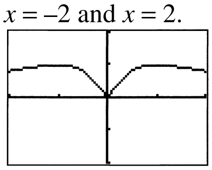
Chapter 5

Applications of Derivatives

Section 5.1 Extreme Values of Functions (pp. 193–201)

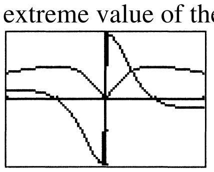
Exploration 1 Finding Extreme Values

1. From the graph we can see that there are three critical points: $x = -1, 0, 1$.
   Critical point values:
   $f(-1) = 0.5,\ f(0) = 0,\ f(1) = 0.5$ Endpoint values: $f(-2) = 0.4,\ f(2) = 0.4$
   Thus f has absolute maximum value of 0.5 at $x = -1$ and $x = 1$, absolute minimum value of 0 at $x = 0$, and local minimum value of 0.4 at $x = -2$ and $x = 2$.



[−2, 2] by [−1, 1]

2. The graph of f' has zeros at $x = -1$ and $x = 1$ where the graph of f has local extreme values. The graph of f' is not defined at $x = 0$, another extreme value of the graph of f.



[−2, 2] by [−1, 1]

3. We can write $f(x) = \begin{cases} \dfrac{-x}{x^2+1} & \text{for } x < 0 \\ \dfrac{x}{x^2+1} & \text{for } x \ge 0 \end{cases}$,

   so the Quotient Rule gives

   $$f'(x) = \begin{cases} -\dfrac{1-x^2}{(x^2+1)^2} & \text{for } x < 0 \\ \dfrac{1-x^2}{(x^2+1)^2} & \text{for } x \ge 0 \end{cases},$$

   which can be written as $f'(x) = \dfrac{|x|}{x} \cdot \dfrac{1-x^2}{(x^2+1)^2}$.

Quick Review 5.1

1. $f'(x) = \dfrac{1}{2\sqrt{4-x}} \cdot \dfrac{d}{dx}(4-x) = \dfrac{-1}{2\sqrt{4-x}}$

2. $f'(x) = \dfrac{d}{dx} 2(9-x^2)^{-1/2}$
   $= -(9-x^2)^{-3/2} \cdot \dfrac{d}{dx}(9-x^2)$
   $= -(9-x^2)^{-3/2}(-2x)$
   $= \dfrac{2x}{(9-x^2)^{3/2}}$

3. $g'(x) = -\sin(\ln x) \cdot \dfrac{d}{dx} \ln x = -\dfrac{\sin(\ln x)}{x}$

4. $h'(x) = e^{2x} \cdot \dfrac{d}{dx} 2x = 2e^{2x}$

5. Graph (c), since this is the only graph that has positive slope at c.

6. Graph (b), since this is the only graph that represents a differentiable function at a and b and has negative slope at c.

7. Graph (d), since this is the only graph representing a function that is differentiable at b but not at a.

8. Graph (a), since this is the only graph that represents a function that is not differentiable at a or b.

9. As $x \to 3^-$, $\sqrt{9-x^2} \to 0^+$. Therefore, $\lim_{x \to 3^-} f(x) = \infty$.

10. As $x \to 3^+$, $\sqrt{9-x^2} \to 0^+$. Therefore, $\lim_{x \to -3^+} f(x) = \infty$.

11. **(a)** $\dfrac{d}{dx}(x^3 - 2x) = 3x^2 - 2$
    $f'(1) = 3(1)^2 - 2 = 1$

    (b) $\dfrac{d}{dx}(x+2) = 1$
    $f'(3) = 1$

(c) Left-hand derivative:

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}$$
$$=\lim_{h\to 0^-}\frac{[(2+h)^3-2(2+h)]-4}{h}$$
$$=\lim_{h\to 0^-}\frac{h^3+6h^2+10h}{h}$$
$$=\lim_{h\to 0^-}(h^2+6h+10)$$
$$=10$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}$$
$$=\lim_{h\to 0^+}\frac{[(2+h)+2]-4}{h}$$
$$=\lim_{h\to 0^+}\frac{h}{h}$$
$$=\lim_{h\to 0^+}1$$
$$=1$$

Since the left-and right-hand derivatives are not equal, $f'(2)$ is undefined.

12. **(a)** The domain is $x \neq 2$. (See the solution for 11.(c)).

(b) $f'(x)=\begin{cases}3x^2-2, & x<2\\ 1, & x>2\end{cases}$

Section 5.1 Exercises

1. Minima at (–2, 0) and (2, 0), maximum at (0, 2)

2. Local minimum at (–1, 0), local maximum at (1, 0)

3. Maximum at (0, 5); note that there is no minimum since the endpoint (2, 0) is excluded from the graph.

4. Local maximum at (–3, 0), local minimum at (2, 0), maximum at (1, 2), minimum at (0, –1)

5. Maximum at $x = b$, minimum at $x = c_2$;
The Extreme Value Theorem applies because f is continuous on $[a, b]$, so both the maximum and minimum exist.

6. Maximum at $x = c$, minimum at $x = b$;
The Extreme Value Theorem applies because f is continuous on $[a, b]$, so both the maximum and minimum exist.

7. Maximum at $x = c$, no minimum; the Extreme Value Theorem does not apply, because the function is not defined on a closed interval.

8. No maximum, no minimum; the Extreme Value Theorem does not apply, because the function is not continuous or defined on a closed interval.

9. Maximum at $x = c$, minimum at $x = a$; the Extreme Value Theorem does not apply, because the function is not continuous.

10. Maximum at $x = a$, minimum at $x = c$; the Extreme Value Theorem does not apply since the function is not continuous.

11. The first derivative $f'(x)=-\frac{1}{x^2}+\frac{1}{x}$ has a zero at $x=1$.
Critical point value: $f(1)=1+\ln 1=1$
Endpoint values: $f(0.5)=2+\ln 0.5\approx 1.307$

$$f(4)=\frac{1}{4}+\ln 4\approx 1.636$$

Maximum value is $\frac{1}{4}+\ln 4$ at $x = 4$;
minimum value is 1 at $x=1$;
local maximum at $\left(\frac{1}{2},\ 2-\ln 2\right)$
Since f' is zero at the only critical point, there are no critical points that are not stationary points.

12. The first derivative $g'(x)=-e^{-x}$ has no zeros, so we need only consider the endpoints.
$g(-1)=e^{-(-1)}=e$
$g(1)=e^{-1}=\frac{1}{e}$
Maximum value is e at $x=-1$;
minimum value is $\frac{1}{e}$ at $x=1$.
Since there are no critical points, there are no critical points that are not stationary points.

13. The first derivative $h'(x)=\frac{1}{x+1}$ has no zeros, so we need only consider the endpoints.
$h(0)=\ln 1=0 \qquad h(3)=\ln 4$
Maximum value is ln 4 at $x=3$; minimum value is 0 at $x = 0$.
Since there are no critical points, there are no critical points that are not stationary points.

14. The first derivative $k'(x) = -2xe^{-x^2}$ has a zero at $x = 0$. Since the domain has no endpoints, any extreme value must occur at $x = 0$.
Since $k(0) = e^{-0^2} = 1$ and $\lim_{x \to \pm\infty} k(x) = 0$, the maximum value is 1 at $x = 0$.
Since k' is zero at the only critical point, there are no critical points that are not stationary points.

15. The first derivative $f'(x) = \cos\left(x + \frac{\pi}{4}\right)$, has zeros at $x = \frac{\pi}{4},\ x = \frac{5\pi}{4}$.

Critical point values: $x = \frac{\pi}{4}$ $f(x) = 1$

$x = \frac{5\pi}{4}$ $f(x) = -1$

Endpoint values: $x = 0$ $f(x) = \frac{1}{\sqrt{2}}$

$x = \frac{7\pi}{4}$ $f(x) = 0$

Maximum value is 1 at $x = \frac{\pi}{4}$;
minimum value is –1 at $x = \frac{5\pi}{4}$;
local minimum at $\left(0, \frac{1}{\sqrt{2}}\right)$;
local maximum at $\left(\frac{7\pi}{4}, 0\right)$
Since f' is zero at both the critical points, there are no critical points that are not stationary points.

16. The first derivative $g'(x) = \sec x \tan x$ has zeros at $x = 0$ and $x = \pi$ and is undefined at $x = \frac{\pi}{2}$.
Since $g(x) = \sec x$ is also undefined at $x = \frac{\pi}{2}$, the critical points occur only at $x = 0$ and $x = \pi$.

Critical point values: $x = 0$ $g(x) = 1$

$x = \pi$ $g(x) = -1$

Since the range of $g(x)$ is $(-\infty, -1] \cup [1, \infty)$, these values must be a local minimum and local maximum, respectively. Local minimum at $(0, 1)$; local maximum at $(\pi, -1)$
Since g' is zero at both the critical points, there are no critical points that are not stationary points.

17. The first derivative $f'(x) = \frac{2}{5}x^{-3/5}$ is never zero but is undefined at $x = 0$.

Critical point value: $x = 0$ $f(x) = 0$

Endpoint value: $x = -3$ $f(x) = (-3)^{2/5} = 3^{2/5} \approx 1.552$

Since $f(x) > 0$ for $x \neq 0$, the critical point at $x = 0$ is a local minimum, and since $f(x) \le (-3)^{2/5}$ for $-3 \le x < 1$, the endpoint value at $x = -3$ is a global maximum.
Maximum value is $3^{2/5}$ at $x = -3$;
minimum value is 0 at $x = 0$.
Since f' is undefined at $x = 0$, the critical point (0, 0) is not a stationary point.

18. The first derivative $f'(x) = \frac{3}{5}x^{-2/5}$ is never zero but is undefined at $x = 0$.

Critical point value: $x = 0$ $f(x) = 0$

Endpoint value: $x = 3$ $f(x) = 3^{3/5} \approx 1.933$

Since $f(x) < 0$ for $x < 0$ and $f(x) > 0$ for $x > 0$, the critical point is not a local minimum or maximum. The maximum value is $3^{3/5}$ at $x = 3$. Since f' is undefined at $x = 0$, the critical point (0, 0) is not a stationary point.

19. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.
$y' = 4x - 8$
The only critical point is $x = 2$. The value $y = 2(2)^2 - 8(2) + 9 = 1$ is the only candidate for an extreme value. As x moves away from 2 on either side, the values of y increase, and the graph rises. We have a minimum value of 1 at $x = 2$.

20. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.
$y' = 3x^2 - 2$
The critical points are $\pm\sqrt{\frac{2}{3}}$. The values

$y = \left(\sqrt{\frac{2}{3}}\right)^3 - 2\sqrt{\frac{2}{3}} + 4 = 4 - \frac{4\sqrt{6}}{9}$ and

$y = \left(-\sqrt{\frac{2}{3}}\right)^3 - 2\left(-\sqrt{\frac{2}{3}}\right) + 4 = 4 + \frac{4\sqrt{6}}{9}$ are the only candidates for extreme values. As x moves away from $-\sqrt{\frac{2}{3}}$ on either side, the values of y decrease. The function has a local maximum at $\left(-\sqrt{\frac{2}{3}}, 4 + \frac{4\sqrt{6}}{9}\right) \approx (-0.816, 5.089)$. As x moves away from $\sqrt{\frac{2}{3}}$ on either side, the values of y increase. The function has a local minimum at $\left(\sqrt{\frac{2}{3}}, 4 - \frac{4\sqrt{6}}{9}\right) \approx (0.816, 2.911)$.

21. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$y' = 3x^2 + 2x - 8 = (3x - 4)(x + 2)$

The critical points are $\frac{4}{3}$ and -2. The values

$y = \left(\frac{4}{3}\right)^3 + \left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) + 5 = -\frac{41}{27}$ and

$y = (-2)^3 + (-2)^2 - 8(-2) + 5 = 17$ are the only candidates for extreme values. As x moves away from $\frac{4}{3}$ on either side, the values of y increase. The function has a local minimum at $\left(\frac{4}{3}, -\frac{41}{27}\right)$. As x moves away from -2 on either side, the values of y decrease. The function has a local maximum at $(-2, 17)$.

22. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$y' = 3x^2 - 6x + 3 = 3(x - 1)^2$

The only critical point is $x = 1$. The value $y = (1)^3 - 3(1)^2 + 3(1) - 2 = -1$ is the only candidate for an extreme value. As x moves away from 1 on the left, the values of y decrease. As x moves away from 1 on the right, the values of y increase. Neither a local maximum nor a local minimum occurs at $x = 1$. There are no local maxima or minima.

23. The domain is $(-\infty, -1] \cup [1, \infty)$.

$y' = \frac{1}{2}(x^2 - 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - 1}}$

The derivative is zero only when $x = 0$, which is not in the domain. The derivative is undefined at $x = \pm 1$, which are also the endpoints. As x moves away from ± 1 within the domain, the values of y increase. The function has a minimum value of 0 at $x = -1$ and at $x = 1$.

24. The domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$y' = -1(x^2 - 1)^{-2}(2x) = -\frac{2x}{(x^2 - 1)^2}$

The derivative is zero only when $x = 0$. The derivative is undefined at $x = \pm 1$, which are not in the domain. The only critical point is $x = 0$. As x moves away from 0 on either side, the values of y decrease. The function has a local maximum value at $(0, -1)$.

25. The domain is $(-1, 1)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$y' = -\frac{1}{2}(1 - x^2)^{-3/2}(-2x) = \frac{x}{(1 - x^2)^{3/2}}$

The derivative is zero only when $x = 0$. The derivative is undefined at $x = \pm 1$, which are not in the domain. The only critical point is $x = 0$. As x moves away from 0 on either side, the values of y increase. The function has a minimum value of 1 at $x = 0$.

26. The domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$y' = -\frac{1}{3}(1 - x^2)^{-4/3}(-2x) = \frac{2x}{3(1 - x^2)^{4/3}}$

The derivative is zero only when $x = 0$. The derivative is undefined at $x = \pm 1$, which are not in the domain. The only critical point is $x = 0$. As x moves away from 0 on either side, the values of y increase. The function has a local minimum value at $(0, 1)$.

27. The domain is $[-1, 3]$.

$y' = \frac{1}{2}(3 + 2x - x^2)^{-1/2}(2 - 2x)$

$= \frac{1 - x}{\sqrt{3 + 2x - x^2}}$

The derivative is zero when $x = 1$. The derivative is undefined at $x = -1$ and at $x = 3$, which are also the endpoints. As x moves away from 1 on either side, the values of y decrease. The function has a maximum value of 2 at $x = 1$. As x moves away from -1 or 3 within the domain, the values of y increase. The function has a minimum value of 0 at $x = -1$ and at $x = 3$.

28. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$$y' = 6x^3 + 12x^2 - 18x = 6x(x+3)(x-1)$$

The critical points are 0, -3, and 1. As x moves away from 0 on either side, the values of y decrease. The function has a local maximum at (0, 10). As x moves away from -3 on either side, the values of y increase. The function has a minimum value of $-\frac{115}{2}$ at $x = -3$. As x moves away from 1 on either side, the values of y increase. The function has a local minimum at $\left(1, \frac{13}{2}\right)$.

29. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$$y' = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

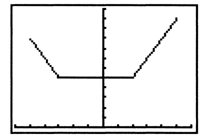
The critical points are -1 and 1. As x moves away from -1 on either side, the values of y increase. The function has a minimum value of $-\frac{1}{2}$ at $x = -1$. As x moves away from 1 on either side, the values of y decrease. The function has a maximum value of $\frac{1}{2}$ at $x = 1$.

30. The domain is $(-\infty, \infty)$. The domain has no endpoints, so all the extreme values must occur at critical points.

$$y' = \frac{(x^2+2x+2)(1) - (x+1)(2x+2)}{(x^2+2x+2)^2} = \frac{-x(x+2)}{(x^2+2x+2)^2}$$

The critical points are -2 and 0. As x moves away from -2 on either side, the values of y increase. The function has a minimum value of $-\frac{1}{2}$ at $x = -2$. As x moves away from 0 on either side, the values of y decrease. The function has a maximum value of $\frac{1}{2}$ at $x = 0$.

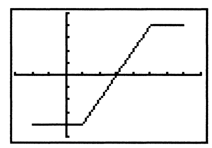
31.



[−6, 6] by [0, 12]

Maximum value is 11 at $x = 5$;
minimum value is 5 on the interval $[-3, 2]$;
local maximum at $(-5, 9)$

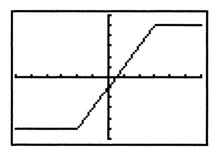
32.



[−3, 8] by [−5, 5]

Maximum value is 4 on the interval $[5, 7]$;
minimum value is -4 on the interval $[-2, 1]$.

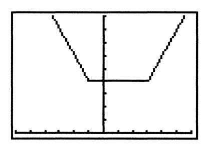
33.



