}, 5) \approx (6.32, 5), (-0.32, 5)$; Foci $= (3 \pm 2, 5) = (5, 5), (1, 5)$

39. Center $(7, -3)$; Vertices: $(7, -3 \pm 9) = (7, 6), (7, -12)$; Foci: $(7, -3 \pm \sqrt{17}) \approx (7, 1.12), (7, -7.12)$

40. Center $(-2, 1)$; Vertices: $(-2, 1 \pm 5) = (-2, -4), (-2, 6)$; Foci: $(-2, 1 \pm 3) = (-2, -2), (-2, 4)$

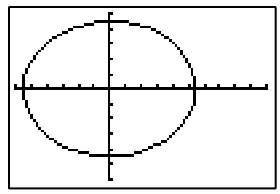
41.



$[-8, 8]$ by $[-6, 6]$

$x = 2\cos t$, $y = 5\sin t$

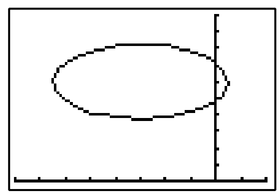
42.



$[-6, 10]$ by $[-6, 5]$

$x = \sqrt{30}\cos t$, $y = 2\sqrt{5}\sin t$

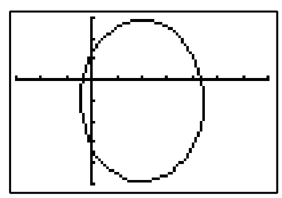
43.



$[-8, 2]$ by $[0, 10]$

$x = 2\sqrt{3}\cos t - 3$, $y = \sqrt{5}\sin t + 6$

44.



$[-3, 7]$ by $[-5, 3]$

$x = \sqrt{6}\cos(t) + 2$, $y = \sqrt{15}\sin(t) - 1$

For #45–48, complete the squares in x and y, then put in standard form. (The first one is done in detail; the others just show the final form.)

45. $9x^2 + 4y^2 - 18x + 8y - 23 = 0$ can be rewritten as $9(x^2 - 2x) + 4(y^2 + 2y) = 23$. This is equivalent to $9(x^2 - 2x + 1) + 4(y^2 + 2y + 1) = 23 + 9 + 4$, or $9(x-1)^2 + 4(y+1)^2 = 36$. Divide both sides by 36 to obtain $\dfrac{(x-1)^2}{4} + \dfrac{(y+1)^2}{9} = 1$. Vertices: $(1, -4)$ and $(1, 2)$ Foci: $(1, -1 \pm \sqrt{5})$. Eccentricity: $\dfrac{\sqrt{5}}{3}$.

46. $\dfrac{(x-2)^2}{5} + \dfrac{(y+3)^2}{3} = 1$. Vertices: $(2 \pm \sqrt{5}, -3)$.
Foci: $(2 \pm \sqrt{2}, -3)$. Eccentricity: $\dfrac{\sqrt{2}}{\sqrt{5}} = \sqrt{\dfrac{2}{5}}$

47. $\dfrac{(x+3)^2}{16} + \dfrac{(y-1)^2}{9} = 1$. Vertices: $(-7, 1)$ and $(1, 1)$.
Foci: $(-3 \pm \sqrt{7}, 1)$. Eccentricity: $\dfrac{\sqrt{7}}{4}$

48. $(x-4)^2 + \frac{(y+8)^2}{4} = 1$. Vertices: $(4, -10)$ and $(4, -6)$.
Foci: $(4, -8 \pm \sqrt{3})$. Eccentricity: $\frac{\sqrt{3}}{2}$

49. The center (h, k) is $(2, 3)$ (given); a and b are half the lengths of the axes (4 and 3, respectively):
$$\frac{(x-2)^2}{16} + \frac{(y-3)^2}{9} = 1$$

50. The center (h, k) is $(-4, 2)$ (given); a and b are half the lengths of the axes (4 and 3, respectively):
$$\frac{(x+4)^2}{16} + \frac{(y-2)^2}{9} = 1$$

51. Consider Figure 8.15(b); call the point $(0, c)$ F_1, and the point $(0, -c)$ F_2. By the definition of an ellipse, any point P (located at (x, y)) satisfies the equation
$\overrightarrow{PF} + \overrightarrow{PF_2} = 2a$ thus, $\sqrt{(x-0)^2 + (y-c)^2} + \sqrt{(x-0)^2 + (y+c)^2} = \sqrt{x^2 + (y-c)^2} + \sqrt{x^2 + (y+c)^2} = 2a$
then $\sqrt{x^2 + (y-c)^2} = 2a - \sqrt{x^2 + (y+c)^2}$
$$x^2 + (y-c)^2 = 4a^2 - 4a\sqrt{x^2 + (y+c)^2} + x^2 + (y+c)^2$$
$$y^2 - 2cy + c^2 = 4a^2 - 4a\sqrt{x^2 + (y+c)^2} + y^2 + 2cy + c^2$$
$$4a\sqrt{x^2 + (y+c)^2} = 4a^2 + 4cy$$
$$a\sqrt{x^2 + (y+c)^2} = a^2 + cy$$
$$a^2(x^2 + (y+c)^2) = a^4 + 2a^2cy + c^2y^2$$
$$a^2x^2 + (a^2 - c^2)y^2 = a^2(a^2 - c^2)$$
$$a^2x^2 + b^2y^2 = a^2b^2$$
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

52. Recall that $e = \frac{c}{a}$ means that $c = ea$, $b = \sqrt{a^2 - c^2}$ and a celestial object's perihelion occurs at $a - c$ for Pluto, $c = ea = (0.2484)(5900) \approx 1456.56$, so its perhelion is $5900 - 1456.56 = 4{,}434.44$ Gm. For Neptune, $c = ea = (0.0050)(4497) \approx 22.49$, so its perihelion is $4497 - 22.49 = 4{,}474.51$ Gm. As a result of its high by eccentric orbit, Pluto comes over 40 Gm closer to the Sun than Neptune.

53. Since the Moon is furthest from the Earth at 252,710 miles and closest at 221,463, we know that $2a = 252{,}710 + 221.463$, or $a = 237{,}086.5$. Since $c + 221{,}463 = a$, we know $c = 15{,}623.5$ and $b = \sqrt{a^2 - c^2} = \sqrt{(237{,}086.5)^2 - (15{,}623.5)^2} \approx 236{,}571$.
From these, we calculate $e = \frac{c}{a} = \frac{15{,}623.5}{237{,}086.5} \approx 0.066$.
The orbit of the Moon is very close to a circle, but still takes the shape of an ellipse.

54. For Mercury, $c = ea = (0.2056)(57.9) \approx 11.90$ Gm and its perihelion $a - c = 57.9 - 11.90 \approx 46$ Gm. Since the diameter of the sin is 1.392 Gm. Mercury gets within $46 - \frac{1.392}{2} \approx 45.3$ Gm of the Sun's surface.

55. For Saturn, $c = ea = (0.0560)(1{,}427) \approx 79.9$ Gm. Saturn's perihelion is $a - c = 1427 - 79.9 \approx 1347$ Gm and its aphelion is $a + c = 1427 + 72.21 \approx 1507$ Gm.

56. Venus: $c = ea = (0.0068)(108.2) \approx 0.74$, so $b = \sqrt{(108.2)^2 - (0.74)^2} \approx 108.2$.
$$\frac{x^2}{11{,}707.24} + \frac{y^2}{11{,}706.70} = 1$$
Mars: $c = ea = (0.0934)(227.9) \approx 21.29$, so $b = \sqrt{(227.91)^2 - (21.29)^2} \approx 226.91$
$$\frac{x^2}{51{,}938} + \frac{y^2}{51{,}485} = 1$$

57. For sungrazers, $a - c < 1.5(1.392) = 2.088$. The eccentricity of their ellipses is very close to 1.

58. $a = \frac{36.18}{2}, b = \frac{9.12}{2}, c = \sqrt{a^2 - b^2}$
$$= \sqrt{\left(\frac{36.18}{2}\right)^2 - \left(\frac{9.12}{2}\right)^2} \approx 17.51 \text{ Au}$$
thus, $e = \frac{17.51}{18.09} \approx 0.97$

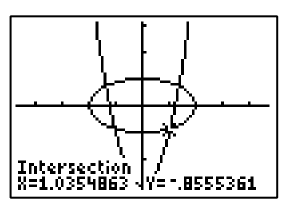
59. $a = 8$ and $b = 3.5$, so $c = \sqrt{a^2 - b^2} = \sqrt{51.75}$. Foci at $(\pm\sqrt{51.75}, 0) \approx (\pm 7.19, 0)$.

60. $a = 13$ and $b = 5$, so $c = \sqrt{a^2 - b^2} = 12$ Place the source and the patient at opposite foci — 12 inches from the center along the major axis.

61. Substitute $y^2 = 4 - x^2$ into the first equation:
$$\frac{x^2}{4} + \frac{4 - x^2}{9} = 1$$
$$9x^2 + 4(4 - x^2) = 36$$
$$5x^2 = 20$$
$$x^2 = 4$$
$$x = \pm 2, y = 0$$
Solution: $(-2, 0), (2, 0)$

62. Substitute $x = 3y - 3$ into the first equation:
$$\frac{(3y-3)^2}{9} + y^2 = 1$$
$$y^2 - 2y + 1 + y^2 = 1$$
$$2y2 - 2y = 0$$
$$2y(y - 1) = 0$$
$$y = 0 \text{ or } y = 1$$
$$x = -3 \quad x = 0$$
Solution: $(-3, 0), (0, 1)$

63. (a)



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

Approximate solutions: $(\pm 1.04, -0.86), (\pm 1.37, 0.73)$

(b) $\left(\frac{\pm\sqrt{94 - 2\sqrt{161}}}{8}, -\frac{1 + \sqrt{161}}{16}\right)$, $\left(\frac{\pm\sqrt{94 + 2\sqrt{161}}}{8}, \frac{-1 + \sqrt{161}}{16}\right)$

64. One possibility: a circle is perfectly "centric": it is an ellipse with both foci at the center. As the foci move off the center and toward the vertices, the ellipse becomes more eccentric as measured by the ratio $= e = c/a$. In everyday life, we say a person is eccentric if he or she deviates from the norm or central tendencies of behavior.

65. False. The distance is $a - c = a(1 - c/a) = a(1 - e)$.

66. True, because $a^2 = b^2 + c^2$ in any ellipse.

67. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$, so $c = \sqrt{a^2 - b^2} = \sqrt{2^2 - 1^2} = \sqrt{3}$. The answer is C.

68. The focal axis runs horizontally through $(2, 3)$. The answer is C.

69. Completing the square produces $\dfrac{(x-4)^2}{4} + \dfrac{(y-3)^2}{9} = 1$. The answer is B.

70. The two foci are a distance $2c$ apart, and the sum of the distances from each of the foci to a point on the ellipse is $2a$. The answer is C.

71. (a) When $a = b = r$, $A = \pi ab = \pi rr = \pi r^2$ and

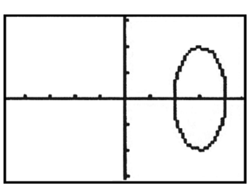
$$P \approx \pi(2r)\left(3 - \frac{\sqrt{(3r + r)(r + 3r)}}{r + r}\right)$$
$$= 2\pi r\left(3 - \frac{\sqrt{16r^2}}{2r}\right) = 2\pi r\left(3 - \frac{4r}{2r}\right)$$
$$= 2\pi r\,(3 - 2) = 2\pi r.$$

(b) One possibility: $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ with $A = 12\pi$ and $P \approx (21 - \sqrt{195})\pi \approx 22.10$, and $\dfrac{x^2}{100} + y^2 = 1$ with $A = 10\pi$ and $P \approx (33 - \sqrt{403})\pi \approx 40.61$.

72. (a) Answers will vary. See Chapter III: The Harmony of Worlds in *Cosmos* by Carl Sagan, Random House, 1980.

(b) Drawings will vary. Kepler's Second Law states that as a planet moves in its orbit around the sun, the line segment from the sun to the planet sweeps out equal areas in equal times.

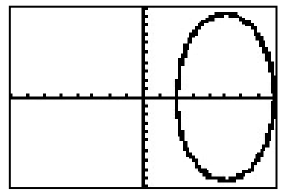
73. (a) Graphing in parametric mode with Tstep $= \dfrac{\pi}{24}$.



[–4.7, 4.7] by [–3.1, 3.1]

(b) The equations $x(t) = 3 + \cos(2t - 5)$ and $y(t) = -2\sin(2t - 5)$ can be rewritten as $\cos(2t - 5) = x - 3$ and $\sin(2t - 5) = -y/2$. Substituting these into the identity $\cos^2(2t - 5) + \sin^2(2t - 5) = 1$ yields the equation $y^2/4 + (x - 3)^2 = 1$. This is the equation of an ellipse with $x = 3$ as the focal axis. The center of the ellipse is $(3, 0)$ and the vertices are $(3, 2)$ and $(3, -2)$. The length of the major axis is 4 and the length of the minor axis is 2.

74. (a) The equations $x(t) = 5 + 3\sin\left(\pi t + \dfrac{\pi}{2}\right)$ and $y(t) = 3\pi\cos\left(\pi t + \dfrac{\pi}{2}\right)$ can be rewritten as $\sin\left(\pi t + \dfrac{\pi}{2}\right) = \dfrac{x - 5}{3}$ and $\cos\left(\pi t + \dfrac{\pi}{2}\right) = \dfrac{y}{3\pi}$. Substituting these into the identity $\cos^2\left(\pi t + \dfrac{\pi}{2}\right) + \sin^2\left(\pi t + \dfrac{\pi}{2}\right) = 1$ yields the equation $\dfrac{y^2}{9\pi^2} + \dfrac{(x-5)^2}{9} = 1$. This is the equation of an ellipse.



[–8, 8] by [–10, 10]

(b) The pendulum begins its swing at $t = 0$ so $x(0) = 5 + 3\sin\left(\dfrac{\pi}{2}\right) = 8$ ft, which is the maximum distance away from the detector. When $t = 1$, $x(1) = 5 + 3\sin\left(\pi + \dfrac{\pi}{2}\right) = 2$ ft, which is the minimum distance from the detector. When $t = 3$, the pendulum is back to the 8-ft position.

As indicated in the table, the maximum velocity (≈ 9.4 ft/sec) happens when the pendulum is at the halfway position of 5 ft from the detector.

T	X_{1T}	Y_{1T}
5.3	3.2366	7.6248
5.4	4.0729	8.9635
5.5	5	9.4248
5.6	5.9271	8.9635
5.7	6.7634	7.6248
5.8	7.4271	5.5397
5.9	7.8532	2.9124

T=5.5

75. Write the equation in standard form by completing the squares and then dividing by the constant on the right-hand side.

$$Ax^2 + Dx + \frac{D^2}{4A} + Cy^2 + Ey + \frac{E^2}{4C} = \frac{D^2}{4A} + \frac{E^2}{4C} - F$$

$$\frac{x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}}{C} + \frac{y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}}{A} = \frac{1}{AC}\left(\frac{D^2}{4A} + \frac{E^2}{4C} - F\right)$$

$$\frac{\left(x + \frac{D}{2A}\right)^2}{C} + \frac{\left(y + \frac{E}{2C}\right)^2}{A} = \frac{CD^2 + AE^2 - 4ACF}{4A^2C^2}$$

$$\left(\frac{4A^2C^2}{CD^2 + AE^2 - 4ACF}\right) \times \left[\frac{\left(x + \frac{D}{2A}\right)^2}{C} + \frac{\left(y + \frac{E}{2C}\right)^2}{A}\right] = 1$$

$$\frac{4A^2C\left(x+\frac{D}{2A}\right)^2}{CD^2+AE^2-4ACF}+\frac{4AC^2\left(y+\frac{E}{2C}\right)^2}{CD^2+AE^2-4ACF}=1$$

Since $AC > 0$, $A \neq 0$ and $C \neq 0$ (we are not dividing by zero). Further, $AC > 0 \Rightarrow 4A^2C > 0$ and $4AC^2 > 0$ (either $A > 0$ and $C > 0$, or $A < 0$ and $C < 0$), so the equation represents an ellipse.

76. Rewrite the equation to $\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2=0$

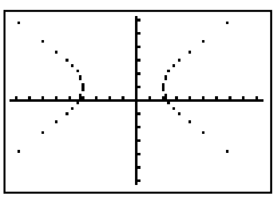
Since that $a \neq 0$ and $b \neq 0$ (otherwise the equation is not defined) we see that the only values of x, y that satisfy the equation are $(x, y) = (h, k)$. In this case, the degenerate ellipse is simply a single point (h, k). The semimajor and semiminor axes both equal 0. See Figure 8.2.

■ Section 8.3 Hyperbolas

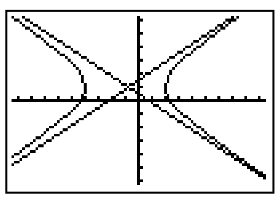
Exploration 1

1. The equations $x = -1 + 3/\cos t = -1 + 3\sec t$ and $y = 1 + 2\tan t$ can be rewritten as $\sec t = \frac{x+1}{3}$ and $\tan t = \frac{y-1}{2}$. Substituting these into the identity $\sec^2 t - \tan^2 t = 1$ yields the equation $\frac{(x+1)^2}{9} - \frac{(y-1)^2}{4} = 1$.

2.



[–9.4, 9.4] by [–6.2, 6.2]

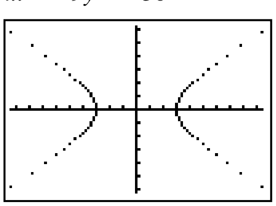


[–9.4, 9.4] by [–6.2, 6.2]

In Connected graphing mode, pseudo-asymptotes appear because the grapher connects computed points by line segments regardless of whether this makes sense. Using Dot mode with a small Tstep will produce the best graphs.

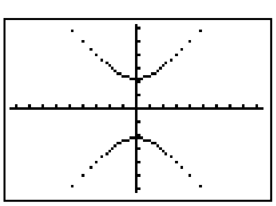
3. Example 1: $x = 3/\cos(t)$, $y = 2\tan(t)$
Example 2: $x = 2\tan(t)$, $y = \sqrt{5}/\cos(t)$
Example 3: $x = 3 + 5/\cos(t)$, $y = -1 + 4\tan(t)$
Example 4: $x = -2 + 3/\cos(t)$, $y = 5 + 7\tan(t)$

4. $4x^2 - 9y^2 = 36$



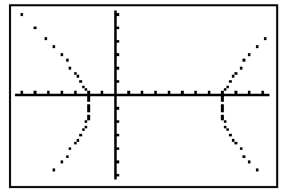
[–9.4, 9.4] by [–6.2, 6.2]

$\frac{y^2}{5} - \frac{x^2}{4} = 1$



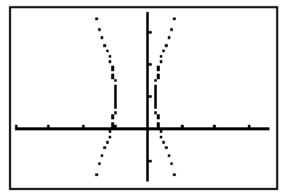
[–9.4, 9.4] by [–6.2, 6.2]

$\frac{(x-3)^2}{25} - \frac{(y+1)^2}{16} = 1$



[–7.4, 11.4] by [–6.2, 6.2]

$\frac{(x+2)^2}{9} - \frac{(y-5)^2}{49} = 1$



[–20, 18] by [–8, 18]

5. Example 1: The equations $x = 3/\cos t = 3\sec t$, $y = 2\tan t$ can be rewritten as $\sec t = \frac{x}{3}$, $\tan t = \frac{y}{2}$, which using the identity $\sec^2 t - \tan^2 t = 1$ yield $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

Example 2: The equations $x = 2\tan t$, $y = \sqrt{5}/\cos t = \sqrt{5}\sec t$ can be rewritten as $\tan t = \frac{x}{2}$, $\sec t = \frac{y}{\sqrt{5}}$, which using $\sec^2 t - \tan^2 t = 1$ yield $\frac{y^2}{5} - \frac{x^2}{4} = 1$.

Example 3: By rewriting $x = 3 + 5/\cos t$, $y = -1 + 4\tan t$ as $\sec t = \frac{x-3}{5}$, $\tan t = \frac{y+1}{4}$ and using $\sec^2 t - \tan^2 t = 1$, we obtain $\frac{(x-3)^2}{25} - \frac{(y+1)^2}{16} = 1$.

Example 4: By rewriting $x = -2 + 3/\cos t$, $y = 5 + 7\tan t$ as $\sec t = \frac{x+2}{3}$, $\tan t = \frac{y-5}{7}$ and using $\sec^2 t - \tan^2 t = 1$, we obtain $\frac{(x+2)^2}{9} - \frac{(y-5)^2}{49} = 1$.

Quick Review 8.3

1. $\sqrt{(-7-4)^2 + (-8-(-3))^2}$
$= \sqrt{(-11)^2 + (-5)^2} = \sqrt{146}$

2. $\sqrt{(b-a)^2 + (c-(-3))^2}$
$= \sqrt{(b-a)^2 + (c+3)^2}$

3.
$$9y^2 - 16x^2 = 144$$
$$9y^2 = 144 + 16x^2$$
$$y = \pm\frac{4}{3}\sqrt{9 + x^2}$$

4.
$$4x^2 - 36y^2 = 144$$
$$36y^2 = 4x^2 - 144$$
$$y = \pm\frac{2}{6}\sqrt{x^2 - 36}$$
$$y = \pm\frac{1}{3}\sqrt{x^2 - 36}$$

5. $\sqrt{3x + 12} = 10 + \sqrt{3x - 8}$
$3x + 12 = 100 + 20\sqrt{3x - 8} + 3x - 8$
$-80 = 20\sqrt{3x - 8}$
$-4 = \sqrt{3x - 8}$ no solution

6. $\sqrt{4x + 12} = 1 + \sqrt{x + 8}$
$4x + 12 = 1 + 2\sqrt{x + 8} + x + 8$
$3x + 3 = 2\sqrt{x + 8}$
$9x^2 + 18x + 9 = 4x + 32$
$9x^2 + 14x - 23 = 0$
$$x = \frac{-14 + \sqrt{196 - 4(9)(-23)}}{18}$$
$$x = \frac{-14 \pm 32}{18}$$
$x = 1$ or $x = -\frac{23}{9}$. When $x = -\frac{23}{9}$,
$\sqrt{4x + 12} - \sqrt{x + 8}$
$= \sqrt{\frac{16}{9}} - \sqrt{\frac{49}{9}} = \frac{4}{3} - \frac{7}{3} = -1$
The only solution is $x = 1$.