[−6, 6] by [−6, 6]

Maximum value is 5 on the interval $[3, \infty)$;
minimum value is -5 on the interval $(-\infty, -2]$.

34.



[−6, 6] by [0, 9]

Minimum value is 4 on the interval $[-1, 3]$

35. $y' = x^{2/3}(1) + \frac{2}{3}x^{-1/3}(x+2) = \frac{5x+4}{3\sqrt[3]{x}}$

crit. pt.	derivative	extremum	value
$x = -\frac{4}{5}$	0	local max	$\frac{12}{25}10^{1/3} \approx 1.034$
$x = 0$	undefined	local min	0

Since y' is undefined at $x = 0$, the critical point (0, 0) is not a stationary point.

36. $y' = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}(x^2 - 4) = \frac{8x^2 - 8}{3\sqrt[3]{x}}$

crit. pt.	derivative	extremum	value
$x = -1$	0	minimum	–3
$x = 0$	undefined	local max	0
$x = 1$	0	minimum	–3

Since y' is undefined at $x = 0$, the critical point (0, 0) is not a stationary point.

37. $y' = x \cdot \frac{1}{2\sqrt{4 - x^2}}(-2x) + (1)\sqrt{4 - x^2}$

$= \frac{-x^2 + (4 - x^2)}{\sqrt{4 - x^2}}$

$= \frac{4 - 2x^2}{\sqrt{4 - x^2}}$

crit. pt.	derivative	extremum	value
$x = -2$	undefined	local max	0
$x = -\sqrt{2}$	0	minimum	–2
$x = \sqrt{2}$	0	maximum	2
$x = 2$	undefined	local min	0

Since y' is undefined at $x = -2$ and $x = 2$, the critical points (–2, 0) and (2, 0) are not stationary points.

38. $y = x^2 \cdot \frac{1}{2\sqrt{3 - x}}(-1) + 2x\sqrt{3 - x}$

$= \frac{-x^2 + 4x(3 - x)}{2\sqrt{3 - x}}$

$= \frac{-5x^2 + 12x}{2\sqrt{3 - x}}$

crit. pt.	derivative	extremum	value
$x = 0$	0	minimum	0
$x = \frac{12}{5}$	0	local max	$\frac{144}{125}15^{1/2}$ ≈ 4.462
$x = 3$	undefined	minimum	0

Since y' is undefined at $x = 3$, the critical point (3, 0) is not a stationary point.

39. $y' = \begin{cases} -2, & x < 1 \\ 1, & x > 1 \end{cases}$

crit. pt.	derivative	extremum	value
$x = 1$	undefined	minimum	2

Since y' is undefined at $x = 1$, the critical point (1, 2) is not a stationary point.

40. $y' = \begin{cases} -1, & x < 0 \\ 2 - 2x, & x > 0 \end{cases}$

crit. pt.	derivative	extremum	value
$x = 0$	undefined	local min	3
$x = 1$	0	local max	4

Since y' is undefined at $x = 0$, the critical point (0, 3) is not a stationary point.

41. $y' = \begin{cases} -2x - 2, & x < 1 \\ -2x + 6, & x > 1 \end{cases}$

crit. pt.	derivative	extremum	value
$x = -1$	0	maximum	5
$x = 1$	undefined	local min	1
$x = 3$	0	maximum	5

Since y' is undefined at $x = 1$, the critical point (1, 1) is not a stationary point.

42. We begin by determining whether $f'(x)$ is defined at $x = 1$, where

$$f(x) = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$$

Left-hand derivative:

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^-} \frac{-\frac{1}{4}(1+h)^2 - \frac{1}{2}(1+h) + \frac{15}{4} - 3}{h}$$

$$= \lim_{h \to 0^-} \frac{-h^2 - 4h}{4h}$$

$$= \lim_{h \to 0^-} \frac{1}{4}(-h - 4)$$

$$= -1$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0^+}\frac{(1+h)^3-6(1+h)^2+8(1+h)-3}{h}$$
$$=\lim_{h\to 0^+}\frac{h^3-3h^2-h}{h}$$
$$=\lim_{h\to 0^+}(h^2-3h-1)$$
$$=-1$$

Thus $f'(x)=\begin{cases}-\frac{1}{2}x-\frac{1}{2}, & x\le 1\\ 3x^2-12x+8, & x>1\end{cases}$

Note that $-\frac{1}{2}x-\frac{1}{2}=0$ when $x=-1$, and

$3x^2-12x+8=0$ when $x=2\pm\frac{2\sqrt{3}}{3}.$

But $2-\frac{2\sqrt{3}}{3}\approx 0.845<1,$ so the only critical points occur at $x=-1$ and

$x=2+\frac{2\sqrt{3}}{3}\approx 3.155.$

crit. pt.	derivative	extremum	value
$x=-1$	0	local max	4
$x\approx 3.155$	0	local min	≈ -3.079

Since y' is zero at both the critical points, there are no critical points that are not stationary points.

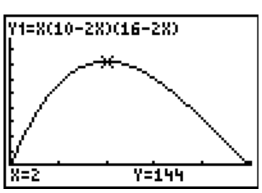
43. **(a)** $V(x)=160x-52x^2+4x^3$

$V'(x)=160-104x+12x^2$
$=4(x-2)(3x-20)$

The only critical point in the interval (0, 5) is at $x=2$. The maximum value of $V(x)$ is 144 at $x=2$.

(b) The largest possible volume of the box is 144 cubic units, and it occurs when $x=2$.

(c)



[0, 5] by [–20, 200]

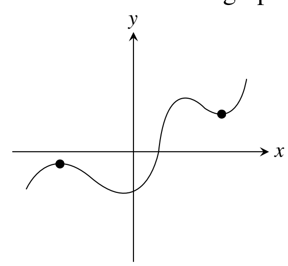
44. **(a)** $P'(x)=2-200x^{-2}$

The only critical point in the interval (0, ∞) is at $x=10$. The minimum value of $P(x)$ is 40 at $x=10$.

(b) The smallest possible perimeter of the rectangle is 40 units and it occurs at $x=10$, which makes the rectangle a 10 by 10 square.

45. False; for example, the maximum could occur at a corner, where $f'(c)$ would not exist.

46. False. Consider the graph below.



47. E; $\frac{d}{dx}(4x-x^2+6)=4-2x$

$4-2x=0$
$x=2$
$f(2)=4(2)-(2)^2+6=10$

48. E; see Theorem 2.

49. B; $\frac{d}{dx}(x^3-6x+5)=3x^2-6$

$3x^2-6=0$
$x=\pm\sqrt{2}$

50. B

51. **(a)** No, since $f'(x)=\frac{2}{3}(x-2)^{-1/3}$, which is undefined at $x=2$.

(b) The derivative is defined and nonzero for all $x\neq 2$. Also, $f(2)=0$ and $f(x)>0$ for all $x\neq 2$.

(c) No, $f(x)$ need not have a global maximum because its domain is all real numbers. Any restriction of f to a closed interval of the form $[a, b]$ would have both a maximum value and a minimum value on the interval.

(d) The answers are the same as (a) and (b) with 2 replaced by *a*.

52. Note that

$$f(x)=\begin{cases}-x^3+9x, & x\le -3 \text{ or } 0\le x<3\\ x^3-9x, & -3<x<0 \text{ or } x\ge 3.\end{cases}$$

Therefore,

$$f'(x)=\begin{cases}-3x^2+9, & x<-3 \text{ or } 0<x<3\\ 3x^2-9, & -3<x<0 \text{ or } x>3.\end{cases}$$

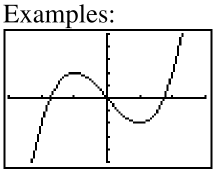
(a) No, since the left- and right-hand derivatives at $x = 0$ are –9 and 9, respectively.

(b) No, since the left- and right-hand derivatives at $x = 3$ are –18 and 18, respectively.

(c) No, since the left- and right-hand derivatives at $x = -3$ are –18 and 18, respectively.

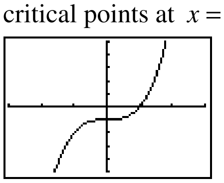
(d) The critical points occur when $f'(x)=0$ $\left(\text{at } x=\pm\sqrt{3}\right)$ and when $f'(x)$ is undefined (at $x = 0$ or $x = \pm 3$). The minimum value is 0 at $x = -3$, at $x = 0$, and at $x = 3$; local maxima occur at $\left(-\sqrt{3}, 6\sqrt{3}\right)$ and $\left(\sqrt{3}, 6\sqrt{3}\right)$.

53. **(a)** $f'(x)=3ax^2+2bx+c$ is a quadratic, so it can have 0, 1, or 2 zeros, which would be the critical points of *f*.
Examples:



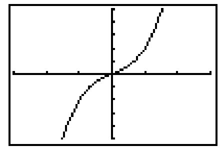
[–3, 3] by [–5, 5]

The function $f(x)=x^3-3x$ has two critical points at $x=-1$ and $x=1$.



[–3, 3] by [–5, 5]

The function $f(x)=x^3-1$ has one critical point at $x = 0$.



[–3, 3] by [–5, 5]

The function $f(x)=x^3+x$ has no critical points.

(b) The function can have either two local extreme values or no extreme values. (If there is only one critical point, the cubic function has no extreme values.)

54. **(a)** By the definition of local maximum value, there is an open interval containing *c* where $f(x)\le f(c)$, so $f(x)-f(c)\le 0$.

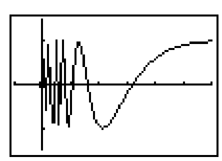
(b) Because $x\to c^+$, we have $(x-c)>0$, and the sign of the quotient must be negative (or zero). This means the limit is nonpositive.

(c) Because $x\to c^-$, we have $(x-c)<0$, and the sign of the quotient must be positive (or zero). This means the limit is nonnegative.

(d) Assuming that $f'(c)$ exists, the one-sided limits in (b) and (c) above must exist and be equal. Since one is nonpositive and one is nonnegative, the only possible common value is 0.

(e) There will be an open interval containing *c* where $f(x)-f(c)\ge 0$. The difference quotient for the left-hand derivative will have to be negative (or zero), and the difference quotient for the right-hand derivative will have to be positive (or zero). Taking the limit, the left-hand derivative will be nonpositive, and the right-hand derivative will be nonnegative. Therefore, the only possible value for $f'(c)$ is 0.

55. **(a)**



[–0.1, 0.6] by [–1.5, 1.5]

$f(0) = 0$ is not a local extreme value because in any open interval containing $x = 0$, there are infinitely many points where $f(x) = 1$ and where $f(x) = -1$.

(b) One possible answer, on the interval [0, 1]:

$$f(x)=\begin{cases}(1-x)\cos\dfrac{1}{1-x}, & 0\le x<1\\ 0, & x=1\end{cases}$$

This function has no local extreme value at $x = 1$. Note that it is continuous on [0, 1].

Section 5.2 Mean Value Theorem (pp. 202–210)

Quick Review 5.2

1. $2x^2-6<0$
$2x^2<6$
$x^2<3$
$-\sqrt{3}<x<\sqrt{3}$
Interval: $(-\sqrt{3},\sqrt{3})$

2. $3x^2-6>0$
$3x^2>6$
$x^2>2$
$x<-\sqrt{2}$ or $x>\sqrt{2}$
Intervals: $\left(-\infty,-\sqrt{2}\right)\cup\left(\sqrt{2},\infty\right)$

3. Domain: $8-2x^2\ge 0$
$8\ge 2x^2$
$4\ge x^2$
$-2\le x\le 2$
The domain is [–2, 2].

4. f is continuous for all x in the domain, or, in the interval [–2, 2].

5. f is differentiable for all x in the interior of its domain, or, in the interval (–2, 2).

6. We require $x^2-1\ne 0$, so the domain is $x\ne\pm 1$.

7. f is continuous for all x in the domain, or, for all $x\ne\pm 1$.

8. f is differentiable for all x in the domain, or, for all $x\ne\pm 1$.

9. $7=-2(-2)+C$
$7=4+C$
$C=3$

10. $-1=(1)^2+2(1)+C$
$-1=3+C$
$C=-4$

Section 5.2 Exercises

1. (a) Yes.

(b) $f'(x)=\dfrac{d}{dx}(x^2+2x-1)=2x+2$

$2c+2=\dfrac{2-(-1)}{1-0}=3$

$c=\dfrac{1}{2}.$

2. (a) Yes.

(b) $f'(x)=\dfrac{d}{dx}x^{2/3}=\dfrac{2}{3}x^{-1/3}$

$\dfrac{2}{3}c^{-1/3}=\dfrac{1-0}{1-0}=1$

$c=\dfrac{8}{27}.$

3. (a) No. There is a vertical tangent at $x = 0$.

4. (a) No. There is a corner at $x = 1$.

5. (a) Yes.

(b) $f'(x)=\dfrac{d}{dx}\sin^{-1}x=\dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{1}{\sqrt{1-c^2}}=\dfrac{(\pi/2)-(-\pi/2)}{1-(-1)}=\dfrac{\pi}{2}$

$\sqrt{1-c^2}=\dfrac{2}{\pi}$

$c=\sqrt{1-4/\pi^2}\approx 0.771.$

6. (a) Yes.

(b) $f'(x)=\dfrac{d}{dx}\ln(x-1)=\dfrac{1}{x-1}$

$\dfrac{1}{c-1}=\dfrac{\ln 3-\ln 1}{4-2}$

$c=\dfrac{4-2}{\ln 3-\ln 1}+1\approx 2.820$

7. (a) No; the function is discontinuous at $x=\dfrac{\pi}{2}.$

8. (a) No; the function is discontinuous at $x = 1$.

9. (a) The secant line passes through $(0.5, f(0.5)) = (0.5, 2.5)$ and $(2, f(2)) = (2, 2.5)$, so its equation is $y = 2.5$.

(b) The slope of the secant line is 0, so we need to find c such that $f'(c) = 0$.

$$1 - c^{-2} = 0$$
$$c^{-2} = 1$$
$$c = 1$$
$$f(c) = f(1) = 2$$

The tangent line has slope 0 and passes through (1, 2), so its equation is $y = 2$.

10. (a) The secant line passes through $(1, f(1)) = (1, 0)$ and $(3, f(3)) = (3, \sqrt{2})$, so its slope is $\frac{\sqrt{2}-0}{3-1} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$.

The equation is $y = \frac{1}{\sqrt{2}}(x-1) + 0$ or $y = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}$, or $y \approx 0.707x - 0.707$.

(b) We need to find c such that $f'(c) = \frac{1}{\sqrt{2}}$.

$$\frac{1}{2\sqrt{c-1}} = \frac{1}{\sqrt{2}}$$
$$2\sqrt{c-1} = \sqrt{2}$$
$$c - 1 = \frac{1}{2}$$
$$c = \frac{3}{2}$$
$$f(c) = f\left(\frac{3}{2}\right) = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

The tangent line has slope $\frac{1}{\sqrt{2}}$ and passes through $\left(\frac{3}{2}, \frac{1}{\sqrt{2}}\right)$. Its equation is

$y = \frac{1}{\sqrt{2}}\left(x - \frac{3}{2}\right) + \frac{1}{\sqrt{2}}$ or

$y = \frac{1}{\sqrt{2}}x - \frac{1}{2\sqrt{2}}$, or $y \approx 0.707x - 0.354$.

11. Because the trucker's average speed was 79.5 mph, and by the Mean Value Theorem, the trucker must have been going that speed at least once during the trip.

12. Let $f(t)$ denote the temperature indicated after t seconds. We assume that $f'(t)$ is defined and continuous for $0 \le t \le 20$. The average rate of change is $10.6°$ F/sec. Therefore, by the Mean Value Theorem, $f'(c) = 10.6°$ F/sec for some value of c in [0, 20]. Since the temperature was constant before $t = 0$, we also know that $f'(0) = 0°$ F/min. But f' is continuous, so by the Intermediate Value Theorem, the rate of change $f'(t)$ must have been 10.6°F/sec at some moment during the interval.

13. Because its average speed was approximately 7.667 knots, and by the Mean Value Theorem, it must been going that speed at least once during the trip.

14. The runner's average speed for the marathon was approximately 11.909 mph. Therefore, by the Mean Value Theorem, the runner must have been going that speed at least once during the marathon. Since the initial speed and final speed are both 0 mph and the runner's speed is continuous, by the Intermediate Value Theorem, the runner's speed must have been 11 mph at least twice.

15. (a) $f'(x) = 5 - 2x$

Since $f'(x) > 0$ on $\left(-\infty, \frac{5}{2}\right)$, $f'(x) = 0$ at $x = \frac{5}{2}$, and $f'(x) < 0$ on $\left(\frac{5}{2}, \infty\right)$, we know that $f(x)$ has a local maximum at $x = \frac{5}{2}$. Since $f\left(\frac{5}{2}\right) = \frac{25}{4}$, the local maximum occurs at the point $\left(\frac{5}{2}, \frac{25}{4}\right)$. (This is also a global maximum.)

(b) Since $f'(x) > 0$ on $\left(-\infty, \frac{5}{2}\right)$, $f(x)$ is increasing on $\left(-\infty, \frac{5}{2}\right]$.

(c) Since $f'(x) < 0$ on $\left(\frac{5}{2}, \infty\right)$, $f(x)$ is decreasing on $\left[\frac{5}{2}, \infty\right)$.

16. (a) $g'(x) = 2x - 1$

Since $g'(x) < 0$ on $\left(-\infty, \frac{1}{2}\right)$, $g'(x) = 0$ at $x = \frac{1}{2}$, and $g'(x) > 0$ on $\left(\frac{1}{2}, \infty\right)$, we know that $g(x)$ has a local minimum at $x = \frac{1}{2}$.

Since $g\left(\frac{1}{2}\right) = -\frac{49}{4}$, the local minimum occurs at the point $\left(\frac{1}{2}, -\frac{49}{4}\right)$. (This is also a global minimum.)

(b) Since $g'(x) > 0$ on $\left(\frac{1}{2}, \infty\right)$, $g(x)$ is increasing on $\left[\frac{1}{2}, \infty\right)$.

(c) Since $g'(x) < 0$ on $\left(-\infty, \frac{1}{2}\right)$, $g(x)$is decreasing on $\left(-\infty, \frac{1}{2}\right]$.

17. (a) $h'(x) = -\frac{2}{x^2}$

Since $h'(x)$ is never zero and is undefined only where $h(x)$ is undefined, there are no critical points. Also, the domain $(-\infty, 0) \cup (0, \infty)$ has no endpoints. Therefore, $h(x)$ has no local extrema.

(b) Since $h'(x)$ is never positive, $h(x)$ is not increasing on any interval.

(c) Since $h'(x) < 0$ on $(-\infty, 0) \cup (0, \infty)$, $h(x)$ is decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

18. (a) $k'(x) = -\frac{2}{x^3}$

Since $k'(x)$ is never zero and is undefined only where $k(x)$ is undefined, there are no critical points. Also, the domain $(-\infty, 0) \cup (0, \infty)$ has no endpoints. Therefore, $k(x)$ has no local extrema.

(b) Since $k'(x) > 0$ on $(-\infty, 0)$, $k(x)$ is increasing on $(-\infty, 0)$.

(c) Since $k'(x) < 0$ on $(0, \infty)$, $k(x)$ is decreasing on $(0, \infty)$.

19. (a) $f'(x) = 2e^{2x}$

Since $f'(x)$ is never zero or undefined, and the domain of $f(x)$ has no endpoints, $f(x)$ has no extrema.

(b) Since $f'(x)$ is always positive, $f(x)$ is increasing on $(-\infty, \infty)$.

(c) Since $f'(x)$ is never negative, $f(x)$ is not decreasing on any interval.

20. (a) $f'(x) = -0.5e^{-0.5x}$

Since $f'(x)$ is never zero or undefined, and the domain of $f(x)$ has no endpoints, $f(x)$ has no extrema.

(b) Since $f'(x)$ is never positive, $f(x)$ is not increasing on any interval.

(c) Since $f'(x)$ is always negative, $f(x)$ is decreasing on $(-\infty, \infty)$.

21. (a) $y' = -\frac{1}{2\sqrt{x+2}}$

In the domain $[-2, \infty)$, y' is never zero and is undefined only at the endpoint $x = -2$. The function y has a local maximum at $(-2, 4)$. (This is also a global maximum.)

(b) Since y' is never positive, y is not increasing on any interval.

(c) Since y' is negative on $(-2, \infty)$, y is decreasing on $[-2, \infty)$.

22. (a) $y' = 4x^3 - 20x = 4x\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)$

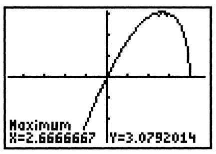
The function has critical points at $x = -\sqrt{5}$, $x = 0$, and $x = \sqrt{5}$. Since $y' < 0$ on $\left(-\infty, -\sqrt{5}\right)$ and $\left(0, \sqrt{5}\right)$ and $y' > 0$ on $\left(-\sqrt{5}, 0\right)$ and $\left(\sqrt{5}, \infty\right)$, the points at $x = \pm\sqrt{5}$ are local minima and the point at $x = 0$ is a local maximum.

Thus, the function has a local maximum at (0, 9) and local minima at $\left(-\sqrt{5}, -16\right)$ and $\left(\sqrt{5}, -16\right)$. (These are also global minima.)

(b) Since $y' > 0$ on $\left(-\sqrt{5}, 0\right)$ and $\left(\sqrt{5}, \infty\right)$, y is increasing on $\left[-\sqrt{5}, 0\right]$ and $\left[\sqrt{5}, \infty\right)$.

(c) Since $y' > 0$ on $\left(-\infty, -\sqrt{5}\right)$ and $\left(0, \sqrt{5}\right)$, y is decreasing on $\left(-\infty, -\sqrt{5}\right)$ and $\left[0, \sqrt{5}\right]$.

23.



[−4.7, 4.7] by [−3.1, 3.1]

(a) $f'(x) = x \cdot \dfrac{1}{2\sqrt{4-x}}(-1) + \sqrt{4-x}$

$= \dfrac{-3x+8}{2\sqrt{4-x}}$

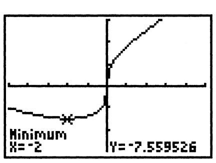
The local extrema occur at the critical point $x = \dfrac{8}{3}$ and at the endpoint $x = 4$.

There is a local (and absolute) maximum at $\left(\dfrac{8}{3}, \dfrac{16}{3\sqrt{3}}\right)$ or approximately (2.67, 3.08), and a local minimum at (4, 0).

(b) Since $f'(x) > 0$ on $\left(-\infty, \dfrac{8}{3}\right)$, $f(x)$ is increasing on $\left(-\infty, \dfrac{8}{3}\right]$.

(c) Since $f'(x) < 0$ on $\left(\dfrac{8}{3}, 4\right)$, $f(x)$ is decreasing on $\left[\dfrac{8}{3}, 4\right]$.

24.



[−5, 5] by [−15, 15]

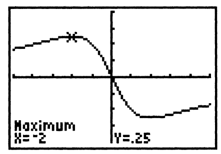
(a) $g'(x) = x^{1/3}(1) + \dfrac{1}{3}x^{-2/3}(x+8) = \dfrac{4x+8}{3x^{2/3}}$

The local extrema can occur at the critical points $x = -2$ and $x = 0$, but the graph shows that no extrema occurs at $x = 0$. There is a local (and absolute) minimum at $(-2, -6\sqrt[3]{2})$ or approximately (−2, −7.56).

(b) Since $g'(x) > 0$ on the intervals $(-2, 0)$ and $(0, \infty)$, and $g(x)$ is continuous at $x = 0$, $g(x)$ is increasing on $[-2, \infty)$.

(c) Since $g'(x) < 0$ on the interval $(-\infty, -2)$, $g(x)$ is decreasing on $(-\infty, -2]$.

25.



[−5, 5] by [−0.4, 0.4]

(a) $h'(x) = \dfrac{(x^2+4)(-1) - (-x)(2x)}{(x^2+4)^2}$

$= \dfrac{x^2-4}{(x^2+4)^2}$

$= \dfrac{(x+2)(x-2)}{(x^2+4)^2}$

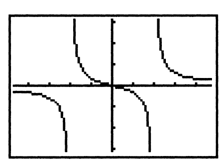
The local extrema occur at the critical points, $x = \pm 2$.

There is a local (and absolute) maximum at $\left(-2, \dfrac{1}{4}\right)$ and a local (and absolute) minimum at $\left(2, -\dfrac{1}{4}\right)$.

(b) Since $h'(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $h(x)$ is increasing on $(-\infty, -2]$ and $[2, \infty)$.

(c) Since $h'(x) < 0$ on $(-2, 2)$, $h(x)$ is decreasing on $[-2, 2]$.

26.



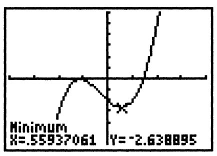
[−4.7, 4.7] by [−3.1, 3.1]

(a) $k'(x) = \dfrac{(x^2-4)(1) - x(2x)}{(x^2-4)^2} = -\dfrac{x^2+4}{(x^2-4)^2}$

Since $k'(x)$ is never zero and is undefined only where $k(x)$ is undefined, there are no critical points. Since there are no critical points and the domain includes no endpoints, $k(x)$ has no local extrema.

(b) Since $k'(x)$ is never positive, $k(x)$ is not increasing on any interval.

(c) Since $k'(x)$ is negative wherever it is defined, $k(x)$ is decreasing on each interval of its domain; on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

27. 

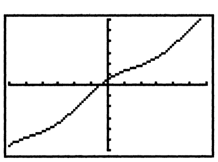
[−4, 4] by [−6, 6]

(a) $f'(x) = 3x^2 - 2 + 2\sin x$

Note that $3x^2 - 2 > 2$ for $|x| \geq 1.2$ and $|2\sin x| \leq 2$ for all x, so $f'(x) > 0$ for $|x| \geq 1.2$. Therefore, all critical points occur in the interval $(-1.2, 1.2)$, as suggested by the graph. Using grapher techniques, there is a local maximum at approximately $(-1.126, -0.036)$, and a local minimum at approximately $(0.559, -2.639)$.

(b) $f(x)$ is increasing on the intervals $(-\infty, -1.126]$ and $[0.559, \infty)$, where the interval endpoints are approximate.

(c) $f(x)$ is decreasing on the interval $[-1.126, 0.559]$, where the interval endpoints are approximate.

28. 

[−6, 6] by [−12, 12]

(a) $g'(x) = 2 - \sin x$

Since $1 \leq g'(x) \leq 3$ for all x, there are no critical points. Since there are no critical points and the domain has no endpoints, there are no local extrema.

(b) Since $g'(x) > 0$ for all x, $g(x)$ is increasing on $(-\infty, \infty)$.

(c) Since $g'(x)$ is never negative, $g(x)$ is not decreasing on any interval.

29. $f(x) = \dfrac{x^2}{2} + C$

30. $f(x) = 2x + C$

31. $f(x) = x^3 - x^2 + x + C$

32. $f(x) = -\cos x + C$

33. $f(x) = e^x + C$

34. $f(x) = \ln(x-1) + C$

35. $f(x) = \dfrac{1}{x} + C,\ x > 0$

$f(2) = 1$

$\dfrac{1}{2} + C = 1$

$C = \dfrac{1}{2}$

$f(x) = \dfrac{1}{x} + \dfrac{1}{2},\ x > 0$

36. $f(x) = x^{1/4} + C$

$f(1) = -2$

$1^{1/4} + C = -2$

$1 + C = -2$

$C = -3$

$f(x) = x^{1/4} - 3$

37. $f(x) = \ln(x+2) + C$

$f(-1) = 3$

$\ln(-1+2) + C = 3$

$0 + C = 3$

$C = 3$

$f(x) = \ln(x+2) + 3$

38. $f(x) = x^2 + x - \sin x + C$

$f(0) = 3$

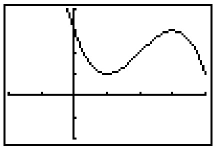
$0 + C = 3$

$C = 3$

$f(x) = x^2 + x - \sin x + 3$

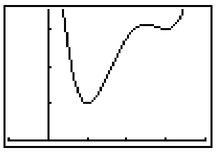
39. Possible answers:

(a)



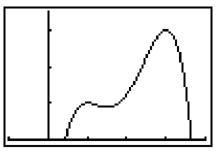
[–2, 4] by [–2, 4]

(b)



[–1, 4] by [0, 3.5]

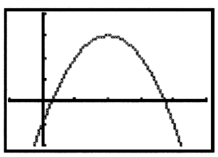
(c)



[–1, 4] by [0, 3.5]

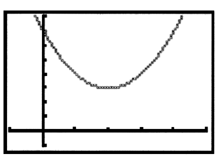
40. Possible answers:

(a)



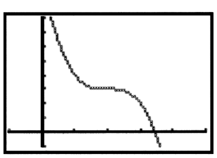
[−1, 5] by [−2, 4]

(b)



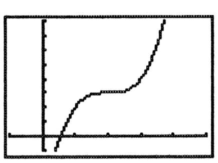
[−1, 5] by [−1, 8]

(c)



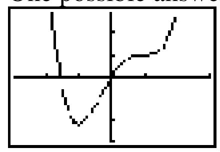
[−1, 5] by [−1, 8]

(d)



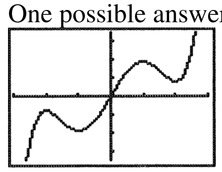
[−1, 5] by [−1, 8]

41. One possible answer:



[−3, 3] by [−15, 15]

42. One possible answer:



[−3, 3] by [−70, 70]

43. **(a)** Since $v'(t) = 1.6$, $v(t) = 1.6t + C$. But $v(0) = 0$, so $C = 0$ and $v(t) = 1.6t$. Therefore, $v(30) = 1.6(30) = 48$. The rock will be going 48 m/sec.

(b) Let $s(t)$ represent position. Since $s'(t) = v(t) = 1.6t$, $s(t) = 0.8t^2 + D$. But $s(0) = 0$, so $D = 0$ and $s(t) = 0.8t^2$. Therefore, $s(30) = 0.8(30)^2 = 720$. The rock travels 720 meters in the 30 seconds it takes to hit bottom, so the bottom of the crevasse is 720 meters below the point of release.

(c) The velocity is now given by $v(t) = 1.6t + C$, where $v(0) = 4$. (Note that the sign of the initial velocity is the same as the sign used for the acceleration, since both act in a downward direction.) Therefore, $v(t) = 1.6t + 4$, and $s(t) = 0.8t^2 + 4t + D$, where $s(0) = 0$ and so $D = 0$. Using $s(t) = 0.8t^2 + 4t$ and the known crevasse depth of 720 meters, we solve $s(t) = 720$ to obtain the positive solution $t \approx 27.604$, and so

$$\begin{aligned} v(t) &= v(27.604) \\ &= 1.6(27.604) + 4 \\ &\approx 48.166. \end{aligned}$$

The rock will hit bottom after about 27.604 seconds, and it will be going about 48.166 m/sec.

44. **(a)** We assume the diving board is located at $s = 0$ and the water at $s = 10$, so that downward velocities are positive. The acceleration due to gravity is 9.8 m/sec^2, so $v'(t) = 9.8$ and $v(t) = 9.8t + C$. Since $v(0) = 0$, we have $v(t) = 9.8t$. Then the position is given by $s(t)$ where $s'(t) = v(t) = 9.8t$, so $s(t) = 4.9t^2 + D$. Since $s(0) = 0$, we have $s(t) = 4.9t^2$. Solving $s(t) = 10$ gives $t^2 = \frac{10}{4.9} = \frac{100}{49}$, so the positive solution is $t = \frac{10}{7}$. The velocity at this time is $v\left(\frac{10}{7}\right) = 9.8\left(\frac{10}{7}\right) = 14$ m/sec.

(b) Again $v(t) = 9.8t + C$, but this time $v(0) = -2$ and so $v(t) = 9.8t - 2$. Then $s'(t) = 9.8t - 2$, so $s(t) = 4.9t^2 - 2t + D$. Since $s(0) = 0$, we have $s(t) = 4.9t^2 - 2t$. Solving $s(t) = 10$ gives the positive solution $t = \frac{2+10\sqrt{2}}{9.8} \approx 1.647$ sec. The velocity at this time is

$$v\left(\frac{2+10\sqrt{2}}{9.8}\right) = 9.8\left(\frac{2+10\sqrt{2}}{9.8}\right) - 2 = 10\sqrt{2} \text{ m/sec}$$

or about 14.142 m/sec.

45. Because the function is not continuous on [0, 1]. The function does not satisfy the hypotheses of the Mean Value Theorem, and so it need not satisfy the conclusion of the Mean Value Theorem.

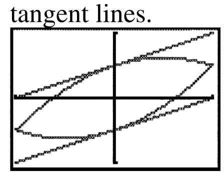
46. Because the Mean Value Theorem applies to the function $y = \sin x$ on any interval, and $y = \cos x$ is the derivative of $\sin x$. So, between any two zeros of $\sin x$, its derivative, $\cos x$, must be zero at least once.

47. $f(x)$ must be zero at least once between a and b by the Intermediate Value Theorem. Now suppose that $f(x)$ is zero twice between a and b. Then by the Mean Value Theorem, $f'(x)$ would have to be zero at least once between the two zeros of $f(x)$, but this can't be true since we are given that $f'(x) \neq 0$ on this interval. Therefore, $f(x)$ is zero once and only once between a and b.