7. $\sqrt{6x^2 + 12} = 1 + \sqrt{6x^2 + 1}$
$6x^2 + 12 = 1 + 2\sqrt{6x^2 + 1} + 6x^2 + 1$
$10 = 2\sqrt{6x^2 + 1}$
$25 = 6x^2 + 1$
$6x^2 - 24 = 0$
$x^2 - 4 = 0$
$x = 2, x = -2$

8. $\sqrt{2x^2 + 12} = -8 + \sqrt{3x^2 + 4}$
$2x^2 + 12 = 64 - 16\sqrt{3x^2 + 4} + 3x^2 + 4$
$x^2 + 56 = 16\sqrt{3x^2 + 4}$
$x^4 + 112x^2 + 3136 = 768x^2 + 1024$
$x^4 - 656x^2 + 2112 = 0$
$x = \{25.55, -25.55\}$ (the other solutions are extraneous)

9. $c = a + 2, (a + 2)^2 - a^2 = \frac{16a}{3}$,
$a^2 + 4a + 4 - a^2 = \frac{16a}{3}, 4a = 12: a = 3, c = 5$

10. $c = a + 1, (a + 1)^2 - a^2 = \frac{25a}{12}$,
$a^2 + 2a + 1 - a^2 = \frac{25a}{12}: a = 12, c = 13$

Section 8.3 Exercises

For #1–6, recall the Pythagorean relation that $c^2 = a^2 + b^2$.

1. $a = 4, b = \sqrt{7}, c = \sqrt{16 + 7} = \sqrt{23}$;
Vertices: $(\pm 4, 0)$; Foci: $(\pm\sqrt{23}, 0)$

2. $a = 5, b = \sqrt{21}, c = \sqrt{25 + 21} = \sqrt{46}$;
Vertices: $(0, \pm 5)$; Foci: $(0, \pm\sqrt{46})$

3. $a = 6, b = \sqrt{13}, c = \sqrt{36 + 13} = 7$;
Vertices: $(0, \pm 6)$; Foci: $(0, \pm 7)$

4. $a = 3, b = 4, c = \sqrt{9 + 16} = 5$;
Vertices: $(\pm 3, 0)$; Foci: $(\pm 5, 0)$

5. $\frac{x^2}{4} - \frac{y^2}{3} = 1; a = 2, b = \sqrt{3}, c = \sqrt{7}$;
Vertices: $(\pm 2, 0)$; Foci: $(\pm\sqrt{7}, 0)$

6. $\frac{x^2}{4} - \frac{y^2}{9} = 1; a = 2, b = 3, c = \sqrt{13}$;
Vertices: $(\pm 2, 0)$; Foci: $(\pm\sqrt{13}, 0)$

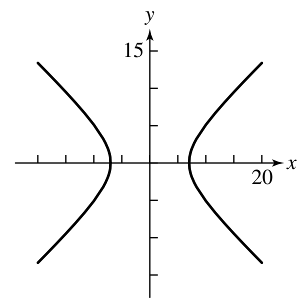
7. (c)

8. (b)

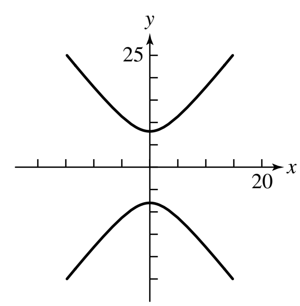
9. (a)

10. (d)

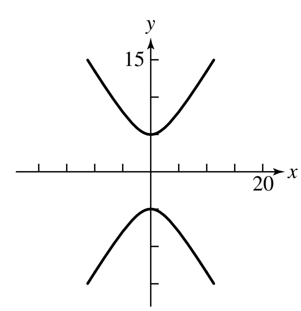
11. Transverse axis from $(-7, 0)$ to $(7, 0)$; asymptotes:
$y = \pm\frac{5}{7}x$,
$y = \pm\frac{5}{7}\sqrt{x^2 - 49}$



12. Transverse axis from $(0, -8)$ to $(0, 8)$; asymptotes:
$y = \pm\frac{8}{5}x$,
$y = \pm\frac{8}{5}\sqrt{x^2 + 25}$



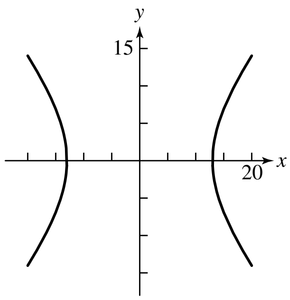
13. Transverse axis from $(0, -5)$ to $(0, 5)$; asymptotes:
$y = \pm\frac{5}{4}x$,
$y = \pm\frac{5}{4}\sqrt{x^2 + 16}$



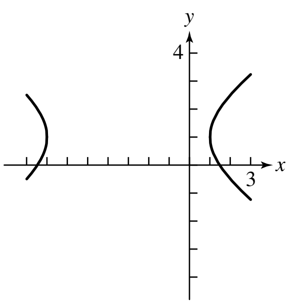
14. Transverse axis from $(-13, 0)$ to $(13, 0)$; asymptotes:

$y = \pm\frac{12}{13}x$,

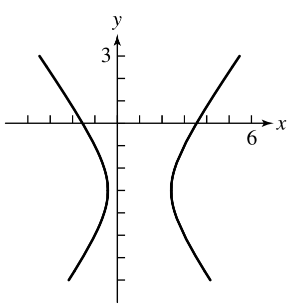
$y = \pm\frac{12}{13}\sqrt{x^2 - 169}$



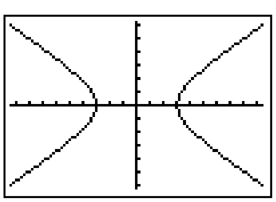
15. The center (h, k) is $(-3, 1)$. Since $a^2 = 16$ and $b^2 = 4$, we have $a = 4$ and $b = 2$. The vertices are at $(-3 \pm 4, 1)$ or $(-7, 1)$ and $(1, 1)$.



16. The center (h, k) is $(1, -3)$. Since $a^2 = 2$ and $b^2 = 4$, we have $a = \sqrt{2}$ and $b = 2$. The vertices are at $(1 \pm \sqrt{2}, -3)$.



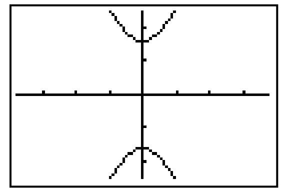
17.



[–18.8, 18.8] by [–12.4, 12.4]

$y = \pm\frac{2}{3}\sqrt{x^2 - 36}$

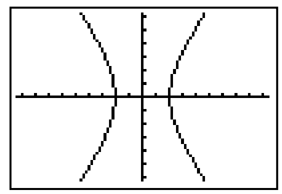
18.



[–18.8, 18.8] by [–12.4, 12.4]

$y = \pm 2\sqrt{x^2 + 16}$

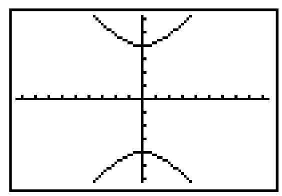
19.



[–9.4, 9.4] by [–6.2, 6.2]

$y = \pm\frac{3}{2}\sqrt{x^2 - 4}$

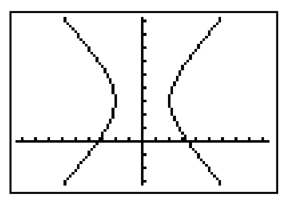
20.



[–9.4, 9.4] by [–6.2, 6.2]

$y = \pm\frac{4}{3}\sqrt{x^2 + 9}$

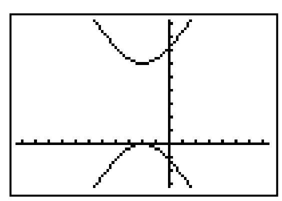
21.



[–9.4, 9.4] by [–3.2, 9.2]

$y = 3 \pm \frac{1}{2}\sqrt{5x^2 - 20}$

22.



[–11.4, 7.4] by [–3.2, 9.2]

$y = 3 \pm \frac{3}{2}\sqrt{x^2 + 4x + 8}$

23. $c = 3$ and $a = 2$, so $b = \sqrt{c^2 - a^2} = \sqrt{5}$: $\frac{x^2}{4} - \frac{y^2}{5} = 1$

24. $c = 3$ and $b = 2$, so $a = \sqrt{c^2 - b^2} = \sqrt{5}$: $\frac{y^2}{4} - \frac{x^2}{5} = 1$

25. $c = 15$ and $b = 4$, so $a = \sqrt{c^2 - b^2} = \sqrt{209}$:

$\frac{y^2}{16} - \frac{x^2}{209} = 1$

26. $c = 5$ and $a = 3/2$, so $b = \sqrt{c^2 - a^2} = \frac{1}{2}\sqrt{91}$:
$\frac{x^2}{2.25} - \frac{y^2}{22.75} = 1$ or $\frac{x^2}{9/4} - \frac{y^2}{91/4} = 1$

27. $a = 5$ and $c = ea = 10$, so $b = \sqrt{100 - 25} = 5\sqrt{5}$:
$\frac{x^2}{25} - \frac{y^2}{75} = 1$

28. $a = 4$ and $c = ea = 6$, so $b = \sqrt{36 - 16} = 2\sqrt{5}$:

$\frac{y^2}{16} - \frac{x^2}{20} = 1$

29. $b = 5, a = \sqrt{c^2 - b^2} = \sqrt{169 - 25} = 12$:
$\frac{y^2}{144} - \frac{x^2}{25} = 1$

30. $c = 6, a = \frac{c}{e} = 3, b = \sqrt{c^2 - a^2} = \sqrt{36 - 9} = 3\sqrt{3}$:
$\frac{x^2}{9} - \frac{y^2}{27} = 1$

31. The center (h, k) is $(2, 1)$ (the midpoint of the transverse axis endpoints); $a = 2$, half the length of the transverse axis. And $b = 3$, half the length of the conjugate axis.
$\frac{(y - 1)^2}{4} - \frac{(x - 2)^2}{9} = 1$

32. The center (h, k) is $(-1, 3)$ (the midpoint of the transverse axis endpoints); $a = 6$, half the length of the transverse axis. And $b = 5$, half the length of the conjugate axis.
$\frac{(x + 1)^2}{36} - \frac{(y - 3)^2}{25} = 1$

33. The center (h, k) is $(2, 3)$ (the midpoint of the transverse axis); $a = 3$, half the length of the transverse axis.
Since $|b/a| = \frac{4}{3}, b = 4$: $\frac{(x - 2)^2}{9} - \frac{(y - 3)^2}{16} = 1$

34. The center (h, k) is $\left(-2, \frac{5}{2}\right)$, the midpoint of the transverse axis); $a = \frac{9}{2}$, half the length of the transverse axis. Since $|a/b| = \frac{4}{3}, b = \frac{27}{8}$:
$\frac{(y - 5/2)^2}{81/4} - \frac{(x + 2)^2}{729/64} = 1$

35. The center (h, k) is $(-1, 2)$, the midpoint of the transverse axis. $a = 2$, half the length of the transverse axis. The center-to-focus distance is $c = 3$, so $b = \sqrt{c^2 - a^2}$ $= \sqrt{5}$: $\frac{(x + 1)^2}{4} - \frac{(y - 2)^2}{5} = 1$

36. The center (h, k) is $\left(-3, -\frac{11}{2}\right)$, the midpoint of the transverse axis. $b = \frac{7}{2}$, half the length of the transverse axis. The center-to-focus distance is $c = \frac{11}{2}$, so
$a = \sqrt{c^2 - b^2} = \sqrt{18}$: $\frac{(y + 5.5)^2}{49/4} - \frac{(x + 3)^2}{18} = 1$

37. The center (h, k) is $(-3, 6)$, the midpoint of the transverse axis. $a = 5$, half the length of the transverse axis. The center-to-focus distance $c = ea$
$= 2 \cdot 5 = 10$, so $b = \sqrt{c^2 - a^2} = \sqrt{100 - 25} = 5\sqrt{5}$
$\frac{(y - 6)^2}{25} - \frac{(x + 3)^2}{75} = 1$

38. The center (h, k) is $(1, -4)$, the midpoint of the transverse axis. $c = 6$, the center-to-focus distance
$a = \frac{c}{e} = \frac{6}{2} = 3, b = \sqrt{c^2 - a^2} = \sqrt{36 - 9} = \sqrt{27}$
$\frac{(x - 1)^2}{9} - \frac{(y + 4)^2}{27} = 1$

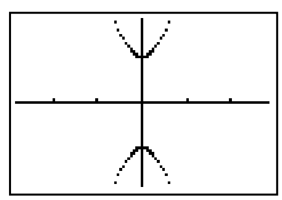
For #39–42, a hyperbola with equation $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$ has center (h, k) vertices $(h \pm a, k)$, and foci $(h \pm c, k)$ where $c = \sqrt{a^2 + b^2}$.
A hyperbola with equation $\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$ has center (h, k), vertices $(h, k \pm a)$, and foci $(h, k \pm c)$ where again $c = \sqrt{a^2 + b^2}$.

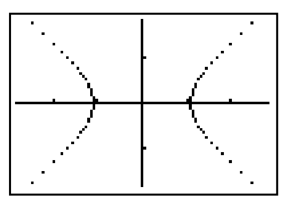
39. Center $(-1, 2)$; Vertices: $(-1 \pm 12, 2) = (11, 2), (-13, 2)$; Foci: $(-1 \pm 13, 2) = (12, 2), (-14, 2)$

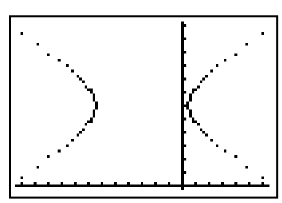
40. Center $(-4, -6)$; Vertices: $(-4 \pm \sqrt{12}, -6)$; Foci: $(-4 \pm 5, -6) = (1, -6), (-9, -6)$

41. Center $(2, -3)$; Vertices: $(2, -3 \pm 8) = (2, 5), (2, -11)$; Foci: $(2, -3 \pm \sqrt{145})$

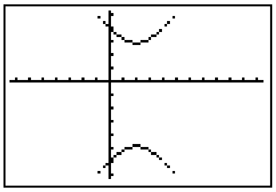
42. Center $(-5, 1)$; Vertices: $(-5, 1 \pm 5) = (-5, -4), (-5, 6)$; Foci: $(-5, 1 \pm 6) = (-5, -5), (-5, 7)$

43.

[–14.1, 14.1] by [–9.3, 9.3]
$y = 5/\cos t, x = 2 \tan t$

44.

[–14.1, 14.1] by [–9.3, 9.3]
$x = \sqrt{30}/\cos t, y = 2\sqrt{5} \tan t$

45.

[–12.4, 6.4] by [–0.2, 12.2]
$x = -3 + 2\sqrt{3}/\cos t, y = 6 + \sqrt{5} \tan t$

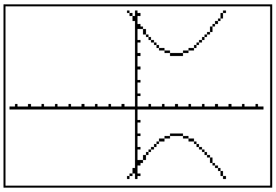
46.



[–7.4, 11.4] by [–7.2, 5.2]

$y = -1 + \sqrt{15}/\cos t,\ x = 2 + \sqrt{6}\tan t$

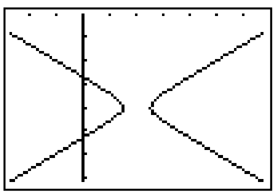
47.



[–9.4, 9.4] by [–5.2, 7.2]

Divide the entire equation by 36. Vertices: $(3, -2)$ and $(3, 4)$, Foci: $(3, 1 \pm \sqrt{13})$, $e = \dfrac{\sqrt{13}}{3}$.

48.



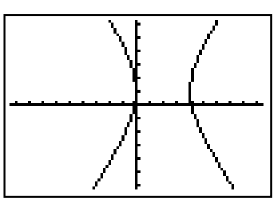
[–2.8, 6.8] by [–7.1, 0]

Vertices: $\left(\dfrac{3}{2}, -4\right)$ and $\left(\dfrac{5}{2}, -4\right)$, Foci: $\left(2 \pm \dfrac{\sqrt{13}}{6}, -4\right)$

$e = \dfrac{\sqrt{(1/4) + (1/9)}}{1/2} = 2\sqrt{\dfrac{9+4}{36}} = \dfrac{\sqrt{13}}{3}$.

For #49–50, complete the squares in x and y, then write the equation in standard form. (The first one is done in detail; the other shows just the final form.) As in the previous problems, the values of h, k, a, and b can be "read" from the equation $\pm\dfrac{(x-h)^2}{a^2} \mp \dfrac{(y-k)^2}{b^2} = 1$. The asymptotes are $y - k = \pm\dfrac{b}{a}(x - h)$. If the x term is positive, the transverse axis endpoints are $(h \pm a, k)$; otherwise the endpoints are $(h, k \pm b)$.

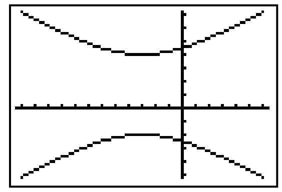
49.



[–9.4, 9.4] by [–6.2, 6.2]

$9x^2 - 4y^2 - 36x + 8y - 4 = 0$ can be rewritten as $9(x^2 - 4x) - 4(y^2 - 2y) = 4$. This is equivalent to $9(x^2 - 4x + 4) - 4(y^2 - 2y + 1) = 4 + 36 - 4$, or $9(x-2)^2 - 4(y-1)^2 = 36$. Divide both sides by 36 to obtain $\dfrac{(x-2)^2}{4} - \dfrac{(y-1)^2}{9} = 1$. Vertices: $(0, 1)$ and $(4, 1)$. Foci: $(2 \pm \sqrt{13}, 1)$, $e = \dfrac{\sqrt{13}}{2}$

50.



[–12.4, 6.4] by [–5.2, 7.2]

$\dfrac{(y-1)^2}{9} - \dfrac{(x+3)^2}{25} = 1$. Vertices: $(-3, -2)$ and $(-3, 4)$. Foci: $(-3, 1 \pm \sqrt{34})$, $e = \dfrac{\sqrt{34}}{3}$

51. $a = 2$, $(h, k) = (0, 0)$ and the hyperbola opens to the left and right, so $\dfrac{x^2}{4} - \dfrac{y^2}{b^2} = 1$. Using $(3, 2)$: $\dfrac{9}{4} - \dfrac{4}{b^2} = 1$, $9b^2 - 16 = 4b^2$, $5b^2 = 16$, $b^2 = \dfrac{16}{5}$; $\dfrac{x^2}{4} - \dfrac{5y^2}{16} = 1$

52. $a = \sqrt{2}$, $(h, k) = (0, 0)$ and the hyperbola opens upward and downward, so $\dfrac{y^2}{2} - \dfrac{x^2}{b^2} = 1$. Using $(2, -2)$: $\dfrac{4}{2} - \dfrac{4}{b^2} = 1$, $\dfrac{4}{b^2} = 1$, $b^2 = 4$; $\dfrac{y^2}{2} - \dfrac{x^2}{4} = 1$

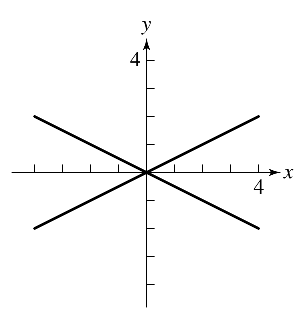
53. Consider Figure 8.24(b). Label $(0, c)$ as point F_1, label $(0, -c)$ as point F_2 and consider any point $P(x, y)$ along the hyperbola. By definition, $PF_1 - PF_2 = \pm 2a$, with $c > a \ge 0$

$$\begin{aligned}
\sqrt{(x-0)^2 + (y-(-c))^2} - \sqrt{(x-0)^2 + (y-c)^2} &= \pm 2a \\
\sqrt{x^2 + (y+c)^2} &= \pm 2a + \sqrt{x^2 + (y-c)^2} \\
x^2 + y^2 + 2cy + c^2 &= 4a^2 \pm 4a\sqrt{x^2 + (y-c)^2} \\
&\quad + x^2 + y^2 - 2cy + c^2 \\
\pm a\sqrt{x^2 + (y-c)^2} &= a^2 - cy \\
a^2(x^2 + y^2 - 2cy + c^2) &= a^4 - 2a^2cy + c^2y^2 \\
-a^2x^2 + (c^2 - a^2)y^2 &= a^2(c^2 - a^2) \\
b^2y^2 - a^2x^2 &= a^2b^2 \\
\frac{y^2}{a^2} - \frac{x^2}{b^2} &= 1
\end{aligned}$$

54. (a) $\dfrac{x^2}{4} - y^2 = 0$

$$y^2 = \frac{x^2}{4}$$

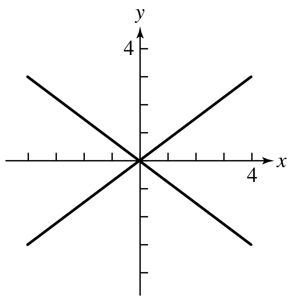
$$y = \pm\frac{x}{2}$$



(b) $\frac{y^2}{9} - \frac{x^2}{16} = 0$

$$y^2 = \frac{9x^2}{16}$$

$$y = \pm\frac{3x}{4}$$



55.
$$\begin{aligned} c - a &= 120,\ b^2 = 250a \\ c^2 - a^2 &= b^2 \\ (a + 120)^2 - a^2 &= 250a \\ a^2 + 240a + 14{,}400 - a^2 &= 250a \\ 10a &= 14{,}400 \\ a &= 1440 \text{ Gm} \end{aligned}$$

$a = 1440$ Gm, $b = 600$ Gm, $c = 1560$, $e = \frac{1560}{1440} = \frac{13}{12}$.

The Sun is centered at focus $(c, 0) = (1560, 0)$.

56.
$$\begin{aligned} c - a &= 140,\ b^2 = 405a \\ c^2 - a^2 &= b^2 \\ (a + 140)^2 - a^2 &= 405a \\ a^2 + 280a + 19{,}600 - a^2 &= 405a \\ 125a &= 19{,}600 \\ a &= 156.8 \end{aligned}$$

$a = 156.8$ Gm, $b = 252$ Gm, $c = 296.8$ Gm, $e = \frac{53}{28}$.

The Sun is centered at focus $(c, 0) = (297, 0)$.

57. The *Princess Ann* is located at the intersection of two hyperbolas: one with foci O and R, and the other with foci O and Q. For the first of these, the center is $(0, 40)$, so the center-to-focus distance is $c = 40$ mi. The transverse axis length is $2b = (323.27\ \mu\text{sec})(980\ \text{ft}/\mu\text{sec}) = 316{,}804.6\ \text{ft} \approx 60$ mi. Then $a \approx \sqrt{40^2 - 30^2} = \sqrt{700}$ mi. For the other hyperbola, $c = 100$ mi, $2a = (646.53\ \mu\text{sec})(980\ \text{ft}/\mu\text{sec}) = 633599.4\ \text{ft} \approx 120$ mi, and $b \approx \sqrt{100^2 - 60^2} = 80$ mi. The two equations are therefore

$$\frac{(y - 40)^2}{900} - \frac{x^2}{700} = 1 \text{ and } \frac{(x - 100)^2}{3600} - \frac{y^2}{6400} = 1.$$

The intersection of the upper branch of the first hyperbola and the right branch of the second hyperbola (found graphically) is approximately (886.67, 1045.83). The ship is located about 887 miles east and 1046 miles north of point O – a bearing and distance of about 40.29° and 1371.11 miles, respectively.