48. Let $f(x) = x^4 + 3x + 1$. Then $f(x)$ is continuous and differentiable everywhere. $f'(x) = 4x^3 + 3$, which is never zero between $x = -2$ and $x = -1$. Since $f(-2) = 11$ and $f(-1) = -1$, exercise 47 applies, and $f(x)$ has exactly one zero between $x = -2$ and $x = -1$.

49. Let $f(x) = x + \ln(x + 1)$. Then $f(x)$ is continuous and differentiable everywhere on [0, 3]. $f'(x) = 1 + \frac{1}{x+1}$, which is never zero on [0, 3]. Now $f(0) = 0$, so $x = 0$ is one solution of the equation. If there were a second solution, $f(x)$ would be zero twice in [0, 3], and by the Mean Value Theorem, $f'(x)$ would have to be zero somewhere between the two zeros of $f(x)$. But this can't happen, since $f'(x)$ is never zero on [0, 3]. Therefore, $f(x) = 0$ has exactly one solution in the interval [0, 3].

50. Consider the function $k(x) = f(x) - g(x)$. $k(x)$ is continuous and differentiable on $[a, b]$, and since $k(a) = f(a) - g(a) = 0$ and $k(b) = f(b) - g(b) = 0$, by the Mean Value Theorem, there must be a point c in (a, b) where $k'(c) = 0$. But since $k'(c) = f'(c) - g'(c)$, this means that $f'(c) = g'(c)$, and c is a point where the graphs of f and g have parallel or identical tangent lines.



(–1, 1) by [–2, 2]

51. False; for example, the function x^3 is increasing on (–1, 1), but $f'(0) = 0$.

52. True; in fact, f is increasing on $[a, b]$ by Corollary 1 to the Mean Value Theorem.

53. A; $f'(x) = \frac{\frac{1}{2} - 1}{\frac{\pi}{3}} = -\frac{3}{2\pi}$.

54. B; $f'(x) = e^{x^3-6x^2+8}(3x^2 - 12x) = e^{x^3-6x^2+8}(3x)(x-4)$, which is negative only when x is between 0 and 4.

55. E; $\frac{d}{dx}(2\sqrt{x} - 10) = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$.

56. D; $x^{3/5}$ is not differentiable at $x = 0$.

57. $\dfrac{f(b)-f(a)}{b-a}=\dfrac{\frac{1}{b}-\frac{1}{a}}{b-a}=-\dfrac{1}{ab}$

$f'(c)=-\dfrac{1}{c^2}$, so $-\dfrac{1}{c^2}=-\dfrac{1}{ab}$ and $c^2=ab$.

Thus, $c=\sqrt{ab}$.

58. $\dfrac{f(b)-f(a)}{b-a}=\dfrac{b^2-a^2}{b-a}=b+a$

$f'(c)=2c$, so $2c=b+a$ and $c=\dfrac{a+b}{2}$.

59. By the Mean Value Theorem, $\sin b-\sin a=(\cos c)(b-a)$ for some c between a and b. Taking the absolute value of both sides and using $|\cos c|\le 1$ gives the result.

60. Since differentiability implies continuity, we can apply the Mean Value Theorem to f on $[a, b]$. Since $f(b)<f(a)$, $\dfrac{f(b)-f(a)}{b-a}$ is negative, and hence $f'(x)$ must be negative at some point between a and b.

61. Let $f(x)$ be a monotonic function defined on an interval D. For any two values in D, we may let x_1 be the smaller value and let x_2 be the larger value, so $x_1<x_2$. Then either $f(x_1)<f(x_2)$ (if f is increasing), or $f(x_1)>f(x_2)$ (if f is decreasing), which means $f(x_1)\ne f(x_2)$. Therefore, f is one-to-one.

62. **(a)** If the maximum occurs at an endpoint, then since $f(a)=f(b)=0$, it follows that $f(x)\le 0$ for all x in (a, b). Similarly, if the maximum occurs at an endpoint, then $f(x)\ge 0$ for all x in (a, b). Thus, if the maximum and minimum only occur at the endpoints, both $f(x)\le 0$ and $f(x)\ge 0$ for all x in (a, b) and the only possibility is that f is the constant function $f(x)=0$. Thus, either $f'(x)=0$ for all x in $[a, b]$ or there is some c, $a<c<b$ where f has a local maximum or minimum.

(b) Since f is given to be differentiable at every point of (a, b), then $f'(c)$ exists for every c in (a, b). If f is the constant function $f(x)=0$, then $f'(c)=0$ for all c in (a, b). If f is not the constant function, then by part (a), f has a local maximum or minimum at some point c in (a, b). Then by Theorem 2, since $f'(c)$ exists, $f'(c)=0$, so there is at least one point c for which $f'(c)=0$.

63. **(a)** At $x=a$, $y=\dfrac{f(b)-f(a)}{b-a}(a-a)+f(a)$
$=0+f(a)$
$=f(a)$.

At $x=b$, $y=\dfrac{f(b)-f(a)}{b-a}(b-a)+f(a)$
$=f(b)-f(a)+f(a)$
$=f(b)$.

Thus, y is the equation of the secant line through $(a, f(a))$ and $(b, f(b))$.

(b) Since $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and y is a linear function, hence continuous and differentiable everywhere, the difference, $g(x)$, is continuous on $[a, b]$ and differentiable on (a, b).
$g(a)=f(a)-f(a)=0$
$g(b)=f(b)-f(b)=0$
Thus, Rolle's Theorem applies and there is one c in (a, b) for which $g'(c)=0$.

(c) $g'(x)=f'(x)-\dfrac{f(b)-f(a)}{b-a}$ so

$g'(c)=f'(c)-\dfrac{f(b)-f(a)}{b-a}$.

Since $g'(c)=0$, $0=f'(c)-\dfrac{f(b)-f(a)}{b-a}$

or $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Thus, there is one point c in (a, b) for which $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Section 5.3 Connecting f' and f'' with the Graph of f (pp. 211–223)

Exploration 1 Finding f from f'

1. Any function $f(x)=x^4-4x^3+C$ where C is a real number. For example, let $C=0, 1, 2$. Their graphs are all vertical shifts of each other.

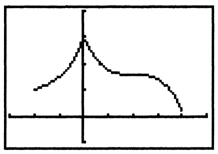
2. Their behavior is the same as the behavior of the function f of Example 7.

Exploration 2 Finding f from f' and f''

1. f has an absolute maximum at $x = 0$ and an absolute minimum of 1 at $x = 4$. We are not given enough information to determine $f(0)$.

2. f has a point of inflection at $x = 2$.

3.



[−3, 5] by [−5, 20]

Quick Review 5.3

1.
$$x^2 - 9 < 0$$
$$(x+3)(x-3) < 0$$

Intervals	$x < -3$	$-3 < x < 3$	$3 < x$
Sign of $(x+3)(x-3)$	+	−	+

Solution set: (–3, 3)

2.
$$x^3 - 4x > 0$$
$$x(x+2)(x-2) > 0$$

Intervals	$x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x$
Sign of $x(x+2)(x-2)$	−	+	−	+

Solution set: $(-2, 0) \cup (2, \infty)$

3. f: all reals
f': all reals, since $f'(x) = xe^x + e^x$

4. f: all reals
f': $x \neq 0$, since $f'(x) = \frac{3}{5}x^{-2/5}$

5. f: $x \neq 2$
f': $x \neq 2$, since $f'(x) = \frac{(x-2)(1)-(x)(1)}{(x-2)^2} = \frac{-2}{(x-2)^2}$

6. f: all reals
f': $x \neq 0$, since $f'(x) = \frac{2}{5}x^{-3/5}$

7. Left end behavior model: 0
Right end behavior model: $-x^2e^x$
Horizontal asymptote: $y = 0$

8. Left end behavior model: x^2e^{-x}
Right end behavior model: 0
Horizontal asymptote: $y = 0$

9. Left end behavior model: 0
Right end behavior model: 200
Horizontal asymptote: $y = 0$, $y = 200$

10. Left end behavior model: 0
Right end behavior model: 375
Horizontal asymptotes: $y = 0$, $y = 375$

Section 5.3 Exercises

1. $y' = 2x - 1$

Intervals	$x < \frac{1}{2}$	$x > \frac{1}{2}$
Sign of y'	–	+
Behavior of y	Decreasing	Increasing

Local (and absolute) minimum at $\left(\frac{1}{2}, -\frac{5}{4}\right)$

2. $y' = -6x^2 + 12x = -6x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y'	–	+	–
Behavior of y	Decreasing	Increasing	Decreasing

Local maximum: (2, 5);
local minimum: (0, –3)

3. $y' = 8x^3 - 8x = 8x(x-1)(x+1)$

Intervals	$x < -1x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	–	+	–	+
Behavior of y	Decreasing	Increasing	Decreasing	Increasing

Local maximum: (0, 1); local (and absolute) minima: (–1, –1) and (1, –1)

4. $y' = xe^{1/x}(-x^{-2}) + e^{1/x} = e^{1/x}\left(1 - \frac{1}{x}\right)$

Intervals	$x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	+	–	+
Behavior of y	Increasing	Decreasing	Increasing

Local minimum: $(1, e)$

5. $y' = x\frac{1}{2\sqrt{8-x^2}}(-2x) + \sqrt{8-x^2}(1) = \frac{8-2x^2}{\sqrt{8-x^2}}$

Intervals	$-\sqrt{8} < x < -2$	$-2 < x < 2$	$2 < x < \sqrt{8}$
Sign of y'	–	+	–
Behavior of y	Decreasing	Increasing	Decreasing

Local maxima: $\left(-\sqrt{8},\ 0\right)$ and (2, 4);

local minima: (–2, –4) and $\left(\sqrt{8},\ 0\right)$

Note that the local extrema at $x = \pm 2$ are also absolute extrema.

6. $y' = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases}$

Intervals	$x < 0$	$x > 0$
Sign of y'	+	+
Behavior of y	Increasing	Increasing

Local minimum: (0, 1)

7. $y' = 12x^2 + 42x + 36$

$y'' = 24x + 42 = 6(4x + 7)$

Intervals	$x < -\frac{7}{4}$	$-\frac{7}{4} < x$
Sign of y''	–	+
Behavior of y	Concave down	Concave up

(a) $\left(-\frac{7}{4}, \infty\right)$

(b) $\left(-\infty, -\frac{7}{4}\right)$

8. $y' = -4x^3 + 12x^2 - 4$

$y'' = -12x^2 + 24x = -12x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y''	−	+	−
Behavior of y	Concave down	Concave up	Concave down

(a) (0, 2)

(b) $(-\infty, 0)$ and $(2, \infty)$

9. $y' = \frac{2}{5}x^{-4/5}$

$y'' = -\frac{8}{25}x^{-9/5}$

Intervals	$x < 0$	$0 < x$
Sign of y''	+	−
Behavior of y	Concave up	Concave down

(a) $(-\infty, 0)$

(b) $(0, \infty)$

10. $y' = -\frac{1}{3}x^{-2/3}$

$y'' = \frac{2}{9}x^{-5/3}$

Intervals	$x < 0$	$0 < x$
Sign of y''	−	+
Behavior of y	Concave down	Concave up

(a) $(0, \infty)$

(b) $(-\infty, 0)$

11. $y' = \begin{cases} 2, & x < 1 \\ -2x, & x > 1 \end{cases}$

$y'' = \begin{cases} 0, & x < 1 \\ -2, & x > 1 \end{cases}$

Intervals	$x < 1$	$1 < x$
Sign of y''	0	–
Behavior of y	Linear	Concave down

(a) None

(b) $(1, \infty)$

12. $y' = e^x$

$y'' = e^x$

Since y' and y'' are both positive on the entire domain, y is increasing and concave up on the entire domain.

(a) $(0, 2\pi)$

(b) None

13. $y = xe^x$

$y' = e^x + xe^x$

$y'' = 2e^x + xe^x$

$y'' = 0$ at $x = -2$

Intervals	$x < -2$	$x > -2$
Sign of y''	–	+
Behavior of y	Concave down	Concave up

Inflection point at $\left(-2, -\frac{2}{e^2}\right)$

14. $y = x\sqrt{9 - x^2}$

$$y' = \sqrt{9 - x^2} - \frac{x^2}{\sqrt{9 - x^2}}$$

$$y'' = -\frac{x}{(9 - x^2)^{1/2}} + \frac{x^3 - 18x}{(9 - x^2)^{3/2}} = \frac{x(2x^2 - 27)}{(9 - x^2)^{3/2}}.$$

On the domain $[-3, 3]$, $y'' = 0$ only at $x = 0$

Intervals	$-3 < x < 0$	$0 < x < 3$
Sign of y''	+	–
Behavior of y	Concave up	Concave down

Inflection point at (0, 0)

15. $y' = \dfrac{1}{1+x^2}$

$y'' = \dfrac{d}{dx}(1+x^2)^{-1}$

$= -(1+x^2)^{-2}(2x)$

$= \dfrac{-2x}{(1+x^2)^2}$

Intervals	$x < 0$	$0 < x$
Sign of y''	+	−
Behavior of y	Concave up	Concave down

Inflection point at (0, 0)

16. $y = x^3(4-x)$

$y' = 12x^2 - 4x^3$

$y'' = 24x - 12x^2$

Intervals	$x < 0$	$0 < x < 2$	$x > 2$
Sign of y''	−	+	−
Behavior of y	Concave down	Concave up	Concave down

Inflection points at $(0, 0)$ and $(2, 16)$

17. $y = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$

$y' = \dfrac{4}{3}x^{1/3} - \dfrac{4}{3}x^{-2/3} = \dfrac{4x-4}{3x^{2/3}}$

$y'' = \dfrac{4}{9}x^{-2/3} + \dfrac{8}{9}x^{-5/3} = \dfrac{4x+8}{9x^{5/3}}$

Intervals	$x < -2$	$-2 < x < 0$	$0 < x$
Sign of y''	+	−	+
Behavior of y	Concave up	Concave down	Concave up

Inflection points at $(-2, 6\sqrt[3]{2}) \approx (-2, 7.56)$ and (0, 0)

18. $y = x^{1/2}(x+3)$

$y' = \dfrac{1}{2}x^{-1/2}(x+3) + x^{1/2}$

$y'' = \dfrac{1}{(x)^{1/2}} - \dfrac{x+3}{4(x)^{3/2}} = 0$

Intervals	$0 < x < 1$	$x > 1$
Sign of y''	+	−
Behavior of y	Concave up	Concave down

Inflection pt at (1, 4)

19. $y' = \dfrac{(x-2)(3x^2-4x+1)-(x^3-2x^2+x-1)(1)}{(x-2)^2}$

$= \dfrac{2x^3-8x^2+8x-1}{(x-2)^2}$

$y'' = \dfrac{(x-2)^2(6x^2-16x+8)-(2x^3-8x^2+8x-1)(2)(x-2)}{(x-2)^4}$

$= \dfrac{(x-2)(6x^2-16x+8)-2(2x^3-8x^2+8x-1)}{(x-2)^3}$

$= \dfrac{2x^3-12x^2+24x-14}{(x-2)^3}$

$= \dfrac{2(x-1)(x^2-5x+7)}{(x-2)^3}$

Note that the discriminant of x^2-5x+7 is $(-5)^2-4(1)(7)=-3$, so the only solution of $y''=0$ is $x=1$.

Intervals	$x < 1$	$1 < x < 2$	$2 < x$
Sign of y''	−	+	−
Behavior of y	Concave up	Concave down	Concave up

Inflection point at (1, 1)

20. $y' = \dfrac{(x^2+1)(1)-x(2x)}{(x^2+1)^2} = \dfrac{-x^2+1}{(x^2+1)^2}$

$y'' = \dfrac{(x^2+1)^2(-2x)-(-x^2+1)(2)(x^2+1)(2x)}{(x^2+1)^4}$

$= \dfrac{(x^2+1)(-2x)-4x(-x^2+1)}{(x^2+1)^3}$

$= \dfrac{2x^3-6x}{(x^2+1)^3} = \dfrac{2x(x^2-3)}{(x^2+1)^3}$

Intervals	$x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$\sqrt{3} < x$
Sign of y'	−	+	−	+
Behavior of y	Concave down	Concave up	Concave down	Concave up

Inflection points at (0, 0), $\left(\sqrt{3}, \dfrac{\sqrt{3}}{4}\right)$, and $\left(-\sqrt{3}, -\dfrac{\sqrt{3}}{4}\right)$

21. **(a)** Zero: $x = \pm 1$;
positive: $(-\infty, -1)$ and $(1, \infty)$;
negative: $(-1, 1)$

(b) Zero: $x = 0$;
positive: $(0, \infty)$;
negative: $(-\infty, 0)$

22. **(a)** Zero: $x \approx 0, \pm 1.25$;
positive: $(-1.25, 0)$ and $(1.25, \infty)$;
negative: $(-\infty, -1.25)$ and $(0, 1.25)$

(b) Zero: $x \approx \pm 0.7$;
positive: $(-\infty, -0.7)$ and $(0.7, \infty)$;
negative: $(-0.7, 0.7)$

23. **(a)** $(-\infty, -2]$ and $[0, 2]$

(b) $[-2, 0]$ and $[2, \infty)$

(c) Local maxima: $x = -2$ and $x = 2$;
local minimum: $x = 0$

24. **(a)** $[-2, 2]$

(b) $(-\infty, -2]$ and $[2, \infty)$

(c) Local maximum: $x = 2$;
local minimum: $x = -2$

25. **(a)** $v(t) = x'(t) = 2t - 4$

(b) $a(t) = v'(t) = 2$

(c) It begins at position 3 moving in a negative direction. It moves to position –1 when $t = 2$, and then changes direction, moving in a positive direction thereafter.

26. **(a)** $v(t) = x'(t) = -2 - 2t$

(b) $a(t) = v'(t) = -2$

(c) It begins at position 6 and moves in the negative direction thereafter.

27. **(a)** $v(t) = x'(t) = 3t^2 - 3$

(b) $a(t) = v'(t) = 6t$

(c) It begins at position 3 moving in a negative direction. It moves to position 1 when $t = 1$, and then changes direction, moving in a positive direction thereafter.

28. **(a)** $v(t) = x'(t) = 6t - 6t^2$

(b) $a(t) = v'(t) = 6 - 12t$

(c) It begins at position 0. It starts moving in the positive direction until it reaches position 1 when $t = 1$, and then it changes direction. It moves in the negative direction thereafter.

29. **(a)** The velocity is zero when the tangent line is horizontal, at approximately $t = 2.2$, $t = 6$ and $t = 9.8$.

(b) The acceleration is zero at the inflection points, approximately $t = 4$, $t = 8$ and $t = 11$.

30. **(a)** The velocity is zero when the tangent line is horizontal, at approximately $t = -0.2$, $t = 4$, and $t = 12$.

(b) The acceleration is zero at the inflection points, approximately $t = 1.5$, $t = 5.2$, $t = 8$, $t = 11$, and $t = 13$.

31. $y = 3x - x^3 + 5$
$y' = 3 - 3x^2$
$y'' = -6x$
$y' = 0$ at ± 1.
$y''(-1) > 0$ and $y''(1) < 0$, so there is a local minimum at (–1, 3) and a local maximum at (1, 7).

32. $y = x^5 - 80x + 100$
$y' = 5x^4 - 80$
$y'' = 20x^3$
$y' = 0$ at ± 2
$y''(-2) < 0$ and $y''(2) > 0$, so there is a local maximum at (–2, 228) and a local minimum at (2, –28).

33. $y = x^3 + 3x^2 - 2$
$y' = 3x^2 + 6x$
$y'' = 6x + 6$
$y' = 0$ at -2 and 0.
$y''(-2) < 0$, $y''(0) > 0$,
$y = x^3 + 3x^2 - 2$

$y = x^3 + 3x^2 - 2$
$y' = 3x^2 + 6x$
$y'' = 6x + 6$
$y' = 0$ at -2 and 0.
$y''(-2) < 0,\ y''(0) > 0,$
so there is a local maximum at (–2, 2) and a local minimum at (0, –2).

34. $y = 3x^5 - 25x^3 + 60x + 20$
$y' = 15x^4 - 75x^2 + 60$
$y'' = 60x^3 - 150x$
$y' = 0$ at ± 1 and ± 2.
$y''(-2) < 0,\ y''(-1) > 0$
$y''(1) < 0,$ and $y''(2) > 0;$
so there are local maxima at (–2, 4) and
(1, 58), and there are local minima at (–1, –18) and (2, 36).

35. $y = xe^x$
$y' = (x+1)e^x$
$y'' = (x+2)e^x$
$y' = 0$ at -1.
$y''(-1) > 0$, so there is a local minimum at $\left(-1, -\frac{1}{e}\right)$.

36. $y = xe^{-x}$
$y' = (1-x)e^{-x}$
$y'' = (x-2)e^{-x}$
$y' = 0$ at 1
$y''(1) < 0,$ so there is a local maximum at $\left(1, \frac{1}{e}\right)$.

37. $y' = (x-1)^2(x-2)$

Intervals	$x < 1$	$1 < x < 2$	$2 < x$
Sign of y'	−	−	+
Behavior of y	Decreasing	Decreasing	Increasing

$y'' = (x-1)^2(1) + (x-2)(2)(x-1)$
$= (x-1)[(x-1) + 2(x-2)]$
$= (x-1)(3x-5)$

Intervals	$x < 1$	$1 < x < \frac{5}{3}$	$\frac{5}{3} < x$
Sign of y''	+	−	+
Behavior of y	Concave up	Concave down	Concave up

(a) There are no local maxima.

(b) There is a local (and absolute) minimum at $x = 2$.

(c) There are points of inflection at $x = 1$ and at $x = \frac{5}{3}$.

38. $y' = (x-1) \bullet (x-2)(x-4)$

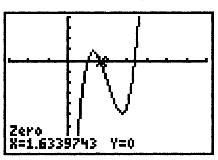
Intervals	$x < 1$	$1 < x < 2$	$2 < x < 4$	$4 < x$
Sign of y'	+	+	–	+
Behavior of y	Increasing	Increasing	Decreasing	Increasing

$$
\begin{aligned}
y'' &= \frac{d}{dx}[(x-1)^2(x^2-6x+8)] \\
&= (x-1)^2(2x-6)+(x^2-6x+8)(2)(x-1) \\
&= (x-1)[(x-1)(2x-6)+2(x^2-6x+8)] \\
&= (x-1)(4x^2-20x+22) \\
&= 2(x-1)(2x^2-10x+11)
\end{aligned}
$$

Note that the zeros of y'' are $x = 1$ and

$$
\begin{aligned}
x &= \frac{10 \pm \sqrt{10^2-4(2)(11)}}{4} \\
&= \frac{10 \pm \sqrt{12}}{4} \\
&= \frac{5 \pm \sqrt{3}}{2} \approx 1.63 \text{ or } 3.37.
\end{aligned}
$$

The zeros of y'' can also be found graphically, as shown.



[−3, 7] by [−8, 4]

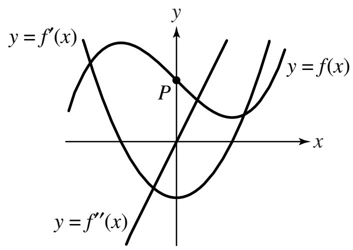
Intervals	$x < 1$	$1 < x < 1.63$	$1.63 < x < 3.37$	$3.37 < x$
Sign of y''	–	+	–	+
Behavior of y	Concave down	Concave up	Concave down	Concave up

(a) Local maximum at $x = 2$

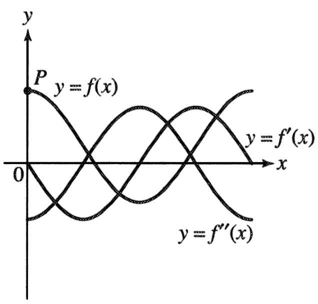
(b) Local minimum at $x = 4$

(c) Points of inflection at $x = 1$, at $x \approx 1.63$, and at $x \approx 3.37$.

39.



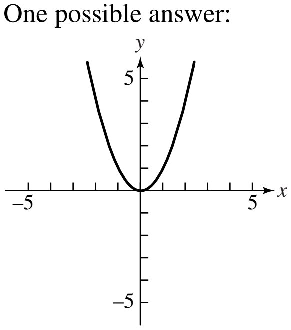
40.



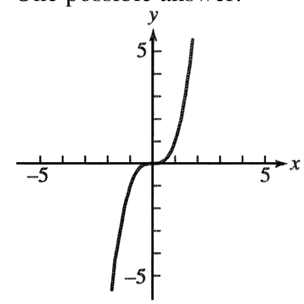
41. No. f must have a horizontal tangent at that point, but f could be increasing (or decreasing), and there would be no local extremum. For example, if $f(x) = x^3$, $f'(0) = 0$ but there is no local extremum at $x = 0$.

42. No; $f''(x)$ could still be positive (or negative) on both sides of $x = c$, in which case the concavity of the function would not change at $x = c$. For example, if $f(x) = x^4$, then $f''(0) = 0$, but f has no inflection point at $x = 0$.

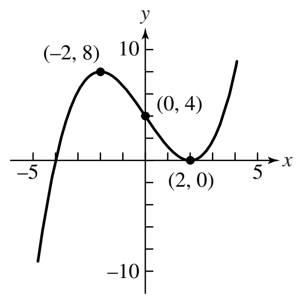
43. One possible answer:



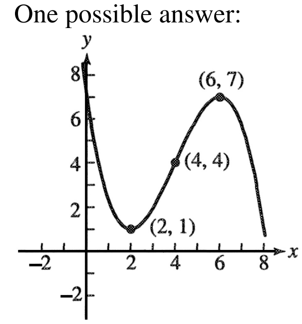
44. One possible answer:



45. One possible answer:



46. One possible answer:



47. **(a)** [0, 1], [3, 4], and [5.5, 6]

(b) [1, 3] and [4, 5.5]

(c) Local maxima: $x = 1$, $x = 4$ (if f is continuous at $x = 4$), and $x = 6$; local minima: $x = 0$, $x = 3$, and $x = 5.5$

48. If f is continuous on the interval [0, 3]:

(a) [0, 3]

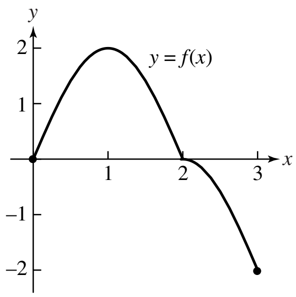
(b) Nowhere

(c) Local maximum: $x = 3$; local minimum: $x = 0$

49. **(a)** Absolute maximum at (1, 2); absolute minimum at (3, –2)

(b) None

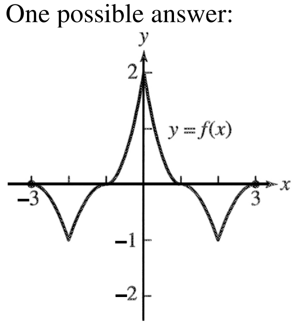
(c) One possible answer:



50. **(a)** Absolute maximum at (0, 2); absolute minimum at (2, –1) and (–2, –1)

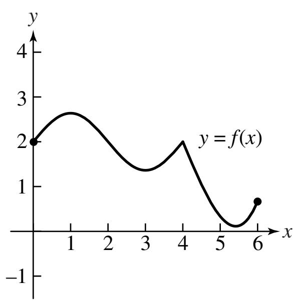
(b) At (1, 0) and (–1, 0)

(c) One possible answer:

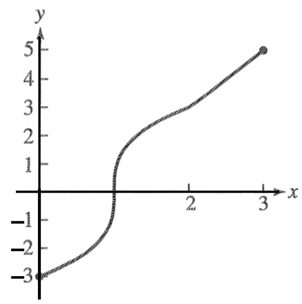


(d) Since f is even, we know $f(3) = f(-3)$. By the continuity of f, since $f(x) < 0$ when $2 < x < 3$, we know that $f(3) \le 0$, and since $f(2) = -1$ and $f'(x) > 0$ when $2 < x < 3$, we know that $f(3) > -1$. In summary, we know that $f(3) = f(-3)$, $-1 < f(3) \le 0$, and $-1 < f(-3) \le 0$.

51.



52.



53. False. For example, consider $f(x) = x^4$ at $c = 0$.

54. True. This is the Second Derivative Test for a local maximum.

55. A; $y = ax^3 + 3x^2 + 4x + 5$ say $a = -2$

$y' = -6x^2 + 6x + 4$

$y'' = -12x + 6$

$y'' = 0$ at $\frac{1}{2}$

Intervals	$x < \frac{1}{2}$	$x > \frac{1}{2}$
Sign of y''	+	–
Behavior of y	Concave up	Concave down

56. E

57. C; $y = x^5 - 5x^4 + 3x + 7$

$y' = 5x^4 - 20x^3 + 3$

$y'' = 20x^3 - 60x^2 = 20x^2(x - 3)$

Note that $y'' = 0$ at $x = 0$ and $x = 3$, but y'' only changes sign at $x = 3$.

Intervals	$x < 3$	$x > 3$
Sign of y''	–	+
Behavior of y	Concave down	Concave up

(3, –146) is an inflection point.

58. A. There is a local maximum of f' at $x = c$.

59. **(a)** In exercise 7, $a = 4$ and $b = 21$, so $-\frac{b}{3a} = -\frac{7}{4}$, which is the x-value where the point of inflection occurs. The local extrema are at $x=-2$ and $x=-\frac{3}{2}$, which are symmetric about $x=-\frac{7}{4}$.

(b) In exercise 2, $a=-2$ and $b=6$, so $-\frac{b}{3a}=1$, which is the x-value where the point of inflection occurs. The local extrema are at $x=0$ and $x=2$, which are symmetric about $x=1$.

(c) $f'(x)=3ax^2+2bx+c$ and $f''(x)=6ax+2b$.
The point of inflection will occur where $f''(x)=0$, which is at $x=-\frac{b}{3a}$.
If there are local extrema, they will occur at the zeros of $f'(x)$. Since $f'(x)$ is quadratic, its graph is a parabola and any zeros will be symmetric about the vertex which will also be where $f''(x)=0$.

60. **(a)** $f'(x)=4ax^3+3bx^2+2cx+d$

$f''(x)=12ax^2+6bx+2c$

Since $f''(x)$ is quadratic, it must have 0, 1, or 2 zeros. A quadratic with 0 or 1 zeros never changes sign, so *f* has no points of infection if $f''(x)$ has 0 or 1 zeros. If $f''(x)$ has 2 zeros, it will change sign twice, and $f(x)$ will have 2 points of inflection.

(b) $f(x)$ has two points of inflection if and only if $3b^2>8ac$.

Quick Quiz Sections 5.1–5.3

1. C; $f'(x)=5(x-2)^4(x+3)^4+4(x-2)^5(x+3)^3=0$

$x=-3, -\frac{7}{9}, 2$

2. D; $f'(x)=(x-3)^2+2(x-2)(x-3)=(x-3)(3x-7)$

$f'(x)=0$ when $x=3$ or $x=\frac{7}{3}$

$f''(x)=6x-16$

$f''(3)=2>0$

$f''\left(\frac{7}{3}\right)=-2<0$

Relative maximum is at $x=\frac{7}{3}$ only.

3. B; $x^2-9=0$

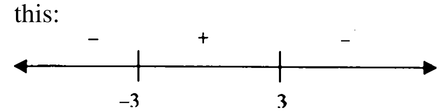
$x=\pm 3$

$f''(x)=2x$

$f''(3)=6>0$

$f''(-3)=-6<0$

$f'(x)=(x^2-9)g(x)$; where $g(x)<0$ for all x. Thus the sign graph for $f'(x)$ looks like this:



By the First Derivative Test, *f* has a relative maximum at $x=-3$ and a relative minimum at $x=3$.

4. **(a)** $\frac{d}{dx}\left(3\ln(x^2+2)-2x\right)=3\frac{2x}{x^2+2}-2=0$

$x=1, 2$

Intervals	$-2<x<1$	$1<x<2$	$2<x<4$
Sign of y'	–	+	–
Behavior of y	Decreasing	Increasing	Decreasing

f has relative minima at $x=1$ and $x=4$, *f* has relative maxima at $x=\pm 2$

(b) $f''(x)=\frac{d}{dx}\left(\frac{6x}{x^2+2}-2\right)$

$f''(x)=\frac{6}{x^2+2}-\frac{12x^2}{(x^2+2)^2}=0$

$x=\pm\sqrt{2}$

f has points of inflection at $x=\pm\sqrt{2}$

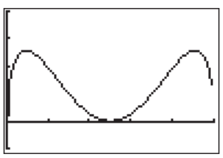
(c) The absolute maximum is at $x=-2$ and $f(x)=3\ln 6+4$.

Section 5.4 Modeling and Optimization (pp. 224–237)

Exploration 1 Constructing Cones

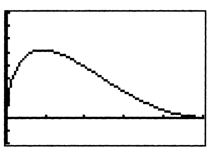
1. The circumference of the base of the cone is the circumference of the circle of radius 4 minus x, or $8\pi - x$. Thus, $r = \frac{8\pi - x}{2\pi}$. Use the Pythagorean Theorem to find h, and the formula for the volume of a cone to find V.

2. The expression under the radical must be nonnegative, that is, $16 - \left(\frac{8\pi - x}{2\pi}\right)^2 \geq 0$. Solving this inequality for x gives: $0 \leq x \leq 16\pi$.