58. The gun is located at the intersection of two hyperbolas: one with foci A and B, and the other with foci B and C. For the first of these, the center is $(0, 2000)$, so the center-to-focus distance is $c = 2000$ mi. The transverse axis length is $2b = (2\ \text{sec})(1100\ \text{ft/sec}) = 2200$ ft. Then $a \approx \sqrt{2000^2 - 1100^2} = 100\sqrt{279}$ ft. For the other hyperbola, $c = 3500$ ft, $2a = (4\ \text{sec})(1100\ \text{ft/sec}) = 4400$ ft, and $b \approx \sqrt{3500^2 - 2200^2} = 100\sqrt{741}$ ft. The two equations are therefore

$$\frac{(y - 2000)^2}{1100^2} - \frac{x^2}{2{,}790{,}000} = 1 \text{ and}$$

$$\frac{(x - 3500)^2}{2200^2} - \frac{y^2}{7{,}410{,}000} = 1.$$

The intersection of the upper branch of the first hyperbola and the right branch of the second hyperbola (found graphically) is approximately (11,714.3, 9792.5). The gun is located about 11,714 ft (2.22 mi) east and 9793 ft (1.85 mi) north of point B – a bearing and distance of about 50.11° and 15,628.2 ft (2.89 mi), respectively.

59.
$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

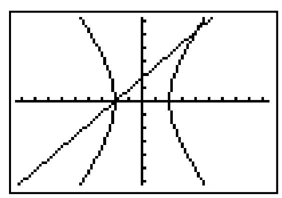
$$x - \frac{2\sqrt{3}}{3}y = -2$$

Solve the second equation for x and substitute into the first equation.

$$x = \frac{2\sqrt{3}}{3}y - 2$$

$$\begin{aligned} \frac{1}{4}\left(\frac{2\sqrt{3}}{3}y - 2\right)^2 - \frac{y^2}{9} &= 1 \\ \frac{1}{4}\left(\frac{4}{3}y^2 - \frac{8\sqrt{3}}{3}y + 4\right) - \frac{y^2}{9} &= 1 \\ \frac{2}{9}y^2 - \frac{2\sqrt{3}}{3}y &= 0 \\ \frac{2}{9}y(y - 3\sqrt{3}) &= 0 \end{aligned}$$

$y = 0$ or $y = 3\sqrt{3}$

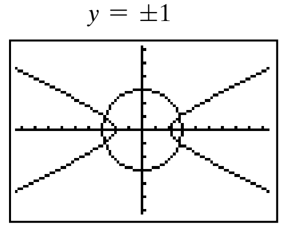


[–9.4, 9.4] by [–6.2, 6.2]

Solutions: $(-2, 0)$, $(4, 3\sqrt{3})$

60. Add:

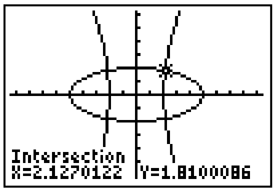
$$\begin{aligned} \frac{x^2}{4} - y^2 &= 1 \\ x^2 + y^2 &= 9 \\ \frac{5x^2}{4} &= 10 \\ x^2 &= 8 \\ x &= \pm 2\sqrt{2} \\ x^2 + y^2 &= 9 \\ 8 + y^2 &= 9 \\ y &= \pm 1 \end{aligned}$$



[–9.4, 9.4] by [–6.2, 6.2]

There are four solutions: $(\pm 2\sqrt{2}, \pm 1)$

61. (a)



[–9.4, 9.4] by [–6.2, 6.2]

There are four solutions: $(\pm 2.13, \pm 1.81)$

(b) The exact solutions are $\left(\pm 10\sqrt{\frac{29}{641}}, \pm 10\sqrt{\frac{21}{641}}\right)$.

62. One possibility: Escape speed is the minimum speed one object needs to achieve in order to break away from the gravity of another object. For example, for a NASA space probe to break away from the Earth's gravity is must meet or exceed the escape speed for Earth $v_E = \sqrt{2GM/r} \approx 11{,}200$ m/s. If this escape speed is exceeded, the probe will follow a hyperbolic path.

63. True. The distance is $c - a = a(c/a - 1) = a(e - 1)$.

64. True. For an ellipse, $b^2 + c^2 = a^2$.

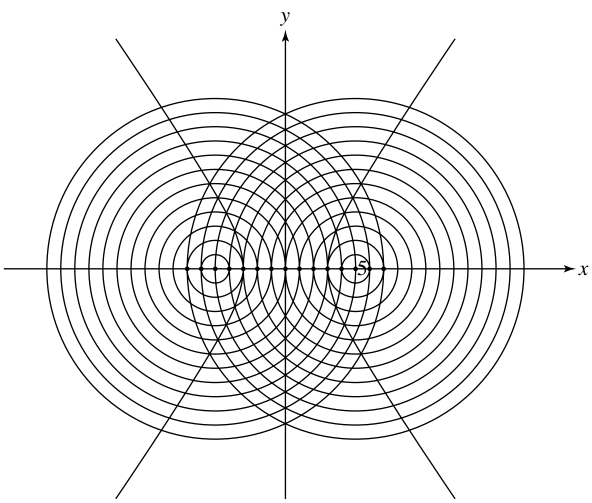
65. $\frac{x^2}{4} - \frac{y^2}{1} = 1$, so $c = \sqrt{4 + 1}$ and the foci are each $\sqrt{5}$ units away horizontally from $(0, 0)$. The answer is B.

66. The focal axis passes horizontally through the center, $(-5, 6)$. The answer is E.

67. Completing the square twice, and dividing to obtain 1 on the right, turns the equation into $\frac{(y + 3)^2}{4} - \frac{(x - 2)^2}{12} = 1$. The answer is B.

68. $a = 2, b = \sqrt{3}$, and the slopes are $\pm b/a$. The answer is C.

69. (a–d)



(e) $a = 3, c = 5, b = 4$;
$x^2/9 - y^2/16 = 1$

70. Assume that the focus for the primary parabolic mirror occurs at F_P and the foci for the hyperbolic mirror occur at F_H and F_H. Assume also that the x-axis extends from the eye piece to the right most F_H, and that the y-axis is perpendicular through the x-axis 60 cm from the eye piece. Then, the center (h, k) of the hyperbolic mirror is $(0, 0)$, the foci $(\pm c, 0) = (\pm 60, 0)$ and the vertices $(\pm a, 0) = (\pm 40, 0)$.
Since $a = 40$, $c = 60$, $b^2 = c^2 - a^2 = 2000$. The equation for the hyperbolic mirror is $\frac{x^2}{1600} - \frac{y^2}{2000} = 1$.

71. From Section 8.2, Question #75, we have $Ax^2 + Cy^2 + Dx + Ey + F = 0$ becomes

$$\frac{4A^2C\left(x + \frac{D}{2A}\right)^2}{CD^2 + AE^2 - 4ACF} + \frac{4AC^2\left(y + \frac{F}{2C}\right)^2}{CD^2 + AE^2 - 4ACF} = 1$$

Since $AC < 0$ means that either $(A < 0$ and $C > 0)$ or $(A > 0$ and $C < 0)$, either $(4A^2C < 0$ and $4AC^2 > 0)$, or $(4A^2C > 0$ and $4AC^2 < 0)$. In the equation above, that means that the $+$ sign will become a $(-)$ sign once all the values A, B, C, D, E, and F are determined, which is exactly the equation of the hyperbola. Note that if $A > 0$ and $C < 0$, the equation becomes:

$$\frac{4AC^2\left(y + \frac{E}{2C}\right)^2}{CD^2 + AE^2 - 4ACF} - \frac{|4A^2C|\left(x + \frac{D}{2A}\right)^2}{CD^2 + AE^2 - 4ACF} = 1$$

If $A < 0$ and $C > 0$, the equation becomes:

$$\frac{4A^2C\left(x + \frac{D}{2A}\right)^2}{CD^2 + AE^2 - 4ACF} - \frac{|4AC^2|\left(y + \frac{E}{2C}\right)^2}{CD^2 + AE^2 - 4ACF} = 1$$

72. With $a \neq 0$ and $b \neq 0$, we have $\left(\frac{x - h}{a}\right)^2 = \left(\frac{y - k}{b}\right)^2$.
Then $\left(\frac{x - h}{a}\right) = \left(\frac{y - k}{b}\right)$ or $\left(\frac{x - h}{a}\right) = -\left(\frac{y - k}{b}\right)$. Solving these two equations, we find that $y = \pm\frac{b}{a}(x - h) + k$. The graph consists of two intersecting slanted lines through (h, k). Its symmetry is like that of a hyperbola. Figure 8.2 shows the relationship between an ordinary hyperbola and two intersecting lines.

73. The asymptotes of the first hyperbola are $y = \pm\frac{b}{a}(x - h) + k$ and the asymptotes of the second hyperbola are $y = \pm\frac{b}{a}(x - h) + k$; they are the same.
[Note that in the second equation, the standard usage of $a + b$ has been revised.] The conjugate axis for hyperbola 1 is $2b$, which is the same as the transverse axis for hyperbola 2. The conjugate axis for hyperbola 2 is $2a$, which is the same as the transverse axis of hyperbola 1.

74. When $x = c$, $\dfrac{c^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$$c^2b^2 - a^2y^2 = a^2b^2$$
$$a^2y^2 = b^2(c^2 - a^2)$$
$$b^2 = c^2 - a^2$$
$$y^2 = \frac{b^4}{a^2}$$
$$y = \pm\frac{b^2}{a}$$

One possible answer: Draw the points $\left(c, \dfrac{b^2}{a}\right)$ and $\left(c, \dfrac{-b^2}{a}\right)$ on a copy of figure 8.24(a). Clearly the points $\left(c, \pm\dfrac{b^2}{a}\right)$ on the hyperbola are the endpoints of a segment perpendicular to the x-axis through the focus $(c, 0)$. Since this is the definition of the focal width used in the construction of a parabola, applying it to the hyperbola also makes sense.

75. The standard forms involved multiples of x, x^2, y, and y^2, as well as constants; therefore they can be rewritten in the general form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ (none of the standard forms we have seen require a Bxy term). For example, rewrite $y = ax^2$ as $ax^2 - y = 0$; this is the general form with $A = a$ and $E = -1$, and all others 0. Similarly, the hyperbola $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$ can be put in standard form with $A = -\dfrac{1}{a^2}$, $C = \dfrac{1}{b^2}$, $F = -1$, and $B = D = E = 0$.

■ Section 8.4 Translation and Rotation of Axes

Quick Review 8.4

1. $\cos 2\alpha = \dfrac{5}{13}$

2. $\cos 2\alpha = \dfrac{8}{17}$

3. $\cos 2\alpha = \dfrac{1}{2}$

4. $\cos 2\alpha = \dfrac{2}{3}$

5. $2\alpha = \dfrac{\pi}{2}$, so $\alpha = \dfrac{\pi}{4}$

6. $2\alpha = \sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$, so $\alpha = \dfrac{\pi}{12}$

7. $\cos 2\alpha = 2\cos^2\alpha - 1 = \dfrac{3}{5}$, $2\cos^2\alpha = \dfrac{8}{5}$, $\cos^2\alpha = \dfrac{4}{5}$,

$\cos\alpha = \dfrac{2}{\sqrt{5}}$

8. $\cos 2\alpha = 2\cos^2\alpha - 1 = \dfrac{3}{4}$, $2\cos^2\alpha = \dfrac{7}{4}$, $\cos^2\alpha = \dfrac{7}{8}$,

$$\cos\alpha = \sqrt{\frac{7}{8}} = \frac{\sqrt{7}}{2\sqrt{2}} = \frac{\sqrt{7}}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{14}}{4}$$

9. $\cos 2\alpha = 1 - 2\sin^2\alpha = \dfrac{5}{6}$, $-2\sin^2\alpha = -\dfrac{1}{6} \Rightarrow$

$\sin^2\alpha = \dfrac{1}{12} \Rightarrow \sin\alpha = \sqrt{\dfrac{1}{12}} \Rightarrow \sin\alpha = \dfrac{1}{\sqrt{12}}$

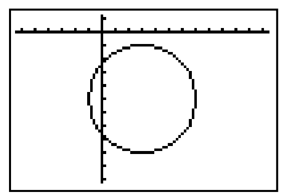
10. $\cos 2\alpha = 1 - 2\sin^2\alpha = \dfrac{45}{53}$, $2\sin^2\alpha = \dfrac{8}{53} \Rightarrow \sin^2\alpha = \dfrac{8}{106} \Rightarrow$

$\sin\alpha = \dfrac{2}{\sqrt{53}}$

Section 8.4 Exercises

1. Use the quadratic formula with $a = 1$, $b = 10$, and $c = x^2 - 6x + 18$. Then $b^2 - 4ac = (10)^2 - 4(x^2 - 6x + 18) = -4x^2 + 24x + 28 = 4(-x^2 + 6x + 7)$, and

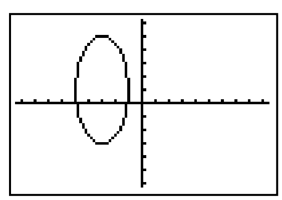
$$y = \frac{-10 \pm \sqrt{4(-x^2 + 6x + 7)}}{2} = -5 \pm \sqrt{-x^2 + 6x + 7}$$



[–6.4, 12.4] by [–11.2, 1.2]

2. Use the quadratic formula with $a = 1$, $b = -2$, and $c = 4x^2 + 24x + 21$. Then $b^2 - 4ac = (-2)^2 - 4(4x^2 + 24x + 21) = -16x^2 - 96x - 80 = 16(-x^2 - 6x - 5)$, and

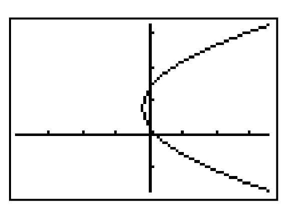
$$y = \frac{2 \pm \sqrt{16(-x^2 - 6x - 5)}}{2} = 1 \pm 2\sqrt{-x^2 - 6x - 5}$$



[–9.4, 9.4] by [–6.2, 6.2]

3. Use the quadratic formula with $a = 1$, $b = -8$, and $c = -8x + 8$. Then $b^2 - 4ac = (-8)^2 - 4(-8x + 8) = 32x + 32 = 32(x + 1)$, and

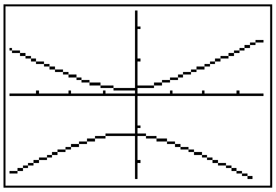
$$y = \frac{8 \pm \sqrt{32(x + 1)}}{2} = 4 \pm 2\sqrt{2x + 2}$$



[–19.8, 17.8] by [–8.4, 16.4]

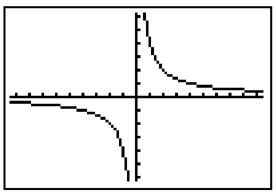
4. Use the quadratic formula with $a = -4$, $b = -40$, and $c = x^2 + 6x + 91$. Then $b^2 - 4ac =$ $(-40)^2 - 4(-4)(x^2 + 6x + 91) = 16x^2 + 96x + 3056$ $= 16(x^2 + 6x + 191)$, and

$$y = \frac{40 \pm \sqrt{16(x^2 + 6x + 191)}}{-8}$$
$$= -5 \pm \frac{1}{2}\sqrt{x^2 + 6x + 191}$$



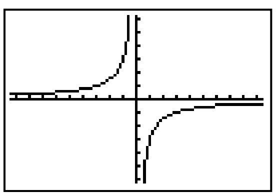
[–37.6, 37.6] by [–24.8, 24.8]

5. $-4xy + 16 = 0 \Rightarrow -4xy = -16 \Rightarrow y = 4/x$



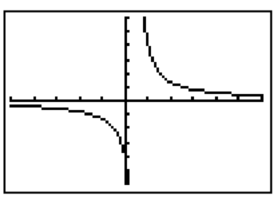
[–9.4, 9.4] by [–6.2, 6.2]

6. $2xy + 6 \Rightarrow 2xy = -6 \Rightarrow y = -3/x$



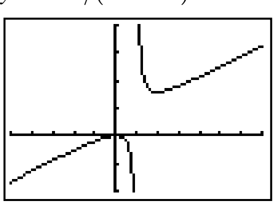
[–9.4, 9.4] by [–6.2, 6.2]

7. $xy - y - 8 = 0 \Rightarrow y(x - 1) = 8 \Rightarrow y = 8/(x - 1)$



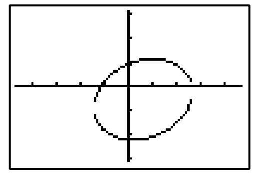
[–10, 12] by [–12, 12]

8. $2x^2 - 5xy + y = 0 \Rightarrow y(1 - 5x) = -2x^2 \Rightarrow$ $y = 2x^2/(5x - 1)$



[–1, 1.4] by [–0.4, 0.8]

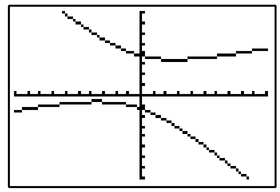
9. Use the quadratic formula with $a = 3$, $b = 4 - x$, and $c = 2x^2 - 3x - 6$. Then $b^2 - 4ac$ $= (4 - x)^2 - 12(2x^2 - 3x - 6) = -23x^2 + 28x + 88$, and $y = \dfrac{x - 4 \pm \sqrt{-23x^2 + 28x + 88}}{6}$



[–4.7, 4.7] by [–3.1, 3.1]

10. Use the quadratic formula with $a = 4$, $b = 3x - 10$, and $c = -x^2 - 5x - 20$. Then $b^2 - 4ac = (3x - 10)^2$ $-16(-x^2 - 5x - 20) = 25x^2 + 20x + 420$, and

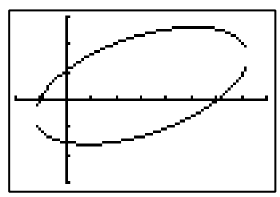
$$y = \frac{1}{8}[10 - 3x \pm \sqrt{25x^2 + 20x + 420}].$$



[−10, 10] by [−8, 8]

11. Use the quadratic formula with $a = 8$, $b = 4 - 4x$, and $c = 2x^2 - 10x - 13$.
Then $b^2 - 4ac = (4 - 4x)^2 - 32(2x^2 - 10x - 13)$ $= -48x^2 + 288x + 432 = 48(-x^2 + 6x + 9)$, and

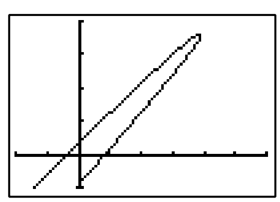
$$y = \frac{4x - 4 \pm \sqrt{48(-x^2 + 6x + 9)}}{16}$$
$$= \frac{1}{4}x - \frac{1}{4} \pm \frac{1}{4}\sqrt{3(-x^2 + 6x + 9)}.$$



[−2, 8] by [−3, 3]

12. Use the quadratic formula with $a = 2$, $b = 6 - 4x$, and $c = 2x^2 - 5x - 15$. Then $b^2 - 4ac = (6 - 4x)^2$ $-8(2x^2 - 5x - 15) = 156 - 8x = 4(39 - 2x)$, and

$$y = \frac{4x - 6 \pm \sqrt{4(39 - 2x)}}{4}$$
$$= x - \frac{3}{2} \pm \frac{1}{2}\sqrt{39 - 2x}.$$



[−10, 30] by [−5, 20]

13. $h = 0$, $k = 0$ and the parabola opens downward, so $4py = x^2$). Using $(2, -1)$: $-4p = 4$, $p = -1$. The standard form is $x^2 = -4y$.

14. $h = 0$, $k = 0$ and the parabola opens to the right, so $4px = y^2$ $(p > 0)$. Using $(2, 4)$: $8p = 16$, $p = 2$. The standard form is $y^2 = 8x$.

15. $h = 0, k = 0$ and the hyperbola opens to the right and left, so $a = 3$, and $b = 4$. The standard form is $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$.

16. $h = 0, k = 0$, and the x-axis is the focal axis, so $a = 4$ and $b = 3$. The standard form is $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$.

For #17–20, recall that $x' = x - h$ and $y' = y - k$.

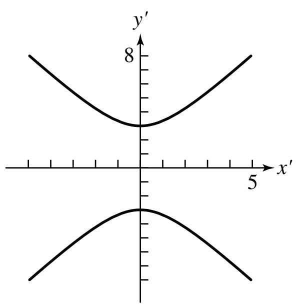
17. $(x', y') = (4, -1)$

18. $(x', y') = (2, 12)$

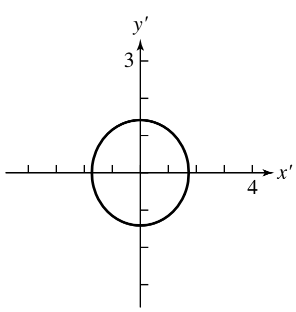
19. $(x', y') = (5, -3 - \sqrt{5})$

20. $(x', y') = (-5 - \sqrt{2}, -1)$

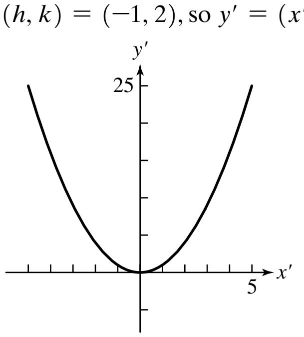
21. $4(y^2 - 2y) - 9(x^2 + 2x) = 41$, so $4(y - 1)^2 - 9(x + 1)^2 = 41 + 4 - 9 = 36$. Then $\dfrac{(y-1)^2}{9} - \dfrac{(x+1)^2}{4} = 1$. This is a hyperbola, with $a = 3, b = 2$, and $c = \sqrt{13}$.
$\dfrac{(y')^2}{9} - \dfrac{(x')^2}{4} = 1$.



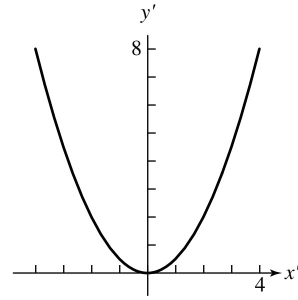
22. $2(x^2 + 6x) + 3(y^2 - 8y) = -60$, so $2(x + 3)^2 + 3(y - 4^2) = -60 + 18 + 48 = 6$. Then $\dfrac{(x+3)^2}{3} + \dfrac{(y-4)^2}{2} = 1$. This is an ellipse with $a = \sqrt{3}, b = \sqrt{2}$, and $c = 1$. $\dfrac{(x')^2}{3} + \dfrac{(y')^2}{2} = 1$.



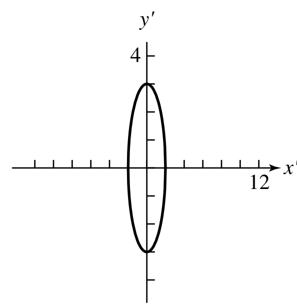
23. $y - 2 = (x + 1)^2$, a parabola. The vertex is $(h, k) = (-1, 2)$, so $y' = (x')^2$.



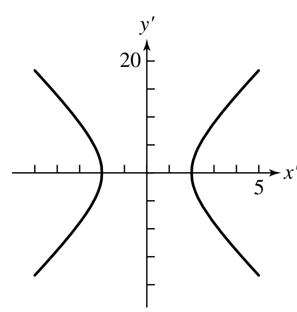
24. $2\left(y - \dfrac{7}{6}\right) = (x - 1)^2$, a parabola. The vertex is $(h, k) = \left(1, \dfrac{7}{6}\right)$, so $2y' = (x')^2$.