[0, 16π] by [−10, 40]

3. The circumference of the original circle of radius 4 is 8π. Thus, $0 \leq x \leq 8\pi$.



[0, 8π] by [−10, 40]

4. The maximum occurs at about $x = 4.61$. The maximum volume is about $V = 25.80$.

5. Start with $\frac{dV}{dx} = \frac{2\pi}{3} rh \frac{dr}{dx} + \frac{\pi}{3} r^2 \frac{dh}{dx}$.
Compute $\frac{dr}{dx}$ and $\frac{dh}{dx}$, substitute these values in $\frac{dV}{dx}$, set $\frac{dV}{dx} = 0$, and solve for x to obtain $x = \frac{8(3-\sqrt{6})\pi}{3} \approx 4.61$. Then $V = \frac{128\pi\sqrt{3}}{27} \approx 25.80$.

Quick Review 5.4

1. $y' = 3x^2 - 12x + 12 = 3(x-2)^2$
Since $y' \geq 0$ for all x (and is increasing on $y' > 0$ for $x \neq 2$), y is increasing on $(-\infty, \infty)$ and there are no local extrema.

2. $y' = 6x^2 + 6x - 12 = 6(x+2)(x-1)$
$y'' = 12x + 6$
The critical points occur at $x = -2$ or $x = 1$, since $y' = 0$ at these points. Since $y''(-2) = -18 < 0$, the graph has a local maximum at $x = -2$. Since $y''(1) = 18 > 0$, the graph has a local minimum at $x = 1$. In summary, there is a local maximum at $(-2, 17)$ and a local minimum at $(1, -10)$.

3. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(5)^2(8) = \frac{200\pi}{3}\text{cm}^3$

4. $V = \pi r^2 h = 1000$
$SA = 2\pi rh + 2\pi r^2 = 600$
Solving the volume equation for h gives $= \frac{1000}{\pi r^2}$. Substituting into the surface area equation gives $\frac{2000}{r} + 2\pi r^2 = 600$. Solving graphically, we have $r \approx -11.14$, $r \approx 4.01$, or $r \approx 7.13$. Discarding the negative value and using $h = \frac{1000}{\pi r^2}$ to find the corresponding values of h, the two possibilities for the dimensions of the cylinder are:
$r \approx 4.01$ cm and $h \approx 19.82$ cm, or,
$r \approx 7.13$ cm and $h \approx 6.26$ cm.

5. Since $y = \sin x$ is an odd function, $\sin(-\alpha) = -\sin \alpha$.

6. Since $y = \cos x$ is an even function, $\cos(-\alpha) = \cos \alpha$.

7. $$\begin{aligned}\sin(\pi - \alpha) &= \sin\pi\cos\alpha - \cos\pi\sin\alpha \\ &= 0\cos\alpha - (-1)\sin\alpha \\ &= \sin\alpha\end{aligned}$$

8. $$\begin{aligned}\cos(\pi - \alpha) &= \cos\pi\cos\alpha + \sin\pi\sin\alpha \\ &= (-1)\cos\alpha + 0\sin\alpha \\ &= -\cos\alpha\end{aligned}$$

9. $x^2 + y^2 = 4$ and $y = \sqrt{3x}$
$$x^2 + \left(\sqrt{3x}\right)^2 = 4$$
$$x^2 + 3x = 4$$
$$x^2 + 3x - 4 = 0$$
$$(x+4)(x-1) = 0$$
$x = -4$ or $x = 1$

Since $y = \sqrt{3x}$, $x = -4$ is an extraneous solution. The only solution is: $x = 1$, $y = \sqrt{3}$. In ordered pair notation, the solution is $\left(1, \sqrt{3}\right)$.

10. $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and $y = x + 3$

$$\frac{x^2}{4} + \frac{(x+3)^2}{9} = 1$$
$$9x^2 + 4(x+3)^2 = 36$$
$$9x^2 + 4x^2 + 24x + 36 = 36$$
$$13x^2 + 24x = 0$$
$$x(13x + 24) = 0$$

$x = 0$ or $x = -\frac{24}{13}$

Since $y = x + 3$, the solutions are:

$x = 0$ and $y = 3$, or, $x = -\frac{24}{13}$ and $y = \frac{15}{13}$.

In ordered pair notation, the solution are (0, 3) and $\left(-\frac{24}{13}, \frac{15}{13}\right)$.

Section 5.4 Exercises

1. Represent the numbers by x and $20 - x$, where $0 \le x \le 20$.

(a) The sum of the squares is given by $f(x) = x^2 + (20 - x)^2 = 2x^2 - 40x + 400$. Then $f'(x) = 4x - 40$. The critical point and endpoints occur at $x = 0$, $x = 10$, and $x = 20$. Then $f(0) = 400$, $f(10) = 200$, and $f(20) = 400$. The sum of the squares is as large as possible for the numbers 0 and 20, and is as small as possible for the numbers 10 and 10.

(b) The sum of one number plus the square root of the other is given by $g(x) = x + \sqrt{20 - x}$. Then $g'(x) = 1 - \frac{1}{2\sqrt{20 - x}}$. The critical point occurs when $2\sqrt{20 - x} = 1$, so $20 - x = \frac{1}{4}$ and $x = \frac{79}{4}$. Testing the endpoints and critical point, we find $g(0) = \sqrt{20} \approx 4.47$, $g\left(\frac{79}{4}\right) = \frac{81}{4} = 20.25$, and $g(20) = 20$. The sum $x + \sqrt{20 - x}$ is as large as possible when the numbers are $x = \frac{79}{4}$ and $20 - x = \frac{1}{4}$. The sum $x + \sqrt{20 - x}$ is as small as possible when the numbers are $x = 0$ and $20 - x = 20$.

2. Let x and y represent the legs of the triangle, and note that $0 < x < 5$. Then $x^2 + y^2 = 25$, so $y = \sqrt{25 - x^2}$ (since $y > 0$). The area is $A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{25 - x^2}$, so

$$\frac{dA}{dx} = \frac{1}{2}x\frac{1}{2\sqrt{25 - x^2}}(-2x) + \frac{1}{2}\sqrt{25 - x^2} = \frac{25 - 2x^2}{2\sqrt{25 - x^2}}.$$

The critical point occurs when $25 - 2x^2 = 0$, which means $x = \frac{5}{\sqrt{2}}$, (since $x > 0$). This value corresponds to the largest possible area, since $\frac{dA}{dx} > 0$ for $0 < x < \frac{5}{\sqrt{2}}$ and $\frac{dA}{dx} < 0$ for $\frac{5}{\sqrt{2}} < x < 5$. When $x = \frac{5}{\sqrt{2}}$, we have $y = \sqrt{25 - \left(\frac{5}{\sqrt{2}}\right)^2} = \frac{5}{\sqrt{2}}$ and $A = \frac{1}{2}xy = \frac{1}{2}\left(\frac{5}{\sqrt{2}}\right)^2 = \frac{25}{4}$. Thus, the largest possible area is $\frac{25}{4}\text{cm}^2$, and the dimensions (legs) are $\frac{5}{\sqrt{2}}$ cm by $\frac{5}{\sqrt{2}}$ cm.

3. Let x represent the length of the rectangle in inches ($x > 0$).

Then the width is $\frac{16}{x}$ and the perimeter is $P(x) = 2\left(x + \frac{16}{x}\right) = 2x + \frac{32}{x}$.

Since $P'(x) = 2 - 32x^{-2} = \frac{2(x^2 - 16)}{x^2}$ this critical point occurs at $x = 4$. Since $P'(x) < 0$ for $0 < x < 4$ and $P'(x) > 0$ for $x > 4$, this critical point corresponds to the minimum perimeter. The smallest possible perimeter is $P(4) = 16$ in., and the rectangle's dimensions are 4 in. by 4 in.

4. Let x represent the length of the rectangle in meters $(0 < x < 4)$. Then the width is $4 - x$ and the area is $A(x) = x(4 - x) = 4x - x^2$. Since $A'(x) = 4 - 2x$, the critical point occurs at $x = 2$. Since $A'(x) > 0$ for $0 < x < 2$ and $A'(x) < 0$ for $2 < x < 4$, this critical point corresponds to the maximum area. The rectangle with the largest area measures 2 m by 4 – 2 = 2m, so it is a square.

5. **(a)** The equation of line AB is $y = -x + 1$, so the y-coordinate of P is $-x + 1$.

(b) $A(x) = 2x(1 - x)$

(c) Since $A'(x) = \frac{d}{dx}(2x - 2x^2) = 2 - 4x$, the critical point occurs at $x = \frac{1}{2}$. Since $A'(x) > 0$ for $0 < x < \frac{1}{2}$ and $A'(x) < 0$ for $\frac{1}{2} < x < 1$, this critical point corresponds to the maximum area. The largest possible area is $A\left(\frac{1}{2}\right) = \frac{1}{2}$ square unit, and the dimensions of the rectangle are $\frac{1}{2}$ unit by 1 unit.

6. If the upper right vertex of the rectangle is located at $(x,\ 12 - x^2)$ for $0 < x < \sqrt{12}$, then the rectangle's dimensions are $2x$ by $12 - x^2$ and the area is A is $(x) = 2x\,(12 - x^2) = 24x - 2\,x^3$. Then $A'(x) = 24 - 6x^2 = 6(4 - x^2)$, so the critical point $\left(\text{for } 0 < x < \sqrt{12}\right)$ occurs at $x = 2$. Since $A'(x) > 0$ for $0 < x < 2$ and $A'(x) < 0$ for $2 < x < \sqrt{12}$, this critical point corresponds to the maximum area. The largest possible area is $A(2) = 32$, and the dimensions are 4 by 8.

7. Let x be the side length of the cut-out square $(0 < x < 4)$. Then the base measures $8 - 2x$ in. by $15 - 2x$ in., and the volume is

$$V(x) = x(8 - 2x)(15 - 2x)$$
$$= 4x^3 - 46x^2 + 120x.$$

Then

$$V'(x) = 12x^2 - 92x + 120 = 4(3x - 5)(x - 6).$$

Then the critical point (in $0 < x < 4$) occurs at $x = \frac{5}{3}$. Since $V'(x) > 0$ for $0 < x < \frac{5}{3}$ and $V'(x) < 0$ for $\frac{5}{3} < x < 4$, the critical point corresponds to the maximum volume. The maximum volume is $V\left(\frac{5}{3}\right) = \frac{2450}{27} \approx 90.74\,\text{in}^3$, and the dimensions are $\frac{5}{3}$ in. by $\frac{14}{3}$ in. by $\frac{35}{3}$ in.

8. Note that the values a and b must satisfy $a^2 + b^2 = 20^2$ and so $b = \sqrt{400 - a^2}$. Then the area is given by $A = \frac{1}{2}ab = \frac{1}{2}a\sqrt{400 - a^2}$ for $0 < a < 20$, and

$$\frac{dA}{da} = \frac{1}{2}a\left(\frac{1}{2\sqrt{400 - a^2}}\right)(-2a) + \frac{1}{2}\sqrt{400 - a^2}$$
$$= \frac{-a^2 + (400 - a^2)}{2\sqrt{400 - a^2}}$$
$$= \frac{200 - a^2}{\sqrt{400 - a^2}}.$$

The critical point occurs when $a^2 = 200$. Since $\frac{dA}{da} > 0$ for $0 < a < \sqrt{200}$ and $\frac{dA}{da} < 0$ for $\sqrt{200} < a < 20$, this critical point corresponds to the maximum area. Furthermore, if $a = \sqrt{200}$ then $b = \sqrt{400 - a^2} = \sqrt{200}$, so the maximum area occurs when $a = b$.

9. Let x be the length in meters of each side that adjoins the river. Then the side parallel to the river measures $800 - 2x$ meters and the area is $A(x) = x(800 - 2x) = 800x - 2x^2$ for $0 < x < 400$. Therefore, $A'(x) = 800 - 4x$ and the critical point occurs at $x = 200$. Since $A'(x) > 0$ for $0 < x < 200$ and $A'(x) < 0$ for $200 < x < 400$, the critical point corresponds to the maximum area. The largest possible area is $A(200) = 80{,}000\text{ m}^2$ and the dimensions are 200 m (perpendicular to the river) by 400 m (parallel to the river).

10. If the subdividing fence measures x meters, then the pea patch measures x m by $\frac{216}{x}$ m and the amount of fence needed is $f(x) = 3x + 2\left(\frac{216}{x}\right) = 3x + 432x^{-1}$. Then $f'(x) = 3 - 432x^{-2}$ and the critical point (for $x > 0$) occurs at $x = 12$. Since $f'(x) < 0$ for $0 < x < 12$ and $f'(x) > 0$ for $x > 12$, the critical point corresponds to the minimum total length of fence. The pea patch will measure 12 m by 18 m (with a 12-m divider), and the total amount of fence needed is $f(12) = 72\,\text{m}$.

11. **(a)** Let x be the length in feet of each side of the square base. Then the height is $\frac{500}{x^2}$ ft and the surface area (not including the open top) is

$$S(x) = x^2 + 4x\left(\frac{500}{x^2}\right) = x^2 + 2000x^{-1}.$$

Therefore,

$$S'(x) = 2x - 2000x^{-2} = \frac{2(x^3 - 1000)}{x^2}$$ and

the critical point occurs at $x = 10$. Since $S'(x) < 0$ for $0 < x < 10$ and $S'(x) > 0$ for $x > 10$, the critical point corresponds to the minimum amount of steel used. The dimensions should be 10 ft by 10 ft by 5 ft, where the height is 5 ft.

(b) Assume that the weight is minimized when the total area of the bottom and the four sides is minimized.

12. **(a)** Note that $x^2 y = 1125$, so $y = \frac{1125}{x^2}$. Then

$$\begin{aligned} c &= 5(x^2 + 4xy) + 10xy \\ &= 5x^2 + 30xy \\ &= 5x^2 + 30x\left(\frac{1125}{x^2}\right) \\ &= 5x^2 + 33{,}750x^{-1} \end{aligned}$$

$$\frac{dc}{dx} = 10x - 33{,}750x^{-2} = \frac{10(x^3 - 3375)}{x^2}$$

The critical point occurs at $x = 15$. Since $\frac{dc}{dx} < 0$ for $0 < x < 15$ and $\frac{dc}{dx} > 0$ for $x > 15$, the critical point corresponds to the minimum cost. The values of x and y are $x = 15$ ft and $y = 5$ ft.

(b) The material for the tank costs 5 dollars/sq ft and the excavation charge is 10 dollars for each square foot of the cross-sectional area of one wall of the hole.

13. Let x be the height in inches of the printed area. Then the width of the printed area is $\frac{50}{x}$ in. and the overall dimensions are $x + 8$ in. by $\frac{50}{x} + 4$ in. The amount of paper used is

$$A(x) = (x+8)\left(\frac{50}{x} + 4\right) = 4x + 82 + \frac{400}{x}\ \text{in}^2.$$

Then $A'(x) = 4 - 400x^{-2} = \frac{4(x^2 - 100)}{x^2}$ and the critical point (for $x > 0$) occurs at $x = 10$. Since $A'(x) < 0$ for $0 < x < 10$ and $A'(x) > 0$ for $x > 10$, the critical point corresponds to the minimum amount of paper. Using $x + 8$ and $\frac{50}{x} + 4$ for $x = 10$, the overall dimensions are 18 in. high by 9 in. wide.

14. **(a)** $s(t) = -16t^2 + 96t + 112$
$v(t) = s'(t) = -32t + 96$
At $t = 0$, the velocity is $v(0) = 96$ ft/sec.

(b) The maximum height occurs when $v(t) = 0$, when $t = 3$. The maximum height is $s(3) = 256$ ft and it occurs at $t = 3$ sec.

(c) Note that

$$\begin{aligned} s(t) &= -16t^2 + 96t + 112 \\ &= -16(t+1)(t-7), \end{aligned}$$

so $s = 0$ at $t = -1$ or $t = 7$. Choosing the positive value, of t, the velocity when $s = 0$ is $v(7) = -128$ ft/sec.

15. We assume that a and b are held constant. Then $A(\theta) = \frac{1}{2}ab \sin\theta$ and $A'(\theta) = \frac{1}{2}ab\cos\theta$. The critical point (for $0 < \theta < \pi$) occurs at $\theta = \frac{\pi}{2}$. Since $A'(\theta) > 0$ for $0 < \theta < \frac{\pi}{2}$ and $A'(\theta) < 0$ for $\frac{\pi}{2} < \theta < \pi$, the critical point corresponds to the maximum area. The angle that maximizes the triangle's area is $\theta = \dfrac{\pi}{2}$ (or 90°).

16. Let the can have radius r cm and height h cm. Then $\pi r^2 h = 1000$, so $h = \dfrac{1000}{\pi r^2}$. The area of material used is

$A = \pi r^2 + 2\pi rh = \pi r^2 + \dfrac{2000}{r}$, so

$\dfrac{dA}{dr} = 2\pi r - 2000r^{-2} = \dfrac{2\pi r^3 - 2000}{r^2}$. The critical point occurs at

$r = \sqrt[3]{\dfrac{1000}{\pi}} = 10\pi^{-1/3}$ cm. Since $\dfrac{dA}{dr} < 0$ for $0 < r < 10\pi^{-1/3}$ and $\dfrac{dA}{dr} > 0$ for $r > 10\pi^{1/3}$, the critical point corresponds to the least amount of material used and hence the lightest possible can. The dimensions are $r = 10\pi^{-1/3} \approx 6.83\text{cm}$ and $h = 10\pi^{-1/3} \approx 6.828\text{cm}$. In Example 4, because of the top of the can, the "best" design is less big around and taller.

17. Note that $\pi r^2 h = 1000$, so $h = \dfrac{1000}{\pi r^2}$. Then

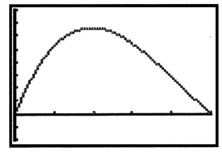
$A = 8r^2 + 2\pi rh = 8r^2 + \dfrac{2000}{r}$, so

$\dfrac{dA}{dr} = 16r - 2000r^{-2} = \dfrac{16(r^3 - 125)}{r^2}$. The critical point occurs at $r = \sqrt[3]{125} = 5$ cm. Since $\dfrac{dA}{dr} < 0$ for $0 < r < 5$ and $\dfrac{dA}{dr} > 0$ for $r > 5$, the critical point corresponds to the least amount of aluminum used or wasted and hence the most economical can. The dimensions are $r = 5$ cm and $h = \dfrac{40}{\pi}$, so the ratio of h to r is $\dfrac{8}{\pi}$ to 1.

18. **(a)** The base measures $10 - 2x$ in. by $\dfrac{15 - 2x}{2}$ in, so the volume formula is

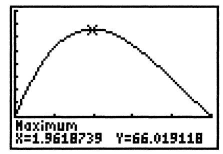
$$V(x) = \frac{x(10-2x)(15-2x)}{2} = 2x^3 - 25x^2 + 75x.$$

(b) We require $x > 0$, $2x < 10$, and $2x < 15$. Combining these requirements, the domain is the interval (0, 5).



[0, 5] by [−20, 80]

(c)



[0, 5] by [−20, 80]

The maximum volume is approximately 66.02 in^3 when $x \approx 1.96$ in.

(d) $V'(x) = 6x^2 - 50x + 75$

The critical point occurs when $V'(x) = 0$,

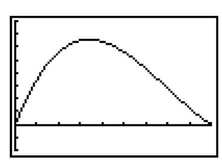
$$\text{at } x = \frac{50 \pm \sqrt{(-50)^2 - 4(6)(75)}}{2(6)} = \frac{50 \pm \sqrt{700}}{12} = \frac{25 \pm 5\sqrt{7}}{6},$$

that is, $x \approx 1.96$ or $x \approx 6.37$. We discard the larger value because it is not in the domain. Since $V''(x) = 12x - 50$, which is negative when $x \approx 1.96$, the critical point corresponds to the maximum volume. The maximum volume occurs when $x = \dfrac{25 - 5\sqrt{7}}{6} \approx 1.96$, which confirms the result in (c).

19. **(a)** The "sides" of the suitcase will measure $24 - 2x$ in. by $18 - 2x$ in. and will be $2x$ in. apart, so the volume formula is

$$V(x) = 2x(24 - 2x)(18 - 2x) = 8x^3 - 168x^2 + 864x.$$

(b) We require $x > 0$, $2x < 18$, and $2x < 24$. Combining these requirements, the domain is the interval (0, 9).

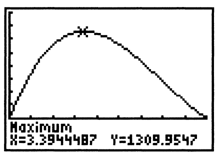


[0, 9] by [–400, 1600]

(c) $V'(x) = 24x^2 - 336x + 864$
$= 24(x^2 - 14x + 36)$

The critical point is at $x = 7 \pm \sqrt{13}$, that is, $x \approx 3.39$ or $x \approx 10.61$. We discard the larger value because it is not in the domain. Since $V''(x) = 24(2x - 14)$, which is negative when $x \approx 3.39$, the critical point corresponds to the maximum volume. The maximum value occurs at $x = 7 - \sqrt{13} \approx 3.39$.

(d)



[0, 9] by [−400, 1600]

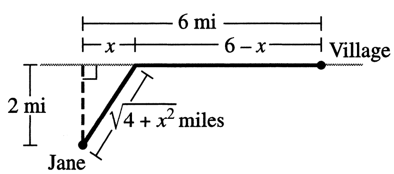
The maximum volume is approximately 1309.95 in^3 when $x \approx 3.39$ in.

(e)
$$8x^3 - 168x^2 + 864x = 1120$$
$$8(x^3 - 21x^2 + 108x - 140) = 0$$
$$8(x-2)(x-5)(x-14) = 0$$

Since 14 is not in the domain, the possible values of x are $x = 2$ in. or $x = 5$ in.

(f) The dimensions of the resulting box are $2x$ in., $(24 – 2x)$ in., and $(18 – 2x)$ in. Each of these measurements must be positive, so that gives the domain of (0, 9)

20.



Let x be the distance from the point on the shoreline nearest Jane's boat to the point where she lands her boat. Then she needs to row $\sqrt{4+x^2}$ mi at 2 mph and walk $6 – x$ mi at 5 mph. The total amount of time to reach the village is $f(x) = \dfrac{\sqrt{4+x^2}}{2} + \dfrac{6-x}{5}$ hours $(0 \le x \le 6)$. Then

$$f'(x) = \frac{1}{2}\frac{1}{2\sqrt{4+x^2}}(2x) - \frac{1}{5} = \frac{x}{2\sqrt{4+x^2}} - \frac{1}{5}.$$

Solving $f'(x) = 0$, we have:

$$\frac{x}{2\sqrt{4+x^2}} = \frac{1}{5}$$
$$5x = 2\sqrt{4+x^2}$$
$$25x^2 = 4(4+x^2)$$
$$21x^2 = 16$$
$$x = \pm\frac{4}{\sqrt{21}}$$

We discard the negative value of x because it is not in the domain. Checking the endpoints and critical point, we have

$f(0) = 2.2$, $f\left(\dfrac{4}{\sqrt{21}}\right) \approx 2.12$, and $f(6) \approx 3.16$.

Jane should land her boat $\dfrac{4}{\sqrt{21}} \approx 0.87$ mile down the shoreline from the point nearest her boat.

21. If the upper right vertex of the rectangle is located at $(x, 4 \cos 0.5x)$ for $0 < x < \pi$, then the rectangle has width $2x$ and height $4 \cos 0.5x$, so the area is $A(x) = 8x \cos 0.5x$.
Then
$A'(x) = 8x(–0.5 \sin 0.5x) + 8(\cos 0.5x)(1)$
$= –4x \sin 0.5x + 8 \cos 0.5x.$
Solving $A'(x)$ graphically for $0 < x < \pi$, we find that $x \approx 1.72$. Evaluating $2x$ and $4 \cos 0.5x$ for $x \approx 1.72$, the dimensions of the rectangle are approximately 3.44 (width) by 2.61 (height), and the maximum area is approximately 8.98.

22. Let the radius of the cylinder be r cm, $0 < r < 10$. Then the height is $2\sqrt{100-r^2}$ and the volume is $V(r) = 2\pi r^2\sqrt{100-r^2}\,\text{cm}^3$.
Then

$$V'(r) = 2\pi r^2\left(\frac{1}{2\sqrt{100-r^2}}\right)(-2r) + (2\pi\sqrt{100-r^2})(2r)$$
$$= \frac{-2\pi r^3 + 4\pi r(100-r^2)}{\sqrt{100-r^2}}$$
$$= \frac{2\pi r(200-3r^2)}{\sqrt{100-r^2}}$$

The critical point for $0 < r < 10$ occurs at $r = \sqrt{\dfrac{200}{3}} = 10\sqrt{\dfrac{2}{3}}$. Since $V'(r) > 0$ for $0 < r < 10\sqrt{\dfrac{2}{3}}$ and

$V'(r) > 0$ for $10\sqrt{\frac{2}{3}} < r < 10$, the critical point corresponds to the maximum volume. The dimensions are $r = 10\sqrt{\frac{2}{3}} \approx 8.16$ cm and $h = \frac{20}{\sqrt{3}} \approx 11.55$ cm, and the volume is $\frac{4000\pi}{3\sqrt{3}} \approx 2418.40\text{ cm}^3$.

23. Set $r'(x) = c'(x)$: $4x^{-1/2} = 4x$. The only positive critical value is $x = 1$, so profit is maximized at a production level of 1000 units. Note that $(r-c)''(x) = -2(x)^{-3/2} - 4 < 0$ for all positive x, so the Second Derivative Test confirms the maximum.

24. Set $r'(x) = c'(x)$: $2x/(x^2+1)^2 = (x-1)^2$. We solve this equation graphically to find that $x \approx 0.294$. The graph of $y = r(x) - c(x)$ shows a minimum at $x \approx 0.294$ and a maximum at $x \approx 1.525$, so profit is maximized at a production level of about 1,525 units.

25. Set $\bar{c}(x) = \frac{c(x)}{x} = x^2 - 10x + 30$. The only positive solution to $\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0$ is $x = 5$, so average cost is minimized at a production level of 5000 units. Note that $\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2 > 0$ for all positive x, so the Second Derivative Test confirms the minimum.

26. Set $\bar{c}(x) = \frac{c(x)}{x} = e^x - 2x$. The only positive solution to $\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0$ is $x = \ln 2$, so average cost is minimized at a production level of 1000 ln 2, which is about 693 units. Note that $\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = e^x > 0$ for all positive x, so the Second Derivative Test confirms the minimum.

27. Revenue: $r(x) = [200 - 2(x-50)]x$
$$= -2x^2 + 300x$$
Cost: $c(x) = 6000 + 32x$

Profit: $p(x) = r(x) - c(x)$
$$= -2x^2 + 268x - 6000,$$
$50 \le x \le 80$

Since $p'(x) = -4x + 268 = -4(x-67)$, the critical point occurs at $x = 67$. This value represents the maximum because $p''(x) = -4$, which is negative for all x in the domain. The maximum profit occurs if 67 people go on the tour.

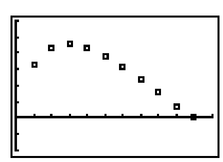
28. **(a)** $f'(x) = x(-e^{-x}) + e^{-x}(1) = e^{-x}(1-x)$

The critical point occurs at $x = 1$. Since $f'(x) > 0$ for $0 \le x < 1$ and $f'(x) < 0$ for $x > 1$, the critical point corresponds to the maximum value of f. The absolute maximum of f occurs at $x = 1$.

(b) To find the values of b, use grapher techniques to solve $xe^{-x} = 0.1e^{-0.1}$, $xe^{-x} = 0.2e^{-0.2}$, and so on. To find the values of A, calculate $(b-a)ae^{-a}$, using the unrounded values of b. (Use the *list* features of the grapher in order to keep track of the unrounded values for part (d).)

a	b	A
0.1	3.71	0.33
0.2	2.86	0.44
0.3	2.36	0.46
0.4	2.02	0.43
0.5	1.76	0.38
0.6	1.55	0.31
0.7	1.38	0.23
0.8	1.23	0.15
0.9	1.11	0.08
1.0	1.00	0.00

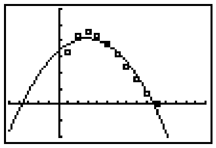
(c)



[0, 1.1] by [–0.2, 0.6]

(d) Quadratic:

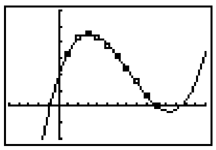
$A \approx -0.91a^2 + 0.54a + 0.34$



[–0.5, 1.5] by [–0.2, 0.6]

Cubic:

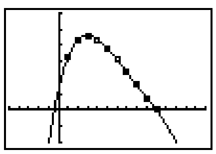
$A \approx 1.74a^3 - 3.78a^2 + 1.86a + 0.19$



[–0.5, 1.5] by [–0.2, 0.6]

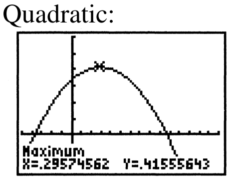
Quartic:

$A \approx -1.92a^4 + 5.96a^3 - 6.87a^2 + 2.71a + 0.12$



[–0.5, 1.5] by [–0.2, 0.6]

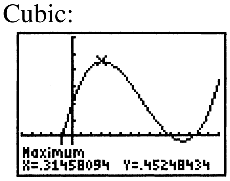
(e) Quadratic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the quadratic regression equation, the maximum area occurs at $a \approx 0.30$ and is approximately 0.42.

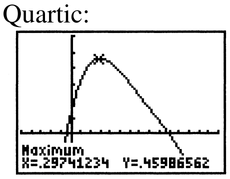
Cubic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the cubic regression equation, the maxiumu area occurs at $a \approx 0.31$ and is approximately 0.45.

Quartic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the quartic regression equation the maximum area occurs at $a \approx 0.30$ and is approximately 0.46.

29. (a) $f'(x)$ is a quadratic polynomial, and as such it can have 0, 1, or 2 zeros. If it has 0 or 1 zeros, then its sign never changes, so $f(f(x))$ has no local extrema.
If $f'(x)$ has 2 zeros, then its sign changes twice, and $f(f(x))$ has 2 local extrema at those points.

(b) Possible answers:
No local extrema: $y = x^3$;
2 local extrema: $y = x^3 - 3x$

30. Let x be the length in inches of each edge of the square end, and let y be the length of the box. Then we require $4x + y \le 108$. Since our goal is to maximize volume, we assume $4x + y = 108$ and so $y = 108 - 4x$. The volume is $V(x) = x^2(108 - 4x) = 108x^2 - 4x^3$, where $0 < x < 27$. Then $V' = 216x - 12x^2 = -12x(x - 18)$, so the critical point occurs at $x = 18$ in. Since $V'(x) > 0$ for $0 < x < 18$ and $V'(x) < 0$ for $18 < x < 27$, the critical point corresponds to the maximum volume. The dimensions of the box with the largest possible volume are 18 in. by 18 in. by 36 in.

31. Since $2x + 2y = 36$, we know that $y = 18 - x$. In part (a), the radius is $\dfrac{x}{2\pi}$ and the height is $18 - x$, and so the volume is given by

$$\pi r^2 h = \pi\left(\frac{x}{2\pi}\right)^2 (18 - x) = \frac{1}{4\pi} x^2 (18 - x).$$

In part (b), the radius is x and the height is $18 - x$, and so the volume is given by $\pi r^2 h = \pi x^2 (18 - x)$. Thus, each problem requires us to find the value of x that maximizes $f(x) = x^2(18 - x)$ in the interval $0 < x < 18$, so the two problems have the same answer.
To solve either problem, note that $f(x) = 18x^2 - x^3$ and so $f'(x) = 36x - 3x^2 = -3x(x - 12)$. The critical point occurs at $x = 12$. Since $f'(x) > 0$ for $0 < x < 12$ and $f'(x) < 0$ for $12 < x < 18$, the critical point corresponds to the maximum value of $f(x)$. To maximize the volume in either part (a) or (b), let $x = 12$ cm and $y = 6$ cm.

32. Note that $h^2 + r^2 = 3$ and so $r = \sqrt{3 - h^2}$. Then the volume is given by
$V = \frac{\pi}{3}r^2h = \frac{\pi}{3}(3 - h^2)h = \pi h - \frac{\pi}{3}h^3$ for
$0 < h < \sqrt{3}$, and so $\frac{dV}{dh} = \pi - \pi h^2 = \pi(1 - h^2)$.
The critical point (for $h > 0$) occurs at $h = 1$. Since $\frac{dV}{dh} > 0$ for $0 < h < 1$ and $\frac{dV}{dh} < 0$ for $1 < h < \sqrt{3}$, the critical point corresponds to the maximum volume. The cone of greatest volume has radius $\sqrt{2}$ m, height 1 m, and volume $\frac{2\pi}{3}\text{m}^3$.

33. **(a)** We require $f(x)$ to have a critical point at $x = 2$. Since $f'(x) = 2x - ax^{-2}$, we have $f'(2) = 4 - \frac{a}{4}$ and so our requirement is that $4 - \frac{a}{4} = 0$. Therefore, $a = 16$. To verify that the critical point corresponds to a local minimum, note that we now have $f'(x) = 2x - 16x^{-2}$ and so $f''(x) = 2 + 32x^{-3}$, so $f''(2) = 6$, which is positive as expected. So, use $a = 16$.

(b) We require $f''(1) = 0$. Since $f'' = 2 + 2ax^{-3}$, we have $f''(1) = 2 + 2a$, so our requirement is that $2 + 2a = 0$. Therefore, $a = -1$. To verify that $x = 1$ is in fact an inflection point, note that we now have $f''(x) = 2 - 2x^{-3}$, which is negative for $0 < x < 1$ and positive for $x > 1$. Therefore, the graph of f is concave down in the interval (0, 1) and concave up in the interval $(1, \infty)$, So, use $a = -1$.