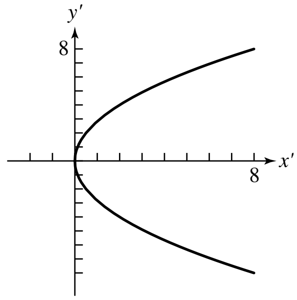
25. $9(x^2 - 2x) + 4(y^2 + 4y) = 11$, so $9(x - 1)^2 + 4(y + 2)^2 = 11 + 9 + 16 = 36$. Then $\dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{9} = 1$.
This is an ellipse, with $a = 2, b = 3$, and $c = \sqrt{5}$. Foci: $(1, -2 \pm \sqrt{5})$. Center $(1, -2)$, so $\dfrac{(x')^2}{4} + \dfrac{(y')^2}{9} = 1$.



26. $16(x^2 - 2x) - (y^2 + 6y) = 57$, so $16(x - 1)^2 - (y + 3)^2 = 57 + 16 - 9 = 64$.
Then $\dfrac{(x-1)^2}{4} - \dfrac{(y+3)^2}{64} = 1$. This is a hyperbola, with $a = 2, b = 8$, and $c = \sqrt{68} = 2\sqrt{17}$. Foci: $(1 \pm 2\sqrt{17}, -3)$. Center $(1, -3)$, so $\dfrac{(x')^2}{4} - \dfrac{(y')^2}{64} = 1$.



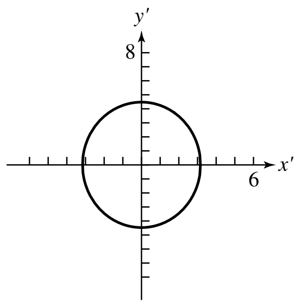
27. $8(x - 2) = (y - 2)^2$, a parabola. The vertex is $(h, k) = (2, 2)$, so $8x' = (y')^2$.



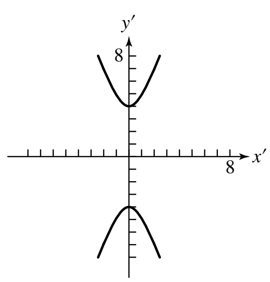
28. $2(x^2 - 2x) + (y^2 - 6y) = 9$, so $2(x - 1)^2 + (y - 3)^2 = 9 + 2 + 9 = 20$. Then $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{20} = 1$.

This is an ellipse, with $a = \sqrt{10}$, $b = \sqrt{20} = 2\sqrt{5}$, and $c = \sqrt{10}$.

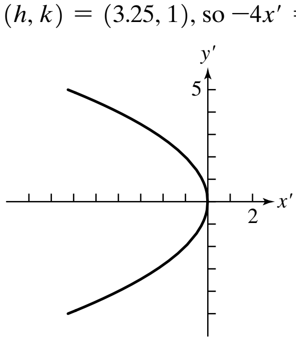
Foci: $(1, 3 \pm \sqrt{10})$. Center $(1, 3)$, so $\dfrac{(x')^2}{10} + \dfrac{(y')^2}{20} = 1$



29. $2(x^2 + 2x) - y^2 = -6$, so $2(x + 1)^2 - y^2 = -6 + 2 = -4$, Then $\dfrac{y^2}{4} - \dfrac{(x+1)^2}{2} = 1$. This is a hyperbola, with $a = \sqrt{2}$, $b = 2$, and $c = \sqrt{6}$. Foci: $(-1, \pm\sqrt{6})$. Center $(-1, 0)$, so $\dfrac{(y')^2}{4} - \dfrac{(x')^2}{2} = 1$.



30. $-4(x - 3.25) = (y - 1)^2$, a parabola. The vertex is $(h, k) = (3.25, 1)$, so $-4x' = (y')^2$



31. The horizontal distance from O to P is $x = h + x' = x' + h$, and the vertical distance from O to P is $y = k + y' = y' + k$.

32. Given $x = x' + h$, subtract h from both sides: $x - h = x'$ or $x' = x - h$. And given $y = y' + k$, subtract k from both sides: $y - k = y'$ or $y' = y - k$.

For #33–36, recall that $x' = x \cos\alpha + y \sin\alpha$ and $y' = -x \sin\alpha + y \cos\alpha$.

33. $(x', y') = \left(-2\cos\frac{\pi}{4} + 5\sin\frac{\pi}{4}, 2\sin\frac{\pi}{4} + 5\cos\frac{\pi}{4}\right) = \left(\frac{3\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$

34. $(x', y') = \left(6\cos\frac{\pi}{3} - 3\sin\frac{\pi}{3}, -6\sin\frac{\pi}{3} - 3\cos\frac{\pi}{3}\right) = \left(\frac{6}{2} - \frac{3\sqrt{3}}{2}, \frac{-6\sqrt{3}}{2} - \frac{3}{2}\right) = \left(\frac{6 - 3\sqrt{3}}{2}, \frac{-6\sqrt{3} - 3}{2}\right) \approx (0.40, -6.70)$

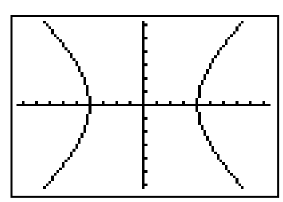
35. $\alpha \approx 1.06$, $(x', y') = (-5\cos(1.06) - 4\sin(1.06), 5\sin(1.06) - 4\cos(1.06)) \approx (-5.94, 2.38)$

36. $\alpha \approx \frac{\pi}{4}$, $(x', y') = \left(2\cos\frac{\pi}{4} + 3\sin\frac{\pi}{4}, -2\sin\frac{\pi}{4} + 3\cos\frac{\pi}{4}\right) = \left(\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

For #37–40, use the discriminant $B^2 - 4AC$ to determine the type of conic. Then use the relationship of $\cot 2\alpha = \dfrac{A - C}{B}$ to determine the angle of rotation.

37. $B^2 - 4AC = 1 > 0$, hyperbola; $\cot 2\alpha = 0$, so $\alpha = \frac{\pi}{4}$.

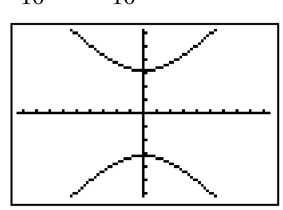
Translating, $\left(\dfrac{x' - y'}{\sqrt{2}}\right)\left(\dfrac{x' + y'}{\sqrt{2}}\right) = 8$, $\dfrac{(x')^2}{16} - \dfrac{(y')^2}{16} = 1$, $y' = \pm\sqrt{(x')^2 - 16}$



[–9.4, 9.4] by [–6.2, 6.2]

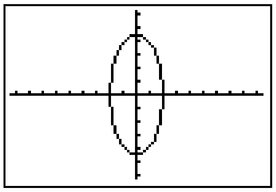
38. $B^2 - 4AC = 9 > 0$, hyperbola; $\cot 2\alpha = 0$, so $\alpha = \frac{\pi}{4}$.

Translating, $3\left(\dfrac{x' - y'}{\sqrt{2}}\right)\left(\dfrac{x' + y'}{\sqrt{2}}\right) + 15 = 0$, $\dfrac{(y')^2}{10} - \dfrac{(x')^2}{10} = 1$, $y' = \pm\sqrt{(x')^2 + 10}$



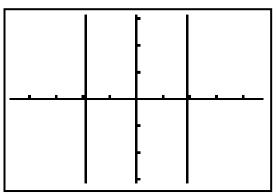
[–9.4, 9.4] by [–6.2, 6.2]

39. $B^2 - 4AC = 3 - 4(2)(1) = -5 < 0$, ellipse;
$\cot 2\alpha = \frac{1}{\sqrt{3}}, \alpha = \frac{\pi}{6}$. Translating,
$x = x' \cos\frac{\pi}{6} - y' \sin\frac{\pi}{6}, y = x' \sin\frac{\pi}{6} + y' \cos\frac{\pi}{6}$, so the equation becomes $\frac{5(x')^2}{2} + \frac{(y')^2}{2} = 10$,
$\frac{(x')^2}{4} + \frac{(y')^2}{20} = 1$



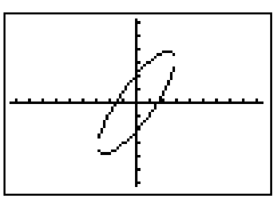
[–9.4, 9.4] by [–6.2, 6.2]

40. $B^2 - 4AC = 12 - 4(3)(1) = 0$, parabola; $\cot 2\alpha = \frac{1}{\sqrt{3}}$,
$\alpha = \frac{\pi}{6}$. Translating, $x = x' \cos\frac{\pi}{6} - y' \sin\frac{\pi}{6}$,
$y = x' \sin\frac{\pi}{6} + y' \cos\frac{\pi}{6}$, so $4(x')^2 = 14, x' = \pm\frac{\sqrt{14}}{2}$.
This is a degenerate form consisting of only two parallel lines.



[–4.7, 4.7] by [–3.1, 3.1]

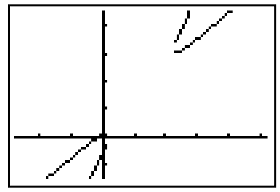
41. $B^2 - 4AC = -176 < 0$, ellipse. Use the quadratic formula with $a = 9, b = -20x$, and $c = 16x^2 - 40$. Then
$b^2 - 4ac = (-20x)^2 - 4(9)(16x^2 - 40)$
$= -176x^2 + 1440 = 16(-11x^2 + 90)$, and
$$y = \frac{20x \pm \sqrt{16(-11x^2 + 90)}}{18} = \frac{10x \pm 2\sqrt{90 - 11x^2}}{9}$$



[–9.4, 9.4] by [–6.2, 6.2]

$\cot 2\alpha = -\frac{7}{20}, \alpha \approx 0.954 \approx 54.65°$

42. $B^2 - 4AC = 4 > 0$, hyperbola. Use the quadratic formula with $a = 2, b = -6x + 10$, and $c = 4x^2 - 3x - 6$. Then
$b^2 - 4ac = (-6x + 10)^2 - 4(2)(4x^2 - 3x - 6)$
$= 4x^2 - 96x + 148 = 4(x^2 - 24x + 37)$, and
$$y = \frac{(6x - 10) \pm \sqrt{4(x^2 - 24x + 37)}}{4} = \frac{(3x - 5) \pm \sqrt{x^2 - 24x + 37}}{2}$$



[–28, 52] by [–15, 45]

$\cot 2\alpha = -\frac{1}{3}, \alpha \approx 0.946 \approx 54.22°$

43. $B^2 - 4AC = 16 - 4(1)(10) = -24 < 0$; ellipse

44. $B^2 - 4AC = 16 - 4(1)(0) = 16 > 0$; hyperbola

45. $B^2 - 4AC = 36 - 4(9)(1) = 0$; parabola

46. $B^2 - 4AC = 1 - 4(0)(3) = 1 > 0$; hyperbola

47. $B^2 - 4AC = 16 - 4(8)(2) = -48 < 0$; ellipse

48. $B^2 - 4AC = 144 - 4(3)(4) = 96 > 0$; hyperbola

49. $B^2 - 4AC = 0 - 4(1)(-3) = 12 > 0$; hyperbola

50. $B^2 - 4AC = 16 - 4(5)(3) = -44 < 0$; ellipse

51. $B^2 - 4AC = 4 - 4(4)(1) = -12 < 0$; ellipse

52. $B^2 - 4AC = 16 - 4(6)(9) = -200 < 0$; ellipse

53. In the new coordinate system, the center $(x', y') = (0, 0)$, the vertices occur at $(\pm 3, 0)$ and the foci are located at $(\pm 3\sqrt{2}, 0)$. We use $x = x' \cos\frac{\pi}{4} - y' \sin\frac{\pi}{4}$, $y = x' \sin\frac{\pi}{4} + y' \cos\frac{\pi}{4}$ to translate "back." Under the "old" coordinate system, the center $(x, y) = (0, 0)$, the vertices occured at $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ and $\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$. and the foci are located at $(3, 3)$ and $(-3, -3)$.

54. **(a)** Reversing the translation and rotation of the parabola, we see that the vertex in the (x'', y'') coordinate system is $V(0, 0)$, with $h = \frac{21}{\sqrt{15}}$, and $k = \frac{3\sqrt{5}}{10}$. This means that the vertex of the parabola in the (x', y') coordinate system is $(x'' + h, y'' + k) = \left(0 + \frac{21}{\sqrt{15}}, 0 + \frac{3\sqrt{5}}{10}\right)$. Since $\cos\alpha = \frac{1}{\sqrt{5}}$ and $\sin\alpha = \frac{2}{\sqrt{5}}$, rotating "back" into the (x, y) coordinate system gives
$(x, y) = (x' \cos\alpha - y' \sin\alpha, x' \sin\alpha + y' \cos\alpha)$
$\left(\frac{21}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} - \frac{3\sqrt{5}}{10} \cdot \frac{2}{15}, \frac{21}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} + \frac{3\sqrt{5}}{10} \cdot \frac{1}{\sqrt{5}}\right)$
$= (3.6, 8.7)$.

(b) See (a).

55. Answers will vary. One possible answer: Using the geometric relationships illustrated, it is clear that
$x = x' \cos\alpha - y' \cos\left(\frac{\pi}{2} - \alpha\right) = x' \cos\alpha - y' \sin\alpha$
and that $y = x' \cos\left(\frac{\pi}{2} - \alpha\right) + y' \cos\alpha$
$= x' \sin\alpha + y' \cos\alpha$.

56. $x' = x\cos\alpha + y\sin\alpha \qquad y' = -x\sin\alpha + y\cos\alpha$
$x'\cos\alpha = x\cos^2\alpha + y\sin\alpha\cos\alpha$
$y'\sin\alpha = -x\sin^2\alpha + y\cos\alpha\sin\alpha$
$x'\cos\alpha - y'\sin\alpha = x\cos^2\alpha + y\sin\alpha\cos\alpha + x\sin^2\alpha - y\sin\alpha\cos\alpha$
$x'\cos\alpha - y'\sin\alpha = x\cos^2\alpha + x\sin^2\alpha$
$x'\cos\alpha - y'\sin\alpha = x$
Similarly,
$x' = \cos\alpha + y\sin\alpha \qquad y' = -x\sin\alpha + y\cos\alpha$
$x'\sin\alpha = x\cos\alpha\sin\alpha + y\sin^2\alpha$
$y'\cos\alpha = -x\sin\alpha\cos\alpha + y\cos^2\alpha$
$x'\sin\alpha + y'\cos\alpha = x\cos\alpha\sin\alpha + y\sin^2\alpha - x\sin\alpha\cos\alpha + y\cos^2\alpha$
$x'\sin\alpha + y'\cos\alpha = y(\sin^2\alpha + \cos^2\alpha)$
$x'\sin\alpha + y'\cos\alpha = y$

57. True. The Bxy term is missing and so the rotation angle α is zero.

58. True. Because the x^2 and y^2 terms have the same coefficient (namely 1), completing the square to put the equation in standard form will produce the same denominator under $(y-k)^2$ as under $(x-h)^2$.

59. Eliminating the cross-product term requires rotation, not translation. The answer is B.

60. Moving the center or vertex to the origin is done through translation, not rotation. The answer is C.

61. Completing the square twice, and dividing to obtain 1 on the right, turns the equation into

$$\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$$

The vertices lie 4 units to the left and right of center $(1, -2)$. The answer is A.

62. The equation is equivalent to $y = 4/x$. The answer is E.

63. (a) The rotated axes pass through the old origin with slopes of ± 1, so the equations are $y = \pm x$.

(b) The location of $(x'', y'') = (0, 0)$ in the xy system can be found by reversing the transformations. In the $x'y'$ system, $(x'', y'') = (0, 0)$ has coordinates $(h, k) = \left(\frac{21}{\sqrt{5}}, \frac{3\sqrt{5}}{10}\right)$. The coordinates of this point in the xy system are then given by the second set of rotation formulas; with $\cos\alpha = \frac{1}{\sqrt{5}}$, $\sin\alpha = \frac{2}{\sqrt{5}}$:

$$x = \frac{21}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\right) - \frac{3\sqrt{5}}{10}\left(\frac{2}{\sqrt{5}}\right) = \frac{18}{5}$$

$$y = \frac{21}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}\right) + \frac{3\sqrt{5}}{10}\left(\frac{1}{\sqrt{5}}\right) = \frac{87}{10}$$

The $x''y''$ axes pass through the point $(x, y) = \left(\frac{18}{5}, \frac{87}{10}\right)$ with slopes of $\frac{2/\sqrt{5}}{1/\sqrt{5}} = 2$ and its negative reciprocal, $-\frac{1}{z}$. Using this information to write linear equations in point-slope form, and then converting to slope-intercept form, we obtain

$$y = 2x + \frac{3}{2}$$

$$y = -\frac{1}{2}x + \frac{21}{2}$$

64. (a) If the translation on $x' = x - h$ and $y' = y - k$ is applied to the equation, we have:
$A(x')^2 + Bx'y' + C(y')^2 + Dx' + Ey' + F = 0$,
so $A(x-h)^2 + B(x-h)(y-k) + C(y-k)^2 + D(x-h) + E(y-k) + F = 0$, which becomes
$Ax^2 + Bxy + Cy^2 + (D - Bk - 2Ah)x + (E - 2ck - Bh)y + (Ah^2 + Ck^2 - Ek - Dh) + Bhk + F = 0$
The discriminants are exactly the same; the coefficients of the x^2, xy, and y^2 terms do not change (no sign change).

(b) If the equation is multiplied by some constant k, we have $kAx^2 + kBxy + kCy^2 + kD + kE + kF = 0$, so the discriminant of the new equation becomes $(kB)^2 - 4(kA)(kC) = k^2B^2 - 4k^2AC = k^2(B^2 - 4AC)$. Since $k^2 > 0$ for $k \neq 0$, no sign change occurs.

65. First, consider the linear terms:

$$\begin{aligned} Dx + Ey &= D(x'\cos\alpha - y'\sin\alpha) \\ &\quad + E(x'\sin\alpha + y'\cos\alpha) \\ &= (D\cos\alpha + E\sin\alpha)x' \\ &\quad + (E\cos\alpha - D\sin\alpha)y' \end{aligned}$$

This shows that $Dx + Ey = D'x' + E'y'$, where $D' = D\cos\alpha + E\sin\alpha$ and $E' = E\cos\alpha - D\sin\alpha$.
Now, consider the quadratic terms:

$$\begin{aligned} &Ax^2 + Bxy + Cy^2 = A(x'\cos\alpha - y'\sin\alpha)^2 + \\ &B(x'\cos\alpha - y'\sin\alpha)(x'\sin\alpha + y'\cos\alpha) + \\ &C(x'\sin\alpha + y'\cos\alpha)^2 \\ &= A(x'^2\cos^2\alpha - 2x'y'\cos\alpha\sin\alpha + y'^2\sin^2\alpha) \\ &\quad + B(x'^2\cos\alpha\sin\alpha + x'y'\cos^2\alpha - x'y'\sin^2\alpha \\ &\quad - y'^2\sin\alpha\cos\alpha) + C(x'^2\sin^2\alpha + 2x'y'\sin\alpha\cos\alpha \\ &\quad + y'^2\cos^2\alpha) \\ &= (A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha)x'^2 \\ &\quad + [B(\cos^2\alpha - \sin^2\alpha) \\ &\quad + 2(C - A)(\sin\alpha\cos\alpha)]x'y' \\ &\quad + (C\cos^2\alpha - B\cos\alpha\sin\alpha + A\sin^2\alpha)y'^2 \\ &= (A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha)x'^2 \\ &\quad + [B\cos 2\alpha + (C - A)\sin 2\alpha]x'y' \\ &\quad + (C\cos^2\alpha - B\cos\alpha\sin\alpha + A\sin^2\alpha)y'^2 \end{aligned}$$

This shows that
$Ax^2 + Bxy + Cy^2 = A'x'^2 + B'x'y' + C'y'^2$, where
$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha$,
$B' = B\cos 2\alpha + (C - A)\sin 2\alpha$, and $C' = C\cos^2\alpha - B\cos\alpha\sin\alpha + A\sin^2\alpha$.

The results above imply that if the formulas for A', B', C', D', and F' are applied, then $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$ is equivalent to $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Therefore, the formulas are correct.

66. This equation is simply a special case of the equation we have used throughout the chapter, where $B = 0$. The discriminant $B^2 - 4AC$, then, reduces simply to $-4AC$. If $-4AC > 0$, we have a hyperbola; $-4AC = 0$, we have a parabola; $-4AC < 0$, we have a ellipse. More simply: a hyperbola if $AC < 0$; a parabola if $AC = 0$; an ellipse if $AC > 0$.

67. Making the substitutions $x = x'\cos\alpha - y'\sin\alpha$ and $y = x'\sin\alpha + y'\cos\alpha$, we find that:

$B'x'y' = (B\cos^2\alpha - B\sin^2\alpha + 2C\sin\alpha\cos\alpha - 2A\sin\alpha\cos\alpha)x'y'$

$Ax'^2 = (A\cos^2\alpha + B\sin\alpha\cos\alpha + C\sin^2\alpha)(x')^2$

$Cy'^2 = (A\sin^2\alpha + C\cos^2\alpha - B\cos\alpha\sin\alpha)(y')^2$

$$B'^2 - 4A'C' = (B\cos(2\alpha) - (A - C)\sin(2\alpha))^2 - 4(A\cos^2\alpha + B\sin\alpha\cos\alpha + C\sin^2\alpha)(A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha)$$

$$= \frac{1}{2}B^2\cos(4\alpha) + \frac{1}{2}B^2 + BC\sin(4\alpha) - BA\sin(4\alpha) + \frac{1}{2}C^2 - \frac{1}{2}C^2\cos(4\alpha) - CA + CA\cos(4\alpha) + \frac{1}{2}A^2 - \frac{1}{2}A^2\cos(4\alpha) - 4\left(\frac{1}{2}A\cos(2\alpha) + \frac{1}{2}A + \frac{1}{2}B\sin(2\alpha) + \frac{1}{2}C - \frac{1}{2}C\cos(2\alpha)\right)\left(\frac{1}{2}A - \frac{1}{2}A\cos(2\alpha) + \frac{1}{2}C\cos(2\alpha) + \frac{1}{2}C - \frac{1}{2}B\sin(2\alpha)\right)$$

$$= \frac{1}{2}B^2\cos(4\alpha) + \frac{1}{2}B^2 + BC\sin(4\alpha) - BA\sin(4\alpha) + \frac{1}{2}C^2 - \frac{1}{2}C^2\cos(4\alpha) - CA + CA\cos(4\alpha) + \frac{1}{2}A^2 - \frac{1}{2}A^2\cos(4\alpha) - BC\sin(4\alpha) + BA\sin(4\alpha) - 3AC - \frac{1}{2}C^2 - \frac{1}{2}A^2 + \frac{1}{2}A^2\cos(4\alpha) + \frac{1}{2}B^2 - \frac{1}{2}B^2\cos(4\alpha) + \frac{1}{2}C^2\cos(4\alpha) - AC\cos(4\alpha)$$

$$= B^2 - 4AC.$$

68. When the rotation is made to the (x', y') coordinate system, the coefficients A', B', C', D', E', and F' become:

$$A' = \frac{A}{2}(1 + \cos(2\alpha)) + \frac{B}{2}\sin(2\alpha) + \frac{C}{2}(1 - \cos(2\alpha))$$

$$B' = B\cos(2\alpha) - (A - C)\sin(2\alpha)$$

$$C' = \frac{A}{2}(1 - \cos(2\alpha)) - \frac{B}{2}\sin(2\alpha) + \frac{C}{2}(\cos(2\alpha) + 1)$$

$D' = D\cos\alpha + E\sin\alpha$

$E' = -D\sin\alpha + E\cos\alpha$

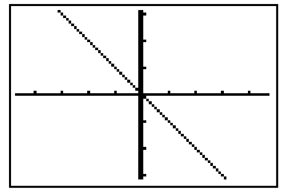
$F' = F$

(a) Since $F' = F$, F is invariant under rotation.

(b) Since $A' + C' = \frac{A}{2}[1 + \cos(2\alpha) + 1 - \cos(2\alpha)] + \frac{B}{2}[\sin(2\alpha) - \sin(2\alpha)] + \frac{C}{2}[1 - \cos(2\alpha) + \cos(2\alpha) + 1] = A + C$, $A + C$ is invariant under rotation.

(c) Since $D'^2 + E'^2 = (D\cos\alpha + E\sin\alpha)^2 + (-D\sin\alpha + E\cos\alpha)^2$
$= D^2\cos^2\alpha + 2DE\cos\alpha\sin\alpha + E^2\sin^2\alpha + D^2\sin^2\alpha - 2DE\cos\alpha\sin\alpha + E^2\cos^2\alpha$
$= D^2(\cos^2\alpha + \sin^2\alpha) + E^2(\sin^2\alpha + \cos^2\alpha)$
$= D^2 + E^2$, $D^2 + E^2$ is invariant under rotation.

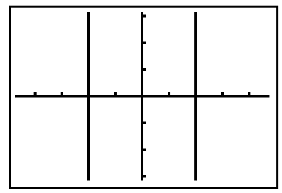
69. Intersecting lines: $x^2 + xy = 0$ can be rewritten as $x = 0$ (the y-axis) and $y = -x$



[–4.7, 4.7] by [–3.1, 3.1]

A plane containing the axis of a cone intersects the cone.

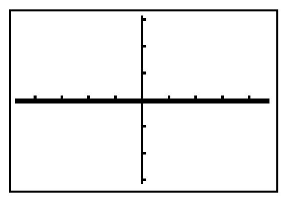
Parallel lines: $x^2 = 4$ can be rewritten as $x = \pm 2$ (a pair of vertical lines)



[–4.7, 4.7] by [–3.1, 3.1]

A degenerate cone is created by a generator that is parallel to the axis, producing a cylinder. A plane parallel to a generator of the cylinder intersects the cylinder and its interior.

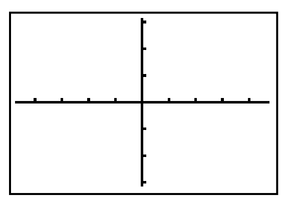
One line: $y^2 = 0$ can be rewritten as $y = 0$ (the x-axis).



[–4.7, 4.7] by [–3.1, 3.1]

A plane containing a generator of a cone intersects the cone.

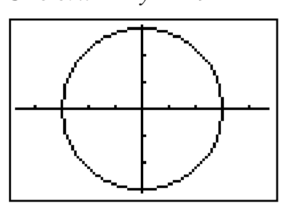
No graph: $x^2 = -1$



[–4.7, 4.7] by [–3.1, 3.1]

A plane parallel to a generator of a cylinder fails to intersect the cylinder.

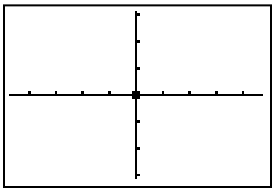
Circle: $x^2 + y^2 = 9$



[–4.7, 4.7] by [–3.1, 3.1]

A plane perpendicular to the axis of a cone intersects the cone but not its vertex.

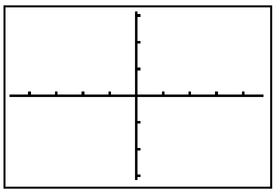
Point: $x^2 + y^2 = 0$, the point $(0, 0)$.



[–4.7, 4.7] by [–3.1, 3.1]

A plane perpendicular to the axis of a cone intersects the vertex of the cone.

No graph: $x^2 + y^2 = -1$



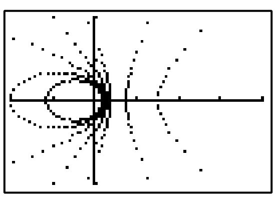
[–4.7, 4.7] by [–3.1, 3.1]

A degenerate cone is created by a generator that is perpendicular to the axis, producing a plane. A second plane perpendicular to the axis of this degenerate cone fails to intersect it.

■ Section 8.5 Polar Equations of Conics

Exploration 1

For $e = 0.7$ and $e = 0.8$, an ellipse; for $e = 1$, a parabola; for $e = 1.5$ and $e = 3$, a hyperbola.



$[-12, 24]$ by $[-12, 12]$

The five graphs all have a common focus, the pole $(0, 0)$, and a common directrix, the line $x = 3$. As the eccentricity e increases, the graphs move away from the focus and toward the directrix.

Quick Review 8.5

1. $r = -3$

2. $r = 2$

3. $\theta = \frac{7\pi}{6}$ or $-\frac{5\pi}{6}$

4. $\theta = -\frac{5\pi}{3}$ or $\frac{\pi}{3}$

5. $h = 0, k = 0, 4p = 16$, so $p = 4$
The focus is $(0, 4)$ and the directrix is $y = -4$.

6. $h = 0, k = 0, 4p = -12$, so $p = -3$
The focus is $(-3, 0)$ and the directrix is $x = 3$.

7. $a = 3, b = 2, c = \sqrt{5}$; Foci: $(\pm\sqrt{5}, 0)$; Vertices: $(\pm 3, 0)$

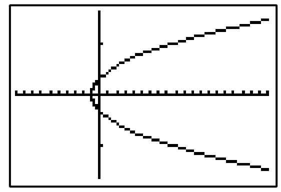
8. $a = 5, b = 3, c = 4$; Foci: $(0, \pm 4)$; Vertices: $(0, \pm 5)$

9. $a = 4, b = 3, c = 5$; Foci: $(\pm 5, 0)$; Vertices: $(\pm 4, 0)$

10. $a = 6, b = 2, c = 4\sqrt{2}$; Foci: $(0, \pm 4\sqrt{2})$;
Vertices: $(0, \pm 6)$

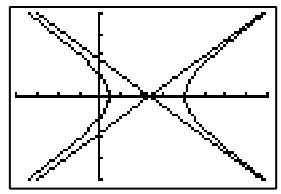
Section 8.5 Exercises

1. $r = \dfrac{2}{1 - \cos\theta}$ — a parabola.



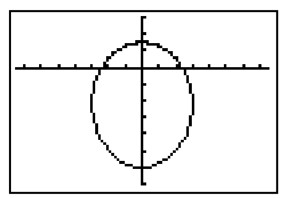
$[-10, 20]$ by $[-10, 10]$

2. $r = \dfrac{5}{1 + \left(\frac{5}{4}\right)\cos\theta} = \dfrac{20}{4 + 5\cos\theta}$ — a hyperbola.



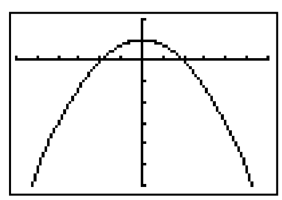
$[-20, 40]$ by $[-20, 20]$

3. $r = \dfrac{\frac{12}{5}}{1 + \left(\frac{3}{5}\right)\sin\theta} = \dfrac{12}{5 + 3\sin\theta}$ — an ellipse.



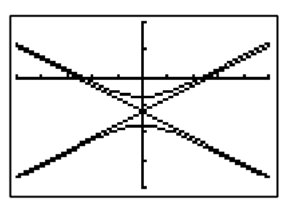
[–7.5, 7.5] by [–7, 3]

4. $r = \dfrac{2}{1 + \sin\theta}$ — a parabola.



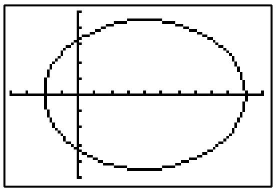
[–6, 6] by [–6, 2]

5. $r = \dfrac{\frac{7}{3}}{1 - \left(\frac{7}{3}\right)\sin\theta} = \dfrac{7}{3 - 7\sin\theta}$ — a hyperbola.



$[-5, 5]$ by $[-4, 2]$

6. $r = \dfrac{\frac{10}{3}}{1 - \left(\frac{2}{3}\right)\cos\theta} = \dfrac{10}{3 - 2\cos\theta}$ — an ellipse.



$[-4, 11]$ by $[-5, 5]$

7. Parabola with $e = 1$ and directrix $x = 2$.

8. Hyperbola with $e = 2$ and directrix $x = 3$.

9. Divide numerator and denominator by 2.
Parabola with $e = 1$ and directrix $y = -\frac{5}{2} = -2.5$.

10. Divide numerator and denominator by 4.
Ellipse with $e = \frac{1}{4} = 0.25$ and directrix $x = -2$.

11. Divide numerator and denominator by 6.
Ellipse with $e = \frac{5}{6}$ and directrix $y = 4$.

12. Divide numerator and denominator by 2.
Hyperbola with $e = \frac{7}{2} = 3.5$ and directrix $y = -6$.

13. Divide numerator and denominator by 5.
Ellipse with $e = \frac{2}{5} = 0.4$ and directrix $x = 3$.

14. Divide numerator and denominator by 2.
Hyperbola with $e = \frac{5}{2} = 2.5$ and directrix $y = 4$.

15. (b) $[-15, 5]$ by $[-10, 10]$

16. (d) $[-5, 5]$ by $[-3, 3]$

17. (f) $[-5, 5]$ by $[-3, 3]$

18. (e) $[-5, 5]$ by $[-3, 5]$

19. (c) $[-10, 10]$ by $[-5, 10]$

20. (a) $[-3, 3]$ by $[-6, 6]$

For #21–28, one must solve two equations $a = \dfrac{ep}{1+e}$ and $b = \dfrac{ep}{1-e}$ for e and p (given two constants a and b). The general solution to this is $e = \dfrac{b-a}{b+a}$ and $p = \dfrac{2ab}{b-a}$.

21. The directrix must be $x = p > 0$, since the right major-axis endpoint is closer to $(0, 0)$ than the left one, so the equation has the form $r = \dfrac{ep}{1 + e\cos\theta}$. Then
$1.5 = \dfrac{ep}{1 + e\cos 0} = \dfrac{ep}{1+e}$ and $6 = \dfrac{ep}{1 + e\cos\pi}$
$= \dfrac{ep}{1-e}$ (so $a = 1.5$ and $b = 6$). Therefore $e = \dfrac{3}{5} = 0.6$
and $p = 4$, so $r = \dfrac{2.4}{1 + (3/5)\cos\theta} = \dfrac{12}{5 + 3\cos\theta}$.

22. The directrix must be $x = -p < 0$, since the left major-axis endpoint is closer to $(0, 0)$ than the right one, so the equation has the form $r = \dfrac{ep}{1 - e\cos\theta}$. Then
$1.5 = \dfrac{ep}{1 - e\cos 0} = \dfrac{ep}{1-e}$ and $1 = \dfrac{ep}{1 - e\cos\pi}$
$= \dfrac{ep}{1+e}$ (so $a = 1$ and $b = 1.5$). Therefore $e = \dfrac{1}{5} = 0.2$
and $p = 6$ (the directrix is $x = -6$) , so
$r = \dfrac{1.2}{1 - (1/5)\cos\theta} = \dfrac{6}{5 - \cos\theta}$.

23. The directrix must be $y = p > 0$, since the upper major-axis endpoint is closer to $(0, 0)$ than the lower one, so the equation has the form $r = \dfrac{ep}{1 + e\cos\theta}$. Then
$1 = \dfrac{ep}{1 + e\sin(\pi/2)} = \dfrac{ep}{1+e}$ and $3 = \dfrac{ep}{1 + e\sin(3\pi/2)}$
$= \dfrac{ep}{1-e}$ (so $a = 1$ and $b = 3$). Therefore $e = \dfrac{1}{2} = 0.5$
and $p = 3$, so $r = \dfrac{1.5}{1 + (1/2)\sin\theta} = \dfrac{3}{2 + \sin\theta}$.

24. The directrix must be $y = -p < 0$, since the lower major-axis endpoint is closer to $(0, 0)$ than the upper one, so the equation has the form $r = \dfrac{ep}{1 - e\cos\theta}$. Then
$3 = \dfrac{ep}{1 - e\sin(\pi/2)} = \dfrac{ep}{1-e}$ and $\dfrac{3}{4} = \dfrac{ep}{1 - e\sin(3\pi/2)}$
$= \dfrac{ep}{1+e}$ (so $a = \dfrac{3}{4}$ and $b = 3$). Therefore $e = \dfrac{3}{5} = 0.6$
and $p = 2$ (the directrix is $y = -2$) , so
$r = \dfrac{1.2}{1 - (3/5)\sin\theta} = \dfrac{6}{5 - 3\sin\theta}$.

25. The directrix must be $x = p > 0$, since both transverse-axis endpoints have positive x coordinates, so the equation has the form $r = \dfrac{ep}{1 + e\cos\theta}$. Then
$3 = \dfrac{ep}{1 + e\cos 0} = \dfrac{ep}{1+e}$ and $-15 = \dfrac{ep}{1 + e\cos\pi}$
$= \dfrac{ep}{1-e}$ (so $a = 3$ and $b = -15$). Therefore $e = \dfrac{3}{2}$
$= 1.5$ and $p = 5$, so $r = \dfrac{7.5}{1 + (3/2)\cos\theta} = \dfrac{15}{2 + 3\cos\theta}$.

26. The directrix must be $x = -p < 0$, since both transverse-axis endpoints have negative x coordinates, so the equation has the form $r = \dfrac{ep}{1 - e\cos\theta}$. Then
$-3 = \dfrac{ep}{1 - e\cos 0} = \dfrac{ep}{1-e}$ and $1.5 = \dfrac{ep}{1 - e\cos\pi}$
$= \dfrac{ep}{1+e}$ (so $a = 1.5$ and $b = -3$). Therefore $e = 3$
and $p = 2$ (the directrix is $x = -2$), so $r = \dfrac{6}{1 - 3\cos\theta}$.

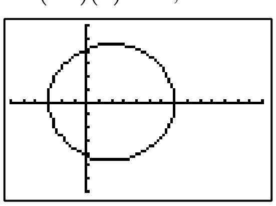
27. The directrix must be $y = p > 0$, since both transverse-axis endpoints have positive y coordinates, so the equation has the form $r = \frac{ep}{1 + e\cos\theta}$. Then $2.4 = \frac{ep}{1 + e\sin(\pi/2)} = \frac{ep}{1 + e}$ and $-12 = \frac{ep}{1 + e\sin(3\pi/2)} = \frac{ep}{1 - e}$ (so $a = 2.4$ and $b = -12$). Therefore $e = \frac{3}{2} = 1.5$ and $p = 4$, so $r = \frac{6}{1 + (3/2)\sin\theta} = \frac{12}{2 + 3\sin\theta}$.

28. The directrix must be $y = -p < 0$, since both transverse-axis endpoints have negative y coordinates, so the equation has the form $r = \frac{ep}{1 - e\cos\theta}$. Then $-6 = \frac{ep}{1 - e\sin(\pi/2)} = \frac{ep}{1 - e}$ and $2 = \frac{ep}{1 - e\sin(3\pi/2)} = \frac{ep}{1 + e}$ (so $a = 2$ and $b = -6$). Therefore $e = 2$ and $p = 3$ (the directrix is $y = -3$), so $r = \frac{6}{1 - 2\sin\theta}$.

29. The directrix must be $x = p > 0$, so the equation has the form $r = \frac{ep}{1 + e\cos\theta}$. Then $0.75 = \frac{ep}{1 + e\cos 0} = \frac{ep}{1 + e}$ and $3 = \frac{ep}{1 + e\cos\pi} = \frac{ep}{1 - e}$ (so $a = 0.75$ and $b = 3$). Therefore $e = \frac{3}{5} = 0.6$ and $p = 2$, so $r = \frac{1.2}{1 + (3/5)\cos\theta} = \frac{6}{5 + 3\cos\theta}$.

30. Since this is a parabola, $e = 1$, and with $y = p > 0$ as the directrix, the equation has the form $r = \frac{p}{1 + \sin\theta}$. Then $1 = \frac{p}{1 + \sin(\pi/2)} = \frac{p}{1 + 1}$, $p = 2$, and therefore $r = \frac{2}{1 + \sin\theta}$. Alternatively, for a parabola, the distance from the focus to the vertex is the same as the distance from the vertex to the directrix (the same is true for *all* points on the parabola). This distance is 1 unit, so we again conclude that the directrix is $y = 2$.

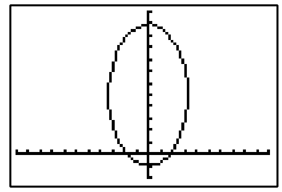
31. $r = \frac{21}{5 - 2\cos\theta} = \frac{4.2}{1 - 0.4\cos\theta}$, so $e = 0.4$. The vertices are $(7, 0)$ and $(3, \pi)$, so $2a = 10$, $a = 5$, $c = ae = (0.4)(5) = 2$, so $b = \sqrt{a^2 - c^2} = \sqrt{25 - 4} = \sqrt{21}$.



$[-6, 14]$ by $[-7, 6]$

$e = 0.4$, $a = 5$, $b = \sqrt{21}$, $c = 2$

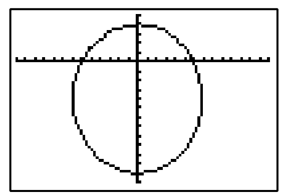
32. $r = \frac{11}{6 - 5\sin\theta} = \frac{11/6}{1 - (5/6)\sin\theta}$, so $e = \frac{5}{6}$. The vertices are $\left(11, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$, so $2a = 12$, $a = 6$. $c = ae = \frac{5}{6}\cdot 6 = 5$, so $b = \sqrt{a^2 - c^2} = \sqrt{36 - 25} = \sqrt{11}$.



$[-11, 10]$ by $[-2, 12]$

$e = \frac{5}{6}$, $a = 6$, $b = \sqrt{11}$, $c = 5$

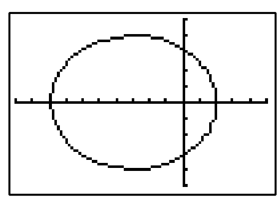
33. $r = \frac{24}{4 + 2\sin\theta} = \frac{6}{1 + (1/2)\sin\theta}$, so $e = \frac{1}{2}$. The vertices are $(4, \frac{\pi}{2})$ and $(12, \frac{3\pi}{2})$, so $2a = 16$, $a = 8$. $c = ae = \frac{1}{2}\cdot 8 = 4$, so $b = \sqrt{a^2 - c^2} = \sqrt{64 - 16} = 4\sqrt{3}$.



$[-13, 14]$ by $[-13, 5]$

$e = \frac{1}{2}$, $a = 8$, $b = 4\sqrt{3}$, $c = 4$

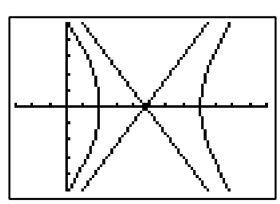
34. $r = \frac{16}{5 + 3\cos\theta} = \frac{16/5}{1 + (3/5)\cos\theta}$, so $e = \frac{3}{5}$. The vertices are $(2, 0)$ and $(8, \pi)$, so $2a = 10$, $a = 5$, $c = ae = 5\left(\frac{3}{5}\right) = 3$, so $b = \sqrt{a^2 - c^2} = \sqrt{25 - 9} = 4$.



$[-10, 5]$ by $[-5, 5]$

$e = 0.6$, $a = 5$, $b = 4$, $c = 3$

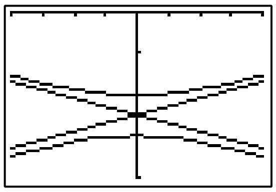
35. $r = \frac{16}{3 + 5\cos\theta} = \frac{16/3}{1 + (5/3)\cos\theta}$, so $e = \frac{5}{3}$. The vertices are $(2, 0)$ and $(-8, \pi)$, so $2a = 6$, $a = 3$, $c = ae = \frac{5}{3}\cdot 3 = 5$ and $b = \sqrt{c^2 - a^2} = \sqrt{25 - 9} = 4$.



$[-3, 12]$ by $[-5, 5]$

$e = \frac{5}{3}$, $a = 3$, $b = 4$, $c = 5$

36. $r = \dfrac{12}{1 - 5\sin\theta}$, so $e = 5$. The vertices are $(-3, \frac{\pi}{2})$ and $\left(2, \frac{3\pi}{2}\right)$, so $2a = 1$, $a = \frac{1}{2}$. $c = ae = 5 \cdot \frac{1}{2} = \frac{5}{2}$ and $b = \sqrt{c^2 - a^2} = \sqrt{\frac{25}{4} - \frac{1}{4}} = \frac{2\sqrt{6}}{2} = \sqrt{6}$.



$[-4, 4]$ by $[-4, 0]$

$e = 5, a = \frac{1}{2}, b = \sqrt{6}, c = \frac{5}{2}$

37. $r = \dfrac{4}{2 - \sin\theta} = \dfrac{2}{1 - (1/2)\sin\theta}$ so $e = \frac{1}{2}$ (an ellipse). The vertices are $\left(4, \frac{\pi}{2}\right)$ and $\left(\frac{4}{3}, \frac{3\pi}{2}\right)$ and the conic is symmetric around $x = 0$, so $x = 0$ is the semi-major axis and $2a = \frac{16}{3}$, so $a = \frac{8}{3}$, $c = ea = \frac{1}{2} \cdot \frac{8}{3} = \frac{4}{3}$ and $b = \sqrt{a^2 - c^2} = \sqrt{\left(\frac{8}{3}\right)^2 - \left(\frac{4}{3}\right)^2} = \frac{4\sqrt{3}}{3}$. The center $(h, k) = \left(0, \frac{12}{3} - \frac{8}{3}\right) = \left(0, \frac{4}{3}\right)$. The equation for the ellipse is

$$\frac{\left(y - \frac{4}{3}\right)^2}{\left(\frac{8}{3}\right)^2} + \frac{(x - 0)^2}{\left(\frac{4\sqrt{3}}{3}\right)^2} = \frac{9\left(y - \frac{4}{3}\right)^2}{64} + \frac{3x^2}{16} = 1$$

38. $r = \dfrac{6}{1 + 2\cos\theta}$, so $e = 2$ (a hyperbola). The vertices are $(2, 0)$ and $(-6, \pi)$ and the function is symmetric about the x-axis, so the semi-major axis runs along $x = 0$. $2a = 4$, $a = 2$, so $c = ea = 2(2) = 4$ and $b = \sqrt{c^2 - a^2} = \sqrt{16 - 4} = 2\sqrt{3}$. The vertex $(h, k) = (4, 0)$. The equation of the hyperbola is

$$\frac{(x - 4)^2}{2^2} - \frac{(y - 0)^2}{(2\sqrt{3})^2} = \frac{(x - 4)^2}{4} - \frac{y^2}{12} = 1$$

39. $r = \dfrac{4}{2 - 2\cos\theta} = \dfrac{2}{1 - \cos\theta}$, so $e = 1$ and $k = \frac{2}{e} = 2$. Since $k = 2p$, $p = 1$ and $4p = 4$, the vertex $(h, k) = (-1, 0)$ and the parabola opens to the right, so the equation is $y^2 = 4(x + 1)$.

40. $r = \dfrac{12}{3 + 3\cos\theta} = \dfrac{4}{1 + \cos\theta}$, so $e = 1$ and $k = \frac{4}{e} = 4$. Since $k = 2p$, $p = 2$ and $4p = 8$, the vertex $(h, k) = (2, 0)$ and the parabola opens to the left, so the equation is $y^2 = -8(x - 2)$.

41. Setting $e = 0.97$ and $a = 18.09$ AU,

$$r = \frac{18.09\,(1 - 0.97^2)}{1 + 0.97\cos\theta}$$

The perihelion of Halley's Comet is

$$r = \frac{18.09\,(1 - 0.97^2)}{1 + 0.97} \approx 0.54 \text{ AU}$$

The aphelion of Halley's Comet is

$$r = \frac{18.09\,(1 - 0.97^2)}{1 - 0.97} \approx 35.64 \text{ AU}$$

42. Setting $e = 0.0461$ and $a = 19.18$,

$$r = \frac{19.18\,(1 - 0.0461^2)}{1 + 0.0461\cos\theta}$$

Uranus' perihelion is $\dfrac{19.18\,(1 - 0.0461^2)}{1 + 0.0461} \approx 18.30$ AU

Uranus' aphelion is $\dfrac{19.18\,(1 - 0.0461^2)}{1 - 0.0461} \approx 20.06$ AU

43. **(a)** The total radius of the orbit is $r = 250 + 1740 = 1990$ km $= 1{,}990{,}000$ m. Then $v \approx \sqrt{2{,}406{,}030} \approx 1551$ m/sec $= 1.551$ km/sec.

(b) The circumference of one orbit is $2\pi r \approx 12503.5$ km; one orbit therefore takes about 8061 seconds, or about 2 hr 14 min.

44. The total radius of the orbit is $r = 1000 + 2100 = 3100$ miles. One mile is about 1.61 km, so $r \approx 4991$ km $= 4{,}991{,}000$ m. Then $v \approx \sqrt{8{,}793{,}800} \approx 2965$ m/sec $= 2.965$ km/sec ≈ 1.843 mi/sec.

45. True. For a circle, $e = 0$. But when $e = 0$, the equation degenerates to $r = 0$, which yields a single point, the pole.

46. True. For a parabola, $e = 1$. But when $e = 1$, the equation degenerates to $r = 0$, which yields a single point, the pole.

47. Conics are defined in terms of the ratio distance to focus : distance to directrix. The answer is D.

48. As the eccentricity increases beginning from zero, the sequence of conics is circle ($e = 0$), ellipse ($e < 1$), parabola ($e = 1$), hyperbola ($e > 1$). The answer is C.

49. Conics written in polar form always have one focus at the pole. The answer is B.

50. $r = 1 + 2\cos\theta$ is a limaçon curve. (See Section 6.5). The answer is A.

51. **(a)** When $\theta = 0$, $\cos\theta = 1$, so $1 + e\cos\theta = 1 + e$. Then $\dfrac{a(1 - e^2)}{1 + e\cos\theta} = \dfrac{a(1 - e^2)}{1 + e} = a(1 - e)$

Similarly, when $\theta = \pi$, $\cos\theta = -1$, so $1 + e\cos\theta = 1 - e$. Then $\dfrac{a(1 - e^2)}{1 + e\cos\theta} = \dfrac{a(1 - e^2)}{1 - e} = a(1 + e)$

(b) $a(1 - e) = a\left(1 - \frac{c}{a}\right) = a - a \cdot \frac{c}{a} = a - c$

$a(1 + e) = a\left(1 + \frac{c}{a}\right) = a + a \cdot \frac{c}{a} = a + c$

(c)

Planet	Perihelion (in Au)	Aphelion
Mercury	0.307	0.467
Venus	0.718	0.728
Earth	0.983	1.017
Mars	1.382	1.665
Jupiter	4.953	5.452
Saturn	9.020	10.090

(d) The difference is greatest for Saturn.

52. $e = 0$ yields a circle (degenerate ellipse); $e = 0.3$ and $e = 0.7$ yield ellipses; $e = 1.5$ and $e = 3$ yield hyperbolas. When $e = 1$, we expect to obtain a parabola. But a has no meaning for a parabola, because a is the center-to-vertex distance and a parabola has no center.
The equation
$r = \dfrac{a(1 - e^2)}{1 + e\cos\theta}$ yields no parabolas. When $e = 1$, $r = 0$.

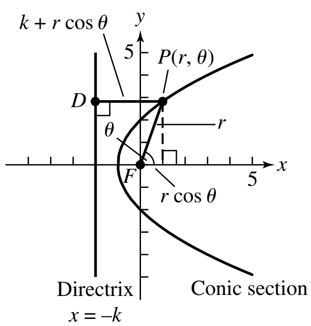
53. If $r < 0$, then the point P can be expressed as the point $(r, \theta + \pi)$ then $PF = r$ and $PD = k - r\cos\theta$.

$$PF = ePD$$
$$r = e(k - r\cos\theta)$$
$$r = \frac{ke}{1 + e\cos\theta}$$

Recall that $P\ (r, \theta)$ can also be expressed as $(-r, \theta - \pi)$ then $PD = -r$ and $PF = -r\cos(\theta - \pi) - k$

$$PD = ePF$$
$$-r = e[-r\cos(\theta - \pi) - k]$$
$$-r = -er\cos(\theta - \pi) - ek$$
$$-r = er\cos\theta - ek$$
$$-r - er\cos\theta = -ek$$
$$r = \frac{ke}{1 + e\cos\theta}$$

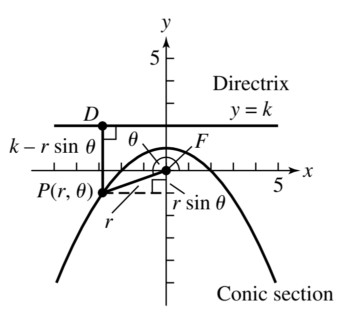
54. (a)



$PF = r$ and $PD = k + r\cos\theta$, so $PF = e\,PD$ becomes

$$r = e(k + r\cos\theta)$$
$$r - er\cos\theta = ek$$
$$r = \frac{ke}{1 - e\cos\theta}$$

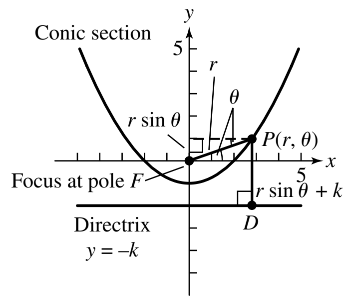
(b)



$PF = r$ and $PD = k - r\sin\theta$, so $PF = e\,PD$ becomes

$$r = e(k - r\sin\theta)$$
$$r + er\sin\theta = ke$$
$$r = \frac{ke}{1 + e\sin\theta}$$

(c)



$PF = r$ and $PD = k + r\sin\theta$, so $PF = e\,PD$ becomes

$$r = e(k + r\sin\theta)$$
$$r - er\sin\theta = ek$$
$$r = \frac{ke}{1 - e\sin\theta}$$

55. Consider the polar equation $r = \dfrac{16}{5 - 3\cos\theta}$. To transform this to a Cartesian equation, rewrite the equation as $5r - 3r\cos\theta = 16$. Then use the substitutions $r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$ to obtain
$5\sqrt{x^2 + y^2} - 3x = 16.$
$5\sqrt{x^2 + y^2} = 3x + 16;\ 25(x^2 + y^2) = 9x^2 + 96x + 256$
$25x^2 + 25y^2 = 9x^2 + 96x + 256$
$16x^2 - 96x + 25y^2 = 256;$
Completing the square on the x term gives
$16(x^2 - 6x + 9) + 25y^2 = 256 + 144$
$16(x - 3)^2 + 25y^2 = 400;$
The Cartesian equation is $\dfrac{(x - 3)^2}{25} + \dfrac{y^2}{16} = 1.$

56. The focal width of a conic is the length of a chord through a focus and perpendicular to the focal axis. If the conic is given by $r = \dfrac{ke}{1 + e\cos\theta}$, the endpoints of the chord occur when $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}$. Thus, the points are $\left(ke, \dfrac{\pi}{2}\right)$ and $\left(ke, \dfrac{3\pi}{2}\right)$ and the length of the chord is $ke + ke = 2ke$.
The focal width of a conic is $2ke$.

57. Apply the formula $e \cdot PD = PF$ to a hyperbola with one focus at the pole and directrix $x = -k$, letting P be the vertex closest to the pole. Then $a + k = c + PD$ and $PF = c - a$. Using $e = \dfrac{c}{a}$, we have:

$$e \cdot PD = PF$$
$$e(a + k - c) = c - a$$
$$e(a + k - ae) = ae - a$$
$$ae + ke - ae^2 = ae - a$$
$$ke - ae^2 = -a$$
$$ke = ae^2 - a$$
$$ke = a(e^2 - 1)$$

Thus, the equation $r = \dfrac{ke}{1 - e\cos\theta}$ becomes $r = \dfrac{a(e^2 - 1)}{1 - e\cos\theta}$.

58. (a) Let $P(x, y)$ be a point on the ellipse. The horizontal distance from P to the point $Q(a^2/c, y)$ on line L is $PQ = a^2/c - x$. The distance to the focus $(c, 0)$ is $PF = \sqrt{(x-c)^2 + y^2} = \sqrt{x^2 - 2cx + c^2 + y^2}$. To confirm that $PF/PQ = c/a$, cross-multiply to get $a\,PF = c\,PQ$; we need to confirm that $a\sqrt{x^2 - 2cx + c^2 + y^2} = a^2 - cx$. Square both sides: $a^2(x^2 - 2cx + c^2 + y^2) = a^4 - 2a^2cx + c^2x^2$. Substitute $a^2 - b^2$ for c^2, multiply out both sides, and cancel out terms, leaving $a^2y^2 - a^2b^2 = -b^2x^2$. Since P is on the ellipse, $x^2/a^2 + y^2/b^2 = 1$, or equivalent $b^2x^2 + a^2y^2 = a^2b^2$; this confirms the equality.

(b) According to the polar definition, the eccentricity is the ratio PF/PQ, which we found to be c/a in (a).

(c) Since $e = c/a$, $a/e = \dfrac{a}{c/a} = a^2/c$ and $ae = c$; the distance from F to L is $a^2/c - c = a/e - ea$ as desired.

59. (a) Let $P(x, y)$ be a point on the hyperbola. The horizontal distance from P to the point $Q(a^2/c, y)$ on line L is $PQ = |a^2/c - x|$. The distance to the focus $(c, 0)$ is $PF = \sqrt{(x-c)^2 + y^2} = \sqrt{x^2 - 2cx + c^2 + y^2}$. To confirm that $PF/PQ = c/a$, cross-multiply to get $a\,PF = c\,PQ$; we need to confirm that $a\sqrt{x^2 - 2cx + c^2 + y^2} = |a^2 - cx|$. Square both sides: $a^2(x^2 - 2cx + c^2 + y^2) = a^4 - 2a^2cx + c^2x^2$. Substitute $a^2 + b^2$ for c^2, multiply out both sides, and cancel out terms, leaving $a^2y^2 + a^2b^2 = b^2x^2$. Since P is on the hyperbola, $x^2/a^2 - y^2/b^2 = 1$, or equivalent $b^2x^2 - a^2y^2 = a^2b^2$; this confirms the equality.

(b) According to the polar definition, the eccentricity is the ratio PF/PQ, which we found to be c/a in (a).

(c) Since $e = c/a$, $a/e = \dfrac{a}{c/a} = a^2/c$ and $ae = c$; the distance from F to L is $c - \dfrac{a^2}{c} = ea - \dfrac{a}{e}$ as desired.

■ Section 8.6 Three-Dimensional Cartesian Coordinates

Quick Review 8.6

1. $\sqrt{(x-2)^2 + (y+3)^2}$

2. $\left(\dfrac{x+2}{2}, \dfrac{y-3}{2}\right)$

3. P lies on the circle of radius 5 centered at $(2, -3)$

4. $|\mathbf{v}| = \sqrt{(-4)^2 + (5)^2} = \sqrt{41}$

5. $\dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{\langle -4, 5\rangle}{\sqrt{41}} = \left\langle \dfrac{-4}{\sqrt{41}}, \dfrac{5}{\sqrt{41}} \right\rangle$

6. $\dfrac{-7 \cdot \mathbf{v}}{|\mathbf{v}|} = \dfrac{\langle 28, -35\rangle}{\sqrt{41}} = \left\langle \dfrac{28}{\sqrt{41}}, \dfrac{-35}{\sqrt{41}} \right\rangle$

7. Circle of radius 5 centered at $(-1, 5)$

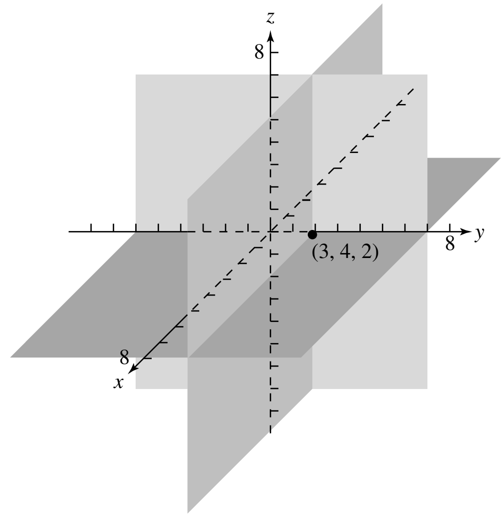
8. A line of slope -2, passing through $(2, -4)$

9. $(x+1)^2 + (y-3)^2 = 4$. Center: $(-1, 3)$, radius: 2

10. $\langle -1 - 2, -4 - 5\rangle, = \langle -3, -9\rangle$

Section 8.6 Exercises

1.



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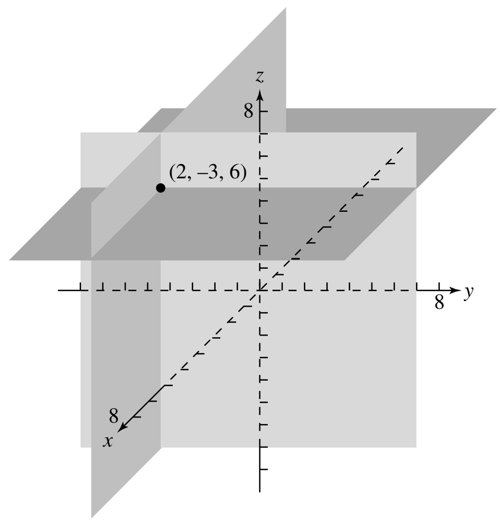

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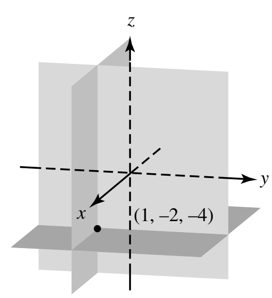

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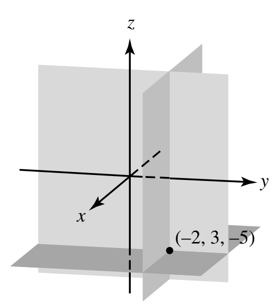


5. $\sqrt{(3-(-1))^2+(-4-2)^2+(6-5)^2}=\sqrt{53}$

6. $\sqrt{(6-2)^2+(-3-(-1))^2+(4-(-8))^2}=2\sqrt{41}$

7. $\sqrt{(a-1)^2+(b-(-3))^2+(c-2)^2}$
$=\sqrt{(a-1)^2+(b+3)^2+(c-2)^2}$

8. $\sqrt{(x-p)^2+(y-q)^2+(z-r)^2}$

9. $\left(\frac{3-1}{2},\frac{-4+2}{2},\frac{6+5}{2}\right)=\left(1,-1,\frac{11}{2}\right)$

10. $\left(\frac{2+6}{2},\frac{-1-3}{2},\frac{-8+4}{2}\right)=(4,-2,-2)$

11. $\left(\frac{2x-2}{2},\frac{2y+8}{2},\frac{2z+6}{2}\right)=(x-1,y+4,z+3)$

12. $\left(\frac{3a-a}{2},\frac{3b-b}{2},\frac{3c-c}{2}\right)=(a,b,c)$

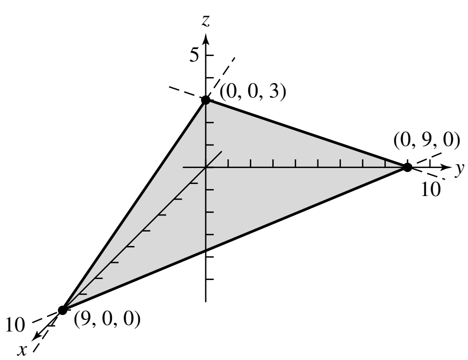
13. $(x-5)^2+(y+1)^2+(z+2)^2=64$

14. $(x+1)^2+(y-5)^2+(z-8)^2=5$

15. $(x-1)^2+(y+3)^2+(z-2)^2=a$

16. $(x-p)^2+(y-q)^2+(z-r)^2=36$

17.



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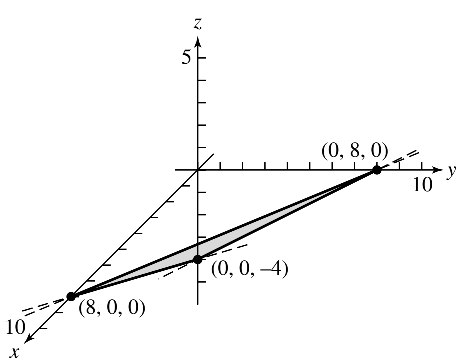

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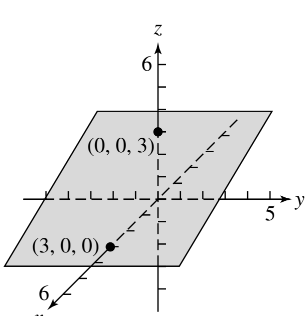

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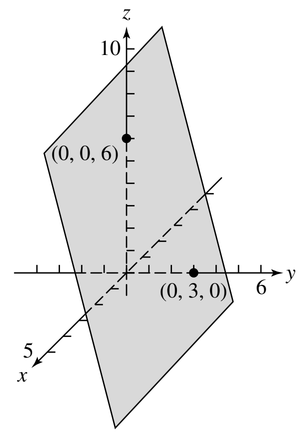

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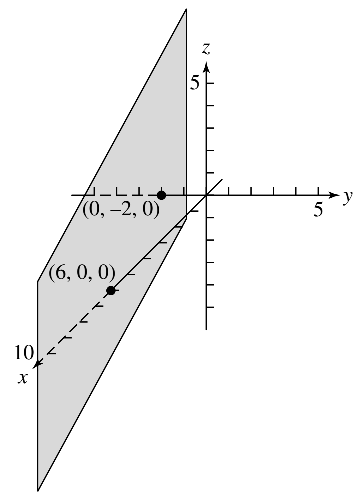

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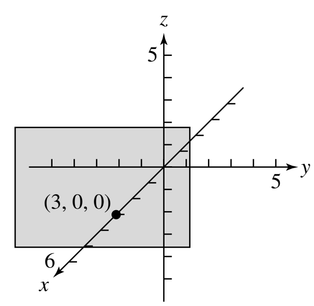


23. $\mathbf{r}+\mathbf{v}=\langle 1,0,-3\rangle+\langle -3,4,-5\rangle=\langle -2,4,-8\rangle$

24. $\mathbf{r}-\mathbf{w}=\langle 1,0,-3\rangle-\langle 4,-3,12\rangle=\langle -3,3,-15\rangle$

25. $\mathbf{v}\cdot\mathbf{w}=-12-12-60=-84$

26. $|\mathbf{w}|=\sqrt{4^2+(-3)^2+12^2}=13$

27. $\mathbf{r}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{r}\cdot(\langle -3,4,-5\rangle+\langle 4,-3,12\rangle)$
$=\langle 1,0,-3\rangle\cdot\langle 1,1,7\rangle=1+0-21=-20$

28. $\mathbf{r}\cdot\mathbf{v}+\mathbf{r}\cdot\mathbf{w}=(-3+0+15)+(4+0-36)=-20$

29. $\frac{\mathbf{w}}{|\mathbf{w}|}=\frac{\langle 4,-3,12\rangle}{\sqrt{4^2+(-3)^2+12^2}}=\left\langle\frac{4}{13},-\frac{3}{13},\frac{12}{13}\right\rangle$

30. $\mathbf{i}\cdot\mathbf{r}=\langle 1,0,0\rangle\cdot\langle 1,0,-3\rangle=1$

31. $\langle \mathbf{i}\cdot\mathbf{v},\mathbf{j}\cdot\mathbf{v},\mathbf{k}\cdot\mathbf{v}\rangle=\langle -3,4,-5\rangle$

32. $(\mathbf{r}\cdot\mathbf{v})\mathbf{w}=(\langle 1,0,-3\rangle\cdot\langle -3,4,-5\rangle)\langle 4,-3,12\rangle$
$=(-3+0+15)\langle 4,-3,12\rangle=\langle 48,-36,144\rangle$

33. The plane's velocity relative to the air is
$\mathbf{v}_1 = -200 \cos 20° \mathbf{i} + 200 \sin 20° \mathbf{k}$
The air's velocity relative to the ground is
$\mathbf{v}_2 = -10 \cos 45° \mathbf{i} - 10 \sin 45° \mathbf{j}$
Adding these two vectors and converting to decimal values rounded to two places produces the plane's velocity relative to the ground:
$\mathbf{v} = -195.01\,\mathbf{i} - 7.07\,\mathbf{j} + 68.40\,\mathbf{k}$

34. The rocket's velocity relative to the air is
$\mathbf{v}_1 = 12{,}000 \cos 80° \mathbf{i} + 12{,}000 \sin 80° \mathbf{k}$
The air's velocity relative to the ground is
$\mathbf{v}_2 = 8 \cos 45° \mathbf{i} + 8 \sin 45° \mathbf{j}$
Adding these two vectors and converting to decimal values rounded to two places produces the rocket's velocity relative to the ground:
$\mathbf{v} = 2089.43\,\mathbf{i} + 5.66\,\mathbf{j} + 11{,}817.69\,\mathbf{k}$

For #35–38, the vector form is $\mathbf{r}_0 + t\mathbf{v}$ with $\mathbf{r}_0\langle x_0, y_0, z_0\rangle$, and the parametric form is $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$ where $\mathbf{v} = \langle a, b, c\rangle$.

35. Vector form: $\mathbf{r} = \langle 2, -1, 5\rangle + t\langle 3, 2, -7\rangle$; parametric form: $x = 2 + 3t$, $y = -1 + 2t$, $z = 5 - 7t$

36. Vector form: $\mathbf{r} = \langle -3, 8, -1\rangle + t\langle -3, 5, 2\rangle$; parametric form: $x = -3 - 3t$, $y = 8 + 5t$, $t = -1 + 2t$

37. Vector form: $\mathbf{r} = \langle 6, -9, 0\rangle + t\langle 1, 0, -4\rangle$; parametric form: $x = 6 + t$, $y = -9$, $z = -4t$

38. Vector form: $\mathbf{r} = \langle 0, -1, 4\rangle + t\langle 0, 0, 1\rangle$; parametric form: $x = 0$, $y = -1$, $z = 4 + t$

39. Midpoint of $\overline{BC}$: $\langle 1, 1, -1\rangle$. Distance from A to midpoint of $\overline{BC}$:
$\sqrt{(-1-1)^2 + (2-1)^2 + (4-(-1))^2} = \sqrt{30}$

40. $\langle 1 - (-1), 1 - 2, -1 - 4\rangle = \langle 2, -1, -5\rangle$

41. Direction vector: $\langle 0 - (-1), 6 - 2, -3 - 4\rangle$
$= \langle 1, 4, -7\rangle$, $\overrightarrow{OA} = \langle -1, 2, 4\rangle$,
$\mathbf{r} = \langle -1, 2, 4\rangle + t\langle 1, 4, -7\rangle$

42. Direction vector: $\langle 2, -1, -5\rangle$ (from #34). The vector equation of the line is $\mathbf{r} = \langle -1, 2, 4\rangle + t\langle 2, -1, -5\rangle$.

43. Direction vector: $\langle 2 - (-1), -4 - 2, 1 - 4\rangle$
$= \langle 3, -6, -3\rangle$, $\overrightarrow{OA} = \langle -1, 2, 4\rangle$, so a vector equation of the line is $\mathbf{r} = \langle -1, 2, 4\rangle + t\langle 3, -6, -3\rangle$
$= \langle -1 + 3t, 2 - 6t, 4 - 3t\rangle$. This can be expressed in parametric form: $x = -1 + 3t$, $y = 2 - 6t$, $z = 4 - 3t$.

44. Direction vector: $\langle 2 - 0, -4 - 6, 1 - (-3)\rangle$
$= \langle 2, -10, 4\rangle$, $\overrightarrow{OB} = \langle 0, 6, -3\rangle$ so a vector equation of the line is $\mathbf{r} = \langle 0, 6, -3\rangle + t\langle 2, -10, 4\rangle$
$= \langle 2t, 6 - 10t, -3 + 4t\rangle$. This can be expressed in parametric form: $x = 2t$, $y = 6 - 10t$, $z = -3 + 4t$.

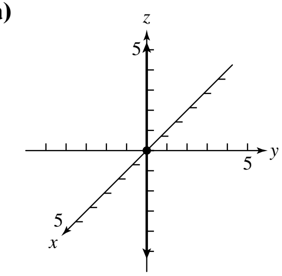
45. Midpoint of $\overline{AC}$: $\left(\frac{1}{2}, -1, \frac{5}{2}\right)$. Direction vector:
$\left\langle \frac{1}{2} - 0, -1 - 6, \frac{5}{2} - (-3)\right\rangle = \left\langle \frac{1}{2}, -7, \frac{11}{2}\right\rangle$,
$\overrightarrow{OB} = \langle 0, -6, -3\rangle$, so a vector equation of the line is
$\mathbf{r} = \langle 0, 6, -3\rangle + t\left\langle \frac{1}{2}, -7, \frac{11}{2}\right\rangle$
$= \left\langle \frac{1}{2}t, 6 - 7t, -3 + \frac{11}{2}t\right\rangle$. This can be expressed in parametric form: $x = \frac{1}{2}t$, $y = 6 - 7t$, $z = -3 + \frac{11}{2}t$.

46. Midpoint of $\overline{AB}$: $\left(-\frac{1}{2}, 4, \frac{1}{2}\right)$. Direction vector:
$\left\langle -\frac{1}{2} - 2, 4 - (-4), \frac{1}{2} - 1\right\rangle = \left\langle -\frac{5}{2}, 8, -\frac{1}{2}\right\rangle$,
$\overrightarrow{OC} = \langle 2, -4, 1\rangle$, so a vector equation of the line is
$\mathbf{r} = \langle 2, -4, 1\rangle + t\left\langle -\frac{5}{2}, 8, -\frac{1}{2}\right\rangle$
$= \left\langle 2 - \frac{5}{2}t, -4 + 8t, 1 - \frac{1}{2}t\right\rangle$. This can be expressed in parametric form: $x = 2 - \frac{5}{2}t$, $y = -4 + 8t$, $z = 1 - \frac{1}{2}t$.

47. The length of $\overline{AB} =$
$\sqrt{(0 - (-1))^2 + (6 - 2)^2 + (-3 - 4)^2} = \sqrt{66}$; the length of $\overline{BC} =$
$\sqrt{(2 - 0)^2 + (-4 - 6)^2 + (1 - (-3))^2} = 2\sqrt{30}$; the length of $\overline{AC} =$
$\sqrt{(2 - (-1))^2 + (-4 - 2)^2 + (1 - 4)^2} = \sqrt{54}$. The triangle ABC is scalene.

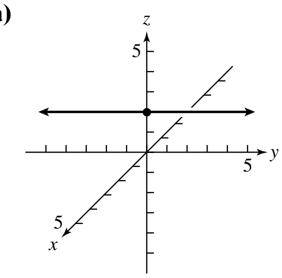
48. $M = (1, 1, -1)$ (from #33). The midpoint of $\overline{AM} = \left(0, \frac{3}{2}, \frac{3}{2}\right)$

49. (a)



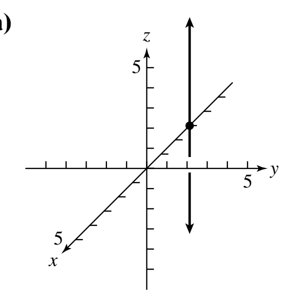
(b) the z-axis; a line through the origin in the direction $\mathbf{k}$.

50. (a)



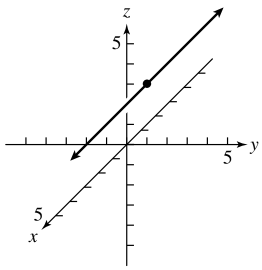
(b) the intersection of the yz plane (at $x = 0$) and xy plane (at $z = 2$); a line parallel to the y-axis through $(0, 0, 2)$

51. (a)



(b) the intersection of the xz plane (at $y = 0$) and yz plane (at $x = -3$); a line parallel to the z-axis through $(-3, 0, 0)$

52. (a)

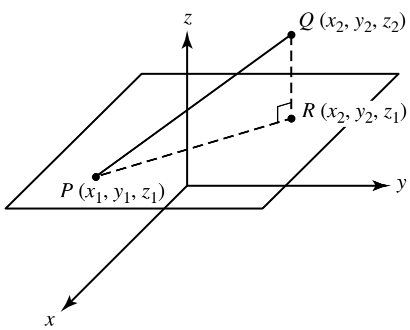


(b) the intersection of the xz plane (at $y = 1$) and xy plane (at $z = 3$); a line through $(0, 1, 3)$ parallel to the x-axis

53. Direction vector: $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$, $\overrightarrow{OP} = \langle x_1, y_2, z_3 \rangle$, so a vector equation of the line is $\mathbf{r} = \langle x_1 + (x_2 - x_1)t, y_1 + (y_2 - y_1)t, z_1 + (z_2 - z_1)t \rangle$.

54. Using the result from Exercise 49, the parametric equations are $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $z = z_1 + (z_2 - z_1)t$.

55.



By the Pythagorean Theorem,

$$d(P, Q) = \sqrt{(d(P, R))^2 + (d(R, Q)^2)}$$

$$= \sqrt{(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2})^2 + (|z_1 - z_2|)^2}$$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

56. Let $\mathbf{u} = \langle x_1, y_1, z_1 \rangle$. Then

$$\mathbf{u} \cdot \mathbf{u} = x_1^2 + y_1^2 + z_1^2$$

$$= (\sqrt{x_1^2 + y_1^2 + z_1^2})^2$$

$$= |\mathbf{u}|^2$$

57. True. This is the equation of a vertical elliptic cylinder. The equation can be viewed as an equation in three variables, where the coefficient of z is zero.

58. False. Because the coefficient of t is always 0, the equations simplify to $x = 1$, $y = 2$, $z = -5$; these represent the point $(1, 2, -5)$.

59. The general form for a first-degree equation in three variables is $Ax + By + Cz + D = 0$. The answer is B.

60. The equation for a plane is first-degree, or linear; there are no squared terms. The answer is A.

61. The dot product of two vectors is a scalar. The answer is C.

62. The conversion to parametric form begins with $x = 2 + 1t$, $y = -3 + 0t$, $z = 0 - 1t$. The answer is E.

63. (a) Each cross-section is its own ellipse.

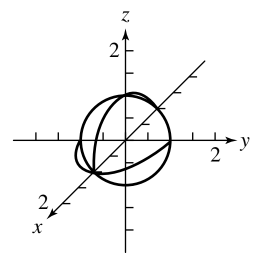
$x = 0$: $\dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$, an ellipse centered at $(0, 0)$ (in the yz plane) of "width" 4 and "height" 8.

$y = 0$: $\dfrac{x^2}{9} + \dfrac{z^2}{16} = 1$, an ellipse centered at $(0, 0)$ (in the xz plane) of "width" 6 and "height" 8.

$z = 0$: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, an ellipse centered at $(0, 0)$ (in the xy plane) of "width" 6 and "height" 4.

(b) Algebraically, $z = \sqrt{1 - x^2 - y^2}$ has only positive values; $0 \le z \le 1$ and the "bottom" half of the sphere is never formed. The equation of the whole sphere is $x^2 + y^2 + z^2 = 1$.

(c)



(d) A sphere is an ellipsoid in which all of the $x = 0$, $y = 0$, and $z = 0$ "slices" (i.e., the cross-sections of the coordinate planes) are circles. Since a circle is a degenerate ellipse, it follows that a sphere is a degenerate ellipsoid.

64. (a) Since **i** points east and **j** points north, we determine that the compass bearing θ is

$$0 = 90° - \tan^{-1}\left(\frac{22.63}{193.88}\right) \approx 90 - 6.66 = 83.34°.$$

(Recall that $\tan\theta$ refers to the x-axis (east) being located at 0°; if the y-axis (north) is 0°, we must adjust our calculations accordingly.)

(b) The speed along the ground is $\sqrt{(193.88)^2 + (22.63)^2} \approx 195.2$ mph

(c) The tangent of the climb angle is the vertical speed divided by the horizontal speed, so

$$\theta \approx \tan^{-1}\frac{125}{195.2}$$

$$\approx 32.63°$$

(d) The overall speed is $\sqrt{(193.88)^2 + (22.63)^2 + (125)^2} \approx 231.8$ mph

65. $\langle 2 - 3, -6 + 1, 1 - 4 \rangle = \langle -1, -5, -3 \rangle$

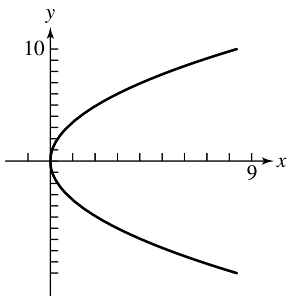
66. $\langle -2 + 6, 2 - 8, -12 + 1 \rangle = \langle 4, -6, -11 \rangle$

67. $\mathbf{i} \times \mathbf{j} = \langle 1, 0, 0 \rangle \times \langle 0, 1, 0 \rangle = \langle 0 - 0, 0 - 0, 1 - 0 \rangle = \langle 0, 0, 1 \rangle = \mathbf{k}$

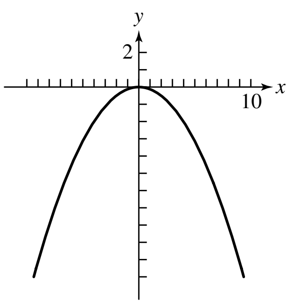
68. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) =$
$\langle u_1, u_2, u_3 \rangle \cdot \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$
$= (u_1u_2v_3 - u_1u_3v_2) + (u_2u_3v_1 - u_1u_2v_3)$
$+ (u_1u_3v_2 - u_2u_3v_1)$
$= 0$
$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) =$
$\langle v_1, v_2, v_3 \rangle \cdot \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$
$= (u_2v_1v_3 - u_3v_1v_2) + (u_3v_1v_2 - u_1v_2v_3)$
$+ (u_1v_2v_3 - u_2v_1v_3)$
$= 0$
So the angles between **u** and **u** × **v**, and **v** and **u** × **v**, both have a cosine of zero by the theorem in Section 6.2. It follows that the angles both measure 90°.

■ Chapter 8 Review

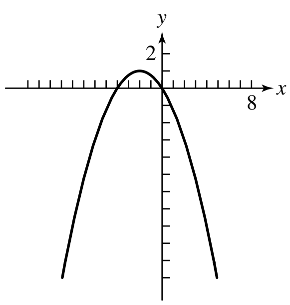
1. $h = 0, k = 0, 4p = 12$, so $p = 3$.
Vertex: (0, 0), focus: (3, 0), directrix: $x = -3$,
focal width: 12



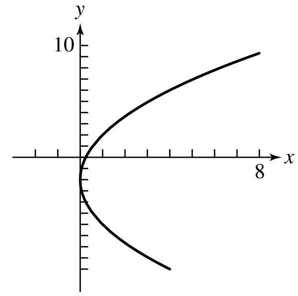
2. $h = 0, k = 0, 4p = -8$, so $p = -2$.
Vertex: (0, 0), focus: (0, −2), directrix: $y = 2$,
focal width: 8



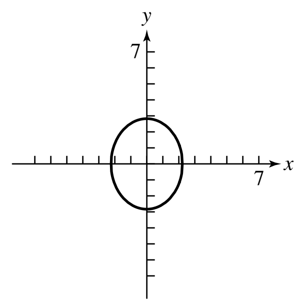
3. $h = -2, k = 1, 4p = -4$, so $p = -1$.
Vertex: (−2, 1), focus: (−2, 0), directrix: $y = 2$,
focal width: 4



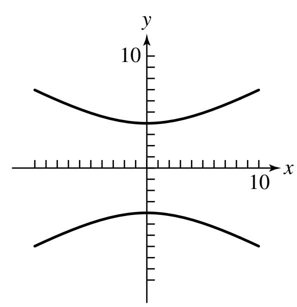
4. $h = 0, k = -2, 4p = 16$, so $p = 4$.
Vertex: (0, −2), focus: (4, −2), directrix: $x = -4$,
focal width: 16



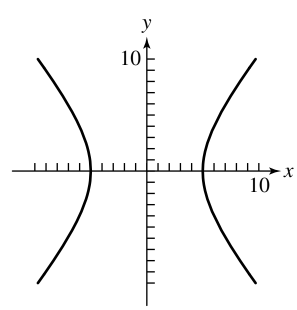
5. Ellipse. Center (0, 0). Vertices: $(0, \pm 2\sqrt{2})$. Foci: $(0, \pm\sqrt{3})$ since $c = \sqrt{8 - 5} = \sqrt{3}$.



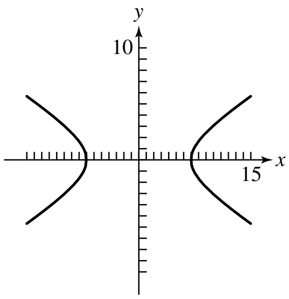
6. Hyperbola. Center: (0, 0). Vertices: (0, ±4).
Foci: $(0, \pm\sqrt{65})$ since $c = \sqrt{16 + 49} = \sqrt{65}$.



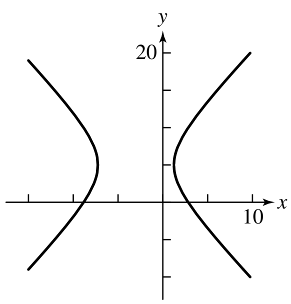
7. Hyperbola. Center: (0, 0). Vertices: (±5, 0),
$c = \sqrt{a^2 + b^2} = \sqrt{25 + 36} = \sqrt{61}$, so the foci are:
$(\pm\sqrt{61}, 0)$



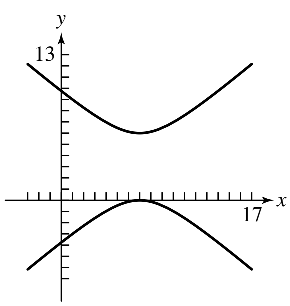
8. Hyperbola. Center: $(0, 0)$. Vertices: $(\pm 7, 0)$, $c = \sqrt{a^2 + 6^2} = \sqrt{49 + 9} = \sqrt{58}$, so the foci are: $(\pm\sqrt{58}, 0)$



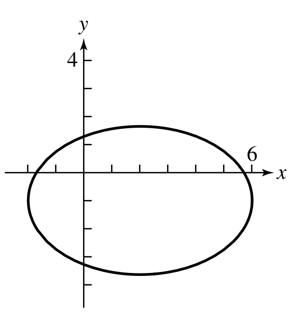
9. Hyperbola. Center: $(-3, 5)$. Vertices: $(-3 \pm 3\sqrt{2}, 5)$, $c = \sqrt{a^2 + b^2} = \sqrt{18 + 28} = \sqrt{46}$, so the foci are: $(-3 \pm \sqrt{46}, 5)$



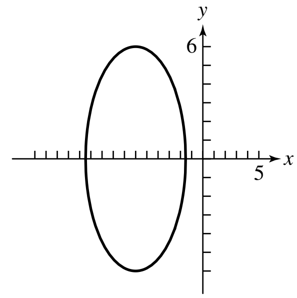
10. Hyperbola. Center: $(7, 3)$. Vertices: $(7, 3 \pm 3) = (7, 0)$ and $(7, 6)$, $c = \sqrt{a^2 + b^2} = \sqrt{9 + 12} = \sqrt{21}$, so the foci are: $(7, 3 \pm \sqrt{21})$



11. Ellipse. Center: $(2, -1)$. Vertices: $(2 \pm 4, -1) = (6, -1)$ and $(-2, -1)$, $c = \sqrt{a^2 - b^2} = \sqrt{16 - 7} = 3$, so the foci are: $(2 \pm 3, -1) = (5, -1)$ and $(-1, -1)$



12. Ellipse. Center: $(-6, 0)$. Vertices: $(-6, \pm 6)$ $c = \sqrt{a^2 - b^2} = \sqrt{36 - 20} = 4$, so the foci are: $(-6, \pm 4)$



13. (b)

14. (g)

15. (h)

16. (e)

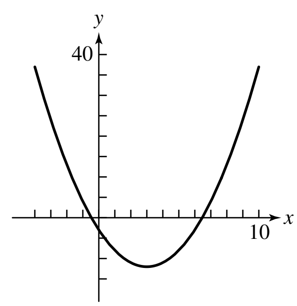
17. (f)

18. (d)

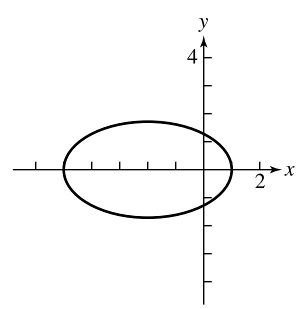
19. (c)

20. (a)

21. $B^2 - 4AC = 0 - 4(1)(0) = 0$, parabola $(x^2 - 6x + 9) = y + 3 + 9$, so $(x - 3)^2 = y + 12$

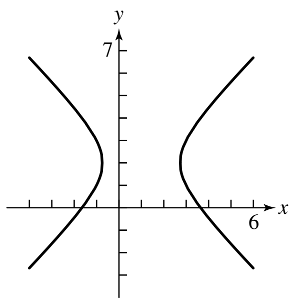


22. $B^2 - 4AC = 0 - 4(1)(3) = -12 < 0$, ellipse $(x^2 + 4x + 4) + 3y^2 = 5 + 4$, so $\dfrac{(x + 2)^2}{9} + \dfrac{y^2}{3} = 1$

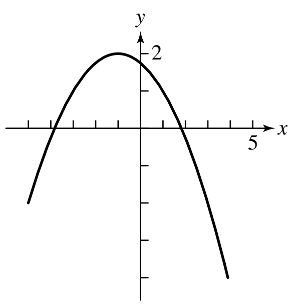


23. $B^2 - 4AC = 0 - 4(1)(-1) = 4 > 0$, hyperbola $(x^2 - 2x + 1) - (y^2 - 4y + 4) = 1 - 4 + 6$

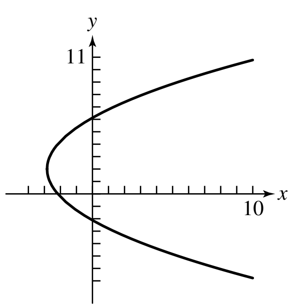
$$\frac{(x-1)^2}{3} - \frac{(y-2)^2}{3} = 1$$



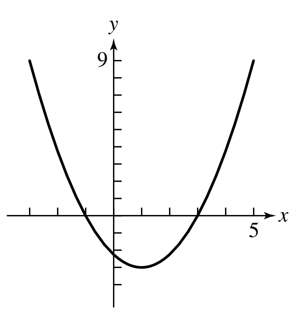
24. $B^2 - 4AC = 0 - 4(1)(0) = 0$, parabola $(x^2 + 2x + 1) = -4y + 7 + 1$, so $(x + 1)^2 = -4(y - 2)$



25. $B^2 - 4AC = 0 - 4(1)(0) = 0$, parabola $(y^2 - 4y + 4) = 6x + 13 + 4$, so $(y-2)^2 = 6\left(x + \frac{17}{6}\right)$

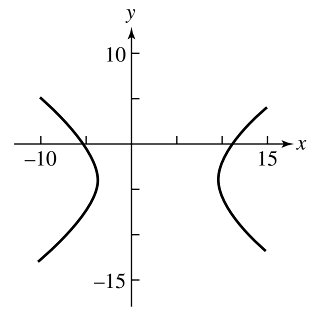


26. $B^2 - 4AC = 0 - 4(3)(0) = 0$, parabola $3(x^2 - 2x + 1) = 4y + 9 + 3$, so $(x - 1)^2 = \frac{4}{3}(y + 3)$



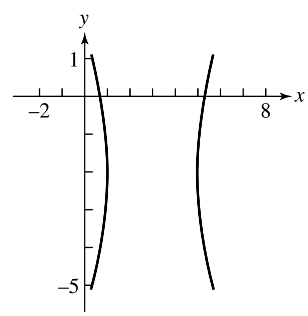
27. $B^2 - 4AC = 0 - 4(2)(-3) = 24 > 0$, hyperbola $2(x^2 - 6x + 9) - 3(y^2 + 8y + 16) = 18 - 48 - 60$, so

$$\frac{(y+4)^2}{30} - \frac{(x-3)^2}{45} = 1$$



28. $B^2 - 4AC = 0 - 4(12)(-4) = 192 > 0$, hyperbola $12(x^2 - 6x + 9) - 4(y^2 + 4y + 4) = 108 - 16 - 44$, so

$$\frac{(x-3)^2}{4} - \frac{(y+2)^2}{12} = 1$$



29. By definition, every point $P(x, y)$ that lies on the parabola is equidistant from the focus to the directrix. The distance between the focus and point P is: $\sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}$, while the distance between the point P and the line $y = -p$ is: $\sqrt{(x-x)^2 + (y+p)^2} = \sqrt{(y+p)^2}$. Setting these equal:

$$\begin{aligned} \sqrt{x^2 + (y-p)^2} &= y + p \\ x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 &= 4py \end{aligned}$$

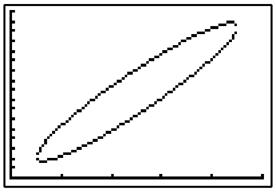
30. Let the point $P(x, y)$ satisfy $y^2 = 4px$. Then we have

$$\begin{aligned} y^2 &= 4px \\ x^2 - 2px + p^2 + y^2 &= x^2 + 2px + p^2 \\ (x-p)^2 + y^2 &= (x+p)^2 + 0 \\ (x-p)^2 + (y-0)^2 &= (x-(-p))^2 + (y-y)^2 \\ \sqrt{(x-p)^2 + (y-0)^2} &= \sqrt{(x-(-p))^2 + (y-y)^2} \end{aligned}$$

distance from $P(x, y)$ to $(p, 0)$ = distance from $P(x, y)$ to $x = -p$

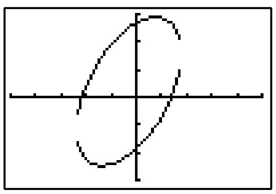
Because $P(x, y)$ is equidistant from the point $(p, 0)$ and the line $x = -p$, by the definition of a parabola, $y^2 = 4px$ is the equation of a parabola with focus $(p, 0)$ and directrix $x = -p$.

31. Use the quadratic formula with $a = 6$, $b = -8x - 5$, and $c = 3x^2 - 5x + 20$. Then $b^2 - 4ac = (-8x - 5)^2 - 24(3x^2 - 5x + 20) = -8x^2 + 200x - 455$, and $y = \frac{1}{12}\left[8x + 5 \pm \sqrt{-8x^2 + 200x - 455}\right]$ – an ellipse



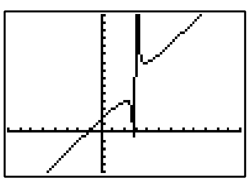
$[0, 25]$ by $[0, 17]$

32. Use the quadratic formula with $a = 6$, $b = -8x - 5$, and $c = 10x^2 + 8x - 30$. Then $b^2 - 4ac = (-8x - 5)^2 - 24(10x^2 + 8x - 30) = -176x^2 - 112x + 745$, and $y = \frac{1}{12}\left[8x + 5 \pm \sqrt{-176x^2 - 112x + 745}\right]$ an ellipse



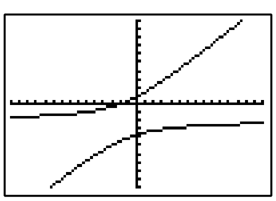
$[-5, 5]$ by $[-3, 3]$

33. This is a *linear* equation in y: $(6 - 2x)y + (3x^2 - 5x - 10) = 0$. Subtract $3x^2 - 5x - 10$ and divide by $6 - 2x$, and we have $y = \frac{3x^2 - 5x - 10}{2x - 6}$ — a hyperbola.



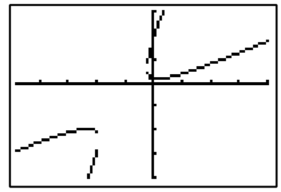
$[-8, 12]$ by $[-5, 15]$

34. Use the quadratic formula with $a = -6$, $b = 5x - 17$, and $c = 10x + 20$. Then $b^2 - 4ac = (5x - 17)^2 + 24(10x + 20) = 25x^2 + 70x + 769$, and $y = \frac{1}{12}\left[5x - 17 \pm \sqrt{25x^2 + 70x + 769}\right]$ a hyperbola.



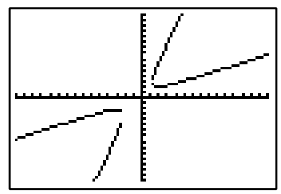
$[-15, 15]$ by $[-10, 10]$

35. Use the quadratic formula with $a = -2$, $b = 7x + 20$, and $c = -3x^2 - x - 15$. Then $b^2 - 4ac = (7x + 20)^2 + 8(-3x^2 - x - 15) = 25x^2 + 272x + 280$, and $y = \frac{1}{4}\left[7x + 20 \pm \sqrt{25x^2 + 272x + 280}\right]$ a hyperbola.



$[-24, 20]$ by $[-20, 15]$

36. Use the quadratic formula with $a = -2$, $b = 7x + 3$, and $c = -3x^2 - 2x - 10$. Then $b^2 - 4ac = (7x + 3)^2 + 8(-3x^2 - 2x - 10) = 25x^2 + 26x - 71$, and $y = \frac{1}{4}\left[7x + 3 \pm \sqrt{25x^2 + 26x - 71}\right]$ a hyperbola.



$[-15, 15]$ by $[-15, 15]$

37. $h = 0$, $k = 0$, $p = 2$, and the parabola opens to the right as $y^2 = 8x$.

38. $h = 0$, $k = 0$, $|4p| = 12$, and the parabola opens downward, so $x^2 = -12y$ $(p = -3)$.

39. $h = -3$, $k = 3$, $p = k - y = 3 - 0 = 3$ (since $y = 0$ is the directrix) the parabola opens upward, so $(x + 3)^2 = 12(y - 3)$.

40. $h = 1$, $k = -2$, $p = 2$ (since the focal length is 2), and the parabola opens to the left, so $(y + 2)^2 = -8(x - 1)$.

41. $h = 0$, $k = 0$, $c = 12$ and $a = 13$, so $b = \sqrt{a^2 - c^2} = \sqrt{169 - 144} = 5$. $\frac{x^2}{169} + \frac{y^2}{25} = 1$

42. $h = 0$, $k = 0$, $c = 2$ and $a = 6$, so $b = \sqrt{a^2 - c^2} = \sqrt{36 - 4} = 4\sqrt{2}$. $\frac{y^2}{36} + \frac{x^2}{32} = 1$

43. $h = 0$, $k = 2$, $a = 3$, $c = 2 - h$ (so $c = 2$) and $b = \sqrt{a^2 - c^2} = \sqrt{9 - 4} = \sqrt{5}$
$\frac{x^2}{9} + \frac{(y - 2)^2}{5} = 1$

44. $h = -3$, $k = -4$, $a = 4$, $0 = -3 \pm c$, $c = 3$, $b = \sqrt{a^2 - c^2} = \sqrt{16 - 9} = \sqrt{7}$, so
$\frac{(x + 3)^2}{16} + \frac{(y + 4)^2}{7} = 1$

45. $h = 0$, $k = 0$, $c = 6$, $a = 5$, $b = \sqrt{c^2 - a^2} = \sqrt{36 - 25} = \sqrt{11}$, so
$\frac{y^2}{25} - \frac{x^2}{11} = 1$

46. $h = 0$, $k = 0$, $a = 2$, $\frac{b}{a} = 2$ $(b = 4)$, so $\frac{x^2}{4} - \frac{y^2}{16} = 1$

47. $h = 2, k = 1, a = 3, \frac{b}{a} = \frac{4}{3} \left(b = \frac{4}{3} \cdot 3 = 4\right)$, so
$$\frac{(x-2)^2}{9} - \frac{(y-1)^2}{16} = 1$$

48. $h = -5, k = 0, c - k = 3\ (c = 3), a - k = 2\ (a = 2)$, $b = \sqrt{c^2 - a^2} = \sqrt{9-4} = \sqrt{5}$, so
$$\frac{y^2}{4} - \frac{(x+5)^2}{5} = 1$$

49. $\frac{x}{5} = \cos t$ and $\frac{y}{2} = \sin t$, so $\frac{x^2}{25} + \frac{y^2}{4} = 1$ — an ellipse.

50. $\frac{x}{4} = \sin t$ and $\frac{y}{6} = \cos t$, so $\frac{x^2}{16} + \frac{y^2}{36} = 1$ — an ellipse.

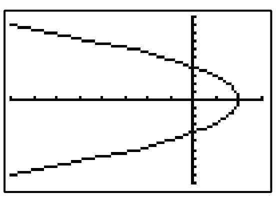
51. $x + 2 = \cos t$ and $y - 4 = \sin t$, so $(x+2)^2 + (y-4)^2 = 1$ — an ellipse (a circle).

52. $\frac{x-5}{3} = \cos t$ and $\frac{y+3}{3} = \sin t$, so $\frac{(x-5)^2}{9} + \frac{(y+3)^2}{9} = 1$, or $(x-5)^2 + (y+3)^2 = 9$ — an ellipse (a circle).

53. $\frac{x}{3} = \sec t$ and $\frac{y}{5} = \tan t$, so $\frac{x^2}{9} - \frac{y^2}{25} = 1$ — a hyperbola.

54. $\frac{x}{4} = \sec t$ and $\frac{y}{3} = \tan t$, so $\frac{x^2}{16} - \frac{y^2}{9} = 1$ — a hyperbola.

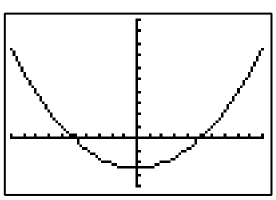
55.



$[-8, 3]$ by $[-10, 10]$

Parabola with vertex at $(2, 0)$, so $h = 2, k = 0, e = 1$. The graph crosses the y-axis, so $(4, \frac{\pi}{2}) = (0, 4)$ lies on the parabola. Substituting $(0, 4)$ into $y^2 = 4p(x-2)$ we have $16 = 4p(-2)$, $p = -2$. $y^2 = -8(x-2)$.

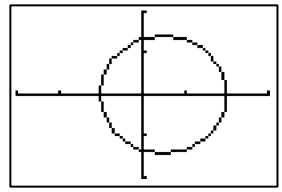
56.



$[-10, 10]$ by $[-4, 10]$

$e = 1$, so a parabola. The vertex is $(h, k) = (0, -\frac{5}{2})$ and the point $(5, 0)$ lies on the curve. Substituting $(5, 0)$ into $x^2 = 4p\left(y + \frac{5}{2}\right)$, we have $25 = 4p\left(\frac{5}{2}\right)$, $p = \frac{5}{2}$.
$$x^2 = 10\left(y + \frac{5}{2}\right)$$

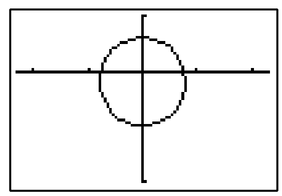
57.



$[-3, 3]$ by $[-2, 2]$

$e = \frac{1}{3}$, so an ellipse. In polar coordinates the vertices are $(2, 0)$ and $(1, \pi)$. Converting to Cartesian we have $(2, 0)$ and $(-1, 0)$, so $2a = 3$, $a = \frac{3}{2}$, $c = ea = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$ and the center $(h, k) = \left(2 - \frac{3}{2}, 0\right) = \left(\frac{1}{2}, 0\right)$ (since it's symmetric about the polar x-axis). Solving for
$$b = \sqrt{a^2 - c^2} = \sqrt{\left(\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{8}{4}} = \sqrt{2}$$
$$\frac{4\left(x - \frac{1}{2}\right)^2}{9} + \frac{y^2}{2} = 1$$

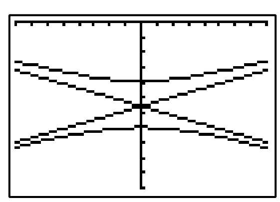
58.



$[-2.3, 2.3]$ by $[-2, 1]$

$e = \frac{1}{4}$, so an ellipse. In polar coordinates the vertices are $\left(\frac{3}{5}, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$. Converting to Cartesian we have $\left(0, \frac{3}{5}\right)$ and $(0, -1)$, so $2a = \frac{8}{5}$, $a = \frac{4}{5}$, $c = ea = \frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$, the center $(h, k) = \left(0, \frac{3}{5} - \frac{4}{5}\right) = \left(0, \frac{-1}{5}\right)$ (since it's symmetric on the y-axis). Solving for
$$b^2 = a^2 - c^2 = \left(\frac{4}{5}\right)^2 - \left(\frac{1}{5}\right)^2 = \frac{15}{25} = \frac{3}{5}.$$
$$\frac{25\left(y + \frac{1}{5}\right)^2}{16} + \frac{5x^2}{3} = 1$$

59.

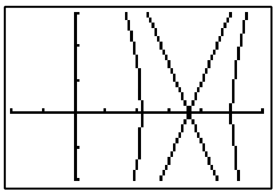


$[-8, 8]$ by $[-11, 0]$

$e = \frac{7}{2}$, so a hyperbola. In polar coordinates the vertices are $\left(-7, \frac{\pi}{2}\right)$ and $\left(\frac{35}{9}, \frac{3\pi}{2}\right)$. Converting to Cartesian we have $(0, -7)$ and $\left(0, \frac{-35}{9}\right)$, so $2a = \frac{28}{9}$, $a = \frac{14}{9}$,

$c = ea = \frac{7}{2}\cdot\frac{14}{9} = \frac{49}{9}$ the center (h, k)
$= \left(0, \frac{-35}{9} - \frac{14}{9}\right) = \left(0, \frac{-49}{9}\right)$ (since it's symmetric on the y-axis). Solving for
$b = \sqrt{c^2 - a^2} = \sqrt{\left(\frac{49}{9}\right)^2 - \left(\frac{14}{9}\right)^2} = \frac{21\sqrt{5}}{9} = \frac{7\sqrt{5}}{3}$

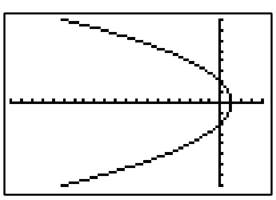
$$\frac{81\left(y + \frac{49}{9}\right)^2}{196} - \frac{9x^2}{245} = 1$$

60. 

$[-2, 6]$ by $[-2, 3]$

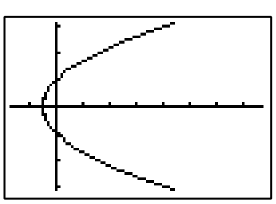
$e = \frac{5}{2}$, so a hyperbola. In polar coordinates the vertices are $\left(\frac{15}{7}, 0\right)$ and $(-5, \pi)$. Converting to Cartesian we have $\left(\frac{15}{7}, 0\right)$ and $(5, 0)$, so $2a = \frac{20}{7}$, $a = \frac{10}{7}$,
$c = ea = \frac{5}{2}\cdot\frac{10}{7} = \frac{25}{7}$ the center (h, k)
$= \left(\frac{5 + 15/7}{2}, 0\right) = \left(\frac{25}{7}, 0\right)$ (since it's symmetric on the y-axis). Solving for $b^2 = a^2 - c^2$
$= \left(\frac{25}{7}\right)^2 - \left(\frac{10}{7}\right)^2 = \frac{525}{49} = \frac{75}{7}$

$$\frac{49\left(x - \frac{25}{7}\right)^2}{100} - \frac{7y^2}{75} = 1$$

61. 

$[-20, 4]$ by $[-8, 8]$

$e = 1$, so a parabola. In polar coordinates, the vertex is $(1, 0)$ and the parabola crosses the y-axis at $\left(2, \frac{\pi}{2}\right)$.
Converting to Cartesian form, we have the vertex $(h, k) = (1, 0)$ and a point on the parabola is $(0, 2)$. Since the parabola opens to the left, $y^2 = 4p(x - 1)$.
Substituting $(0, 2)$, we have $4 = -4p$, $p = -1$
$y^2 = -4(x - 1)$

62. 

$[-1.7, 7.7]$ by $[-3.1, 3.1]$

$e = 1$, so this is a parabola. In polar coordinates, the vertex is $\left(\frac{1}{2}, \pi\right)$ and the parabola crosses the y-axis at $\left(1, \frac{\pi}{2}\right)$. Converting to Cartesian form, we have the vertex $(h, k) = \left(-\frac{1}{2}, 0\right)$ and a point on the parabola is $(0, 1)$. Since the parabola opens to the right,
$y^2 = 4p\left(x + \frac{1}{2}\right)$. Substituting $(0, 1)$, we have $1 = 2p$,
$p = \frac{1}{2}$. $y^2 = 2\left(x + \frac{1}{2}\right)$

63. $\sqrt{(3 - (-1))^2 + (-2 - 0)^2 + (-4 - 3)^2}$
$= \sqrt{16 + 4 + 49} = \sqrt{69}$

64. $\left(\frac{3 - 1}{2}, \frac{-2 + 0}{2}, \frac{-4 + 3}{2}\right) = \left(1, -1, -\frac{1}{2}\right)$

65. $\mathbf{v} + \mathbf{w} = \langle -3, 1, -2\rangle + \langle 3, -4, 0\rangle = \langle 0, -3, -2\rangle$

66. $\mathbf{v} - \mathbf{w} = \langle -3, 1, -2\rangle - \langle 3, -4, 0\rangle = \langle -6, 5, -2\rangle$

67. $\mathbf{v}\cdot\mathbf{w} = \langle -3, 1, -2\rangle\cdot\langle 3, -4, 0\rangle = -9 - 4 + 0 = -13$

68. $|\mathbf{v}| = \sqrt{(-3)^2 + 1^2 + (-2)^2} = \sqrt{9 + 1 + 4} = \sqrt{14}$

69. $\frac{\mathbf{w}}{|\mathbf{w}|} = \frac{\langle 3, -4, 0\rangle}{\sqrt{3^2 + (-4)^2 + 0^2}} = \left\langle\frac{3}{5}, -\frac{4}{5}, 0\right\rangle$

70. $(\mathbf{v}\cdot\mathbf{w})(\mathbf{v} + \mathbf{w}) = -13\,\langle 0, -3, -2\rangle = \langle 0, 39, 26\rangle$

71. $(x + 1)^2 + y^2 + (z - 3)^2 = 16$

72. The direction vector $\overrightarrow{PQ}$ is $\langle 3 - (-1), -2 - 0, -4 - 3\rangle = \langle 4, -2, -7\rangle$. Since the line l through P in the direction of $\overrightarrow{PQ}$ is $l = (-1, 0, 3) + t\,(4, -2, -7)$, the parametric equations are: $x = -1 + 4t$, $y = -2t$, $z = 3 - 7t$.

73. The direction vector is $\langle -3, 1, -2\rangle$ so the vector equation of a line in the direction of $\mathbf{v}$ through P is
$\mathbf{r} = \langle -1, 0, 3\rangle + t\langle -3, 1\ -2\rangle$

74. The mid-point M of $\overrightarrow{PQ}$ is: $\left(1, -1, -\frac{1}{2}\right)$ (from Exercise #64) so $\overrightarrow{OM} = \left\langle 1, -1, -\frac{1}{2}\right\rangle$. The direction vector is $\mathbf{w} = \langle 3, -4, 0\rangle$, so a vector equation of the line is $\mathbf{v} = \left\langle 1 + 3t, -1 - 4t, -\frac{1}{2}\right\rangle$. This can be expresses in parametric form: $x = 1 + 3t$, $y = -1 - 4t$, $z = -\frac{1}{2}$.

75. $4p = 18$, so $p = 4.5$; the focus is at $(0, 4.5)$.

76. $4p = 15$, so $p = 3.75$; the focus is at $(3.75, 0)$.

77. **(a)** The "shark" should aim for the other spot on the table, since a ball that passes through one focus will end up passing through the other focus if nothing gets in the way.

(b) Let $a = 3$, $b = 2$, and $c = \sqrt{5}$. Then the foci are at $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$. These are the points at which to aim.

78. The total radius of the orbit is $r = 0.500 + 6380 = 6380.5$ km, or 6,380,500 m.

(a) $v = 7908$ m/sec $= 7.908$ km/sec

(b) The circumference of the one orbit is $2\pi r \approx 40{,}090$ km; one orbit therefore takes about 5070 seconds, or about 1 hr 25 min.

79. The major axis length is 18,000 km, plus 170 km, plus the diameter of the earth, so $a \approx 15{,}465$ km $= 15{,}465{,}000$ m. At apogee, $r = 18{,}000 + 6380 = 24{,}380$ km, so $v \approx 2633$ m/sec. At perigee, $r = 6380 + 170 = 6550$ km, so $v \approx 9800$ m/sec.

80. Kepler's third law: $T^2 = a^3$, T is in Earth years and a is in AU.

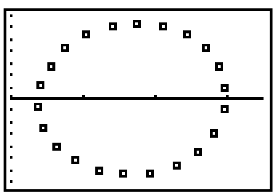
$$a = T^{2/3} = \left(\frac{409 \text{ days}}{365.2 \text{ days/year}}\right)^{2/3} \approx 1.08 \text{ AU} = 161 \text{ Gm}$$

$c = ae = (161 \text{ Gm})(0.83) \approx 134$ Gm
perihelion: $a - c = 161 \text{ Gm} - 134 \text{ Gm} = 27$ Gm
aphelion: $a + c = 161 \text{ Gm} + 134 \text{ Gm} = 295$ Gm

Chapter 8 Project

Answers are based on the sample data provided in the table.

1.



[0.4, 0.75] by [–0.7, 0.7]

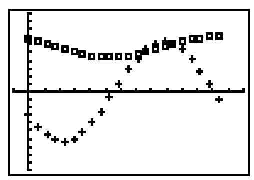
2. The endpoints of the major and minor axes lie at approximately $(0.438, 0)$, $(0.700, 0)$, $(0.569, 0.640)$ and $(0.569, -0.640)$. The ellipse is taller than it is wide, even though the reverse appears to be true on the graphing calculator screen. The semimajor axis length is 0.640, and the semiminor axis length is $(0.700 - 0.438)/2 = 0.131$. The equation is

$$\frac{(y - 0)^2}{(0.640)^2} + \frac{(x - 0.569)^2}{(0.131)^2} = 1$$

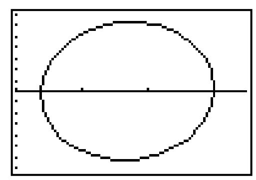
3. With respect to the graph of the ellipse, the point (h, k) represents the center of the ellipse. The value a is the length of the semimajor axis, and b is the length of the semiminor axis.

4. Physically, $h = 0.569$ m is the pendulum's average distance from the CBR, and $k = 0$ m/sec is the pendulum's average velocity. The value $a = 0.64$ m/sec is the maximum velocity, and $b = 0.131$ m is the maximum displacement of the pendulum from its average position.

5. The parametric equations for the sample data set (using sinusoidal regression) are
$x_{1T} \approx 0.131 \sin(4.80T + 2.10) + 0.569$ and
$y_{1T} \approx 0.639 \sin(4.80T - 2.65)$.



[–0.1, 1.4] by [–1, 1]



[0.4, 0.75] by [–0.7, 0.7]