34. $f'(x) = 2x - ax^{-2} = \frac{2x^3 - a}{x^2}$, so the only sign change in $f'(x)$ occurs at $x = \left(\frac{a}{2}\right)^{1/3}$, where the sign changes from negative to positive. This means there is a local minimum at that point, and there are no local maxima.

35. **(a)** Note that $f'(x) = 3x^2 + 2ax + b$. We require $f'(-1) = 0$ and $f'(3) = 0$, which give $3 - 2a + b = 0$ and $27 + 6a + b = 0$. Subtracting the first equation from the second, we have $24 + 8a = 0$ and so $a = -3$.
Substituting into the first equation, we have $9 + b = 0$, so $b = -9$. Therefore, our equation for $f(x)$ is $f(x) = x^3 - 3x^2 - 9x$.
To verify that we have a local maximum at $x = -1$ and a local minimum at $x = 3$, note that
$f'(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)$,
which is positive for $x < -1$, negative for $-1 < x < 3$, and positive for $x > 3$. So, use $a = -3$ and $b = -9$.

(b) Note that $f'(x) = 3x^2 + 2ax + b$ and $f''(x) = 6x + 2a$. We require $f'(4) = 0$ and $f''(1) = 0$, which give $48 + 8a + b = 0$ and $6 + 2a = 0$. By the second equation, $a = -3$, and so the first equation becomes $48 - 24 + b = 0$. Thus $b = -24$. To verify that we have a local minimum at $x = 4$, and an inflection point *at* $x = 1$, note that we now have $f''(x) = 6x - 6$. Since f'' changes sign at $x = 1$ and is positive at $x = 4$, the desired conditions are satisfied. So, use $a = -3$ and $b = -24$.

36. Refer to the illustration in the problem statement. Since $x^2 + y^2 = 9$, we have $x = \sqrt{9 - y^2}$. Then the volume of the cone is given by

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi x^2 (y + 3)$$
$$= \frac{1}{3}\pi(9 - y^2)(y + 3)$$
$$= \frac{\pi}{3}(-y^3 - 3y^2 + 9y + 27),$$

for $-3 < y < 3$.

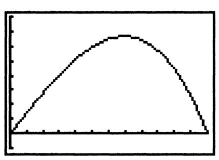
Thus $\frac{dV}{dy} = \frac{\pi}{3}(-3y^2 - 6y + 9)$
$$= -\pi(y^2 + 2y - 3)$$
$$= -\pi(y + 3)(y - 1),$$
so the critical point in the interval (–3, 3) is $y = 1$. Since $\frac{dV}{dy} > 0$ for $-3 < y < 1$ and

$\frac{dV}{dy} < 0$ for $1 < y < 3$, the critical point does correspond to the maximum value, which is $V(1) = \frac{32\pi}{3}$ cubic units.

37. **(a)** Note that $w^2 + d^2 = 12^2$, so $d = \sqrt{144 - w^2}$. Then we may write $S = kwd^2 = kw(144 - w^2) = 144kw - kw^3$ for $0 < w < 12$, so

$\frac{dS}{dw} = 144k - 3kw^2 = -3k(w^2 - 48)$. The critical point (for $0 < w < 12$) occurs at $w = \sqrt{48} = 4\sqrt{3}$. Since $\frac{dS}{dw} > 0$ for $0 < w < 4\sqrt{3}$ and $\frac{dS}{dw} < 0$ for $4\sqrt{3} < w < 12$, the critical point corresponds to the maximum strength. The dimensions are $4\sqrt{3}$ in. wide by $4\sqrt{6}$ in. deep.

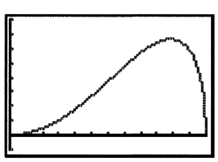
(b)



[0, 12] by [−100, 800]

The graph of $S = 144w - w^3$ is shown. The maximum strength shown in the graph occurs at $w = 4\sqrt{3} \approx 6.9$, which agrees with the answer to part (a).

(c)



[0, 12] by [−100, 800]

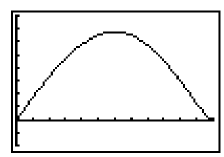
The graph of $S = d^2\sqrt{144 - d^2}$ is shown. The maximum strength shown in the graph occurs at $d = 4\sqrt{6} \approx 9.8$, which agrees with the answer to part (a), and its value is the same as the maximum value found in part (b), as expected.
Changing the value of k changes the maximum strength, but not the dimensions of the strongest beam. The graphs for different values of k look the same except that the vertical scale is different.

38. **(a)** Note that $w^2 + d^2 = 12^2$, so $d = \sqrt{144 - w^2}$. Then we may write $S = kwd^3 = kw(144 - w^2)^{3/2}$, so

$$\frac{dS}{dw} = kw \cdot \frac{3}{2}(144 - w^2)^{1/2}(-2w) + k(144 - w^2)^{3/2}$$
$$= (k\sqrt{144 - w^2})(-3w^2 + 144 - w^2)$$
$$= (-4k\sqrt{144 - w^2})(w^2 - 36)$$

The critical point (for $0 < w < 12$) occurs at $w = 6$. Since $\frac{dS}{dw} > 0$ for $0 < w < 6$ and $\frac{dS}{dw} < 0$ for $6 < w < 12$, the critical point corresponds to the maximum stiffness. The dimensions are 6 in. wide by $6\sqrt{3}$ in. deep.

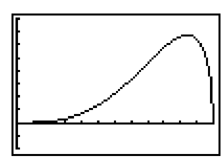
(b)



[0, 12] by [–2000, 8000]

The graph of $S = w(144 - w^2)^{3/2}$ is shown. The maximum stiffness shown in the graph occurs at $w = 6$, which agrees with the answer to part (a).

(c)



[0, 12] by [–2000, 8000]

The graph of $S = d^3\sqrt{144 - d^2}$ is shown. The maximum stiffness shown in the graph occurs at $d = 6\sqrt{3} \approx 10.4$ agrees with the answer to part (a), and its value is the same as the maximum value found in part (b), as expected.
Changing the value of k changes the maximum stiffness, but not the dimensions of the stiffest beam. The graphs for different values of k look the same except that the vertical scale is different.

39. **(a)** $v(t) = s'(t) = -10\pi \sin \pi t$

The speed at time t is $10\pi|\sin \pi t|$. The maximum speed is 10π cm/sec and it occurs at $t = \frac{1}{2},\ t = \frac{3}{2},\ t = \frac{5}{2}$, and $t = \frac{7}{2}$ sec. The position at these times is $s = 0$ cm (rest position), and the acceleration $a(t) = v'(t) = -10\pi^2 \cos \pi t$ is 0 cm/sec^2 at these times.

(b) Since $a(t) = -10\pi^2 \cos \pi t$, the greatest magnitude of the acceleration occurs at $t = 0,\ t = 1,\ t = 2,\ t = 3$, and $t = 4$. At these times, the position of the cart is either $s = -10$ cm or $s = 10$ cm, and the speed of the cart is 0 cm/sec.

40. Since $\frac{di}{dt} = -2\sin t + 2\cos t$, the largest magnitude of the current occurs when $-2 \sin t + 2 \cos t = 0$, or $\sin t = \cos t$. Squaring both sides gives $\sin^2 t = \cos^2 t$, and we know that $\sin^2 t + \cos^2 t = 1$, so $\sin^2 t = \cos^2 t = \frac{1}{2}$. Thus the possible values of t are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and so on. Eliminating extraneous solutions, the solutions of $\sin t = \cos t$ are $t = \frac{\pi}{4} + k\pi$ for integers k, and at these times $|i| = |2\cos t + 2\sin t| = 2\sqrt{2}$. The peak current is $2\sqrt{2}$ amps.

41. The square of the distance is

$$D(x) = \left(x - \frac{3}{2}\right)^2 + \left(\sqrt{x} - 0\right)^2 = x^2 - 2x + \frac{9}{4},$$

so $D'(x) = 2x - 2$ and the critical point occurs at $x = 1$. Since $D'(x) < 0$ for $x < 1$ and $D'(x) > 0$ for $x > 1$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(1)} = \frac{\sqrt{5}}{2}$.

42. Calculus method:
The square of the distance from the point $\left(1, \sqrt{3}\right)$ to $\left(x, \sqrt{16 - x^2}\right)$ is given by

$$\begin{aligned} D(x) &= (x-1)^2 + (\sqrt{16 - x^2} - \sqrt{3})^2 \\ &= x^2 - 2x + 1 + 16 - x^2 - 2\sqrt{48 - 3x^2} + 3 \\ &= -2x + 20 - 2\sqrt{48 - 3x^2}. \text{ Then} \end{aligned}$$

$$\begin{aligned} D'(x) &= -2 - \frac{2}{2\sqrt{48 - 3x^2}}(-6x) \\ &= -2 + \frac{6x}{\sqrt{48 - 3x^2}}. \end{aligned}$$

Solving $D'(x) = 0$, we have:

$$\begin{aligned} 6x &= 2\sqrt{48 - 3x^2} \\ 36x^2 &= 4(48 - 3x^2) \\ 9x^2 &= 48 - 3x^2 \\ 12x^2 &= 48 \\ x &= \pm 2 \end{aligned}$$

We discard $x = -2$ as an extraneous solution, leaving $x = 2$. Since $D'(x) < 0$ for $-4 < x < 2$ and $D'(x) > 0$ for $2 < x < 4$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(2)} = 2$.

Geometry method:
The semicircle is centered at the origin and has radius 4.
The distance from the origin to $\left(1, \sqrt{3}\right)$ is $\sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$. The shortest distance from the point to the semicircle is the distance along the radius containing the point $\left(1, \sqrt{3}\right)$. That distance is $4 - 2 = 2$.

43. No. Since $f(x)$ is a quadratic function and the coefficient of x^2 is positive, it has an absolute minimum at the point where $f'(x) = 2x - 1 = 0$, and the point is $\left(\frac{1}{2}, \frac{3}{4}\right)$.

44. **(a)** Because $f(x)$ is periodic with period 2π.

(b) No; since $f(x)$ is continuous on $[0, 2\pi]$, its absolute minimum occurs at a critical point or endpoint.
Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(x) = -4\sin x - 2\sin 2x &= 0 \\ -4\sin x - 4\sin x \cos x &= 0 \\ -4(\sin x)(1 + \cos x) &= 0 \\ \sin x &= 0 \text{ or } \cos x = -1 \\ x &= 0, \pi, 2\pi \end{aligned}$$

The critical points (and endpoints) are $(0, 8)$, $(\pi, 0)$, and $(2\pi, 8)$. Thus, $f(x)$ has an absolute minimum at $(\pi, 0)$ and it is never negative.

45. (a)

$$\begin{aligned} 2\sin t &= \sin 2t \\ 2\sin t &= 2\sin t\cos t \\ 2(\sin t)(1-\cos t) &= 0 \\ \sin t &= 0 \text{ or } \cos t = 1 \end{aligned}$$

$t = k\pi$, where k is an integer.

The masses pass each other whenever t is an integer multiple of π seconds.

(b) The vertical distance between the objects is the absolute value of $f(x) = \sin 2t - 2\sin t$.

Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(x) = 2\cos 2t - 2\cos t &= 0 \\ 2(2\cos^2 t - 1) - 2\cos t &= 0 \\ 2(2\cos^2 t - \cos t - 1) &= 0 \\ 2(2\cos t + 1)(\cos t - 1) &= 0 \\ \cos t &= -\frac{1}{2} \text{ or } \cos t = 1 \\ t &= \frac{2\pi}{3}, \frac{4\pi}{3}, 0, 2\pi \end{aligned}$$

The critical points (and endpoints) are $(0, 0)$, $\left(\frac{2\pi}{3}, -\frac{3\sqrt{3}}{2}\right)$, $\left(\frac{4\pi}{3}, \frac{3\sqrt{3}}{2}\right)$, and $(2\pi, 0)$

The distance is greatest when $t = \frac{2\pi}{3}$ sec and when $t = \frac{4\pi}{3}$ sec. The distance at those times is $\frac{3\sqrt{3}}{2}$ meters.

46. (a)

$$\sin t = \sin\left(t + \frac{\pi}{3}\right)$$

$$\sin t = \sin t \cos\frac{\pi}{3} + \cos t \sin\frac{\pi}{3}$$

$$\sin t = \frac{1}{2}\sin t + \frac{\sqrt{3}}{2}\cos t$$

$$\frac{1}{2}\sin t = \frac{\sqrt{3}}{2}\cos t$$

$$\tan t = \sqrt{3}$$

Solving for t, the particles meet at $t = \frac{\pi}{3}$ sec and at $t = \frac{4\pi}{3}$ sec.

(b) The distance between the particles is the absolute value of $f(t) = \sin\left(t + \frac{\pi}{3}\right) - \sin t$

$$= \frac{\sqrt{3}}{2}\cos t - \frac{1}{2}\sin t.$$

Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(t) = -\frac{\sqrt{3}}{2}\sin t - \frac{1}{2}\cos t &= 0 \\ -\frac{\sqrt{3}}{2}\sin t &= \frac{1}{2}\cos t \\ \tan t &= -\frac{1}{\sqrt{3}} \end{aligned}$$

The solutions are $t = \frac{5\pi}{6}$ and $t = \frac{11\pi}{6}$, so the critical points are at $\left(\frac{5\pi}{6}, -1\right)$ and $\left(\frac{11\pi}{6}, 1\right)$, and the interval endpoints are at $\left(0, \frac{\sqrt{3}}{2}\right)$, and $\left(2\pi, \frac{\sqrt{3}}{2}\right)$. The particles are farthest apart at $t = \frac{5\pi}{6}$ sec and at $t = \frac{11\pi}{6}$ sec, and the maximum distance between the particles is 1 m.

(c) We need to maximize $f'(t)$, so we solve $f''(t) = 0$.

$$\begin{aligned} f''(t) = -\frac{\sqrt{3}}{2}\cos t + \frac{1}{2}\sin t &= 0 \\ \frac{1}{2}\sin t &= \frac{\sqrt{3}}{2}\cos t \end{aligned}$$

This is the same equation we solved in part (a), so the solutions are $t = \frac{\pi}{3}$ sec and $t = \frac{4\pi}{3}$ sec.

For the function $y = f'(t)$, the critical points occur at $\left(\frac{\pi}{3}, -1\right)$ and $\left(\frac{4\pi}{3}, 1\right)$, and the interval endpoints are at $\left(0, -\frac{1}{2}\right)$ and $\left(2\pi, -\frac{1}{2}\right)$.

Thus, $|f'(t)|$ is maximized at $t = \frac{\pi}{3}$ and $t = \frac{4\pi}{3}$. But these are the instants when the particles pass each other, so the graph of $y = |f(t)|$ has corners at these points and $\frac{d}{dt}|f(t)|$ is undefined at these instants. We cannot say that the distance is changing the fastest at any particular instant, but we can say that near $t = \frac{\pi}{3}$ or $t = \frac{4\pi}{3}$ the distance is changing faster than at any other time in the interval.

47. The trapezoid has height $(\cos\theta)$ft and the trapezoid bases measure 1 ft and $(1 + 2\sin\theta)$ft, so the volume is given by

$$V(\theta) = \frac{1}{2}(\cos\theta)(1 + 1 + 2\sin\theta)(20)$$
$$= 20(\cos\theta)(1 + \sin\theta).$$

Find the critical points for $0 \le \theta < \frac{\pi}{2}$:

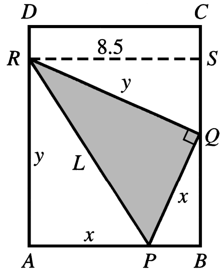
$$V'(\theta)$$
$$= 20(\cos\theta)(\cos\theta) + 20(1 + \sin\theta)(-\sin\theta)$$
$$= 0$$

$$20\cos^2\theta - 20\sin\theta - 20\sin^2\theta = 0$$
$$20(1 - \sin^2\theta) - 20\sin\theta - 20\sin^2\theta = 0$$
$$-20(2\sin^2\theta + \sin\theta - 1 = 0$$
$$-20(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$$\sin\theta = \frac{1}{2} \text{ or } \sin\theta = -1$$
$$\theta = \frac{\pi}{6}$$

The critical point is at $\left(\frac{\pi}{6}, 15\sqrt{3}\right)$. Since $V'(\theta) > 0$ for $0 \le \theta < \frac{\pi}{6}$ and $V'(\theta) < 0$ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$, the critical point corresponds to the maximum possible trough volume. The volume is maximized when $\theta = \frac{\pi}{6}$.

48.



Sketch segment RS as shown, and let y be the length of segment QR. Note that $PB = 8.5 - x$, and so

$$QB = \sqrt{x^2 - (8.5 - x)^2} = \sqrt{8.5(2x - 8.5)}.$$

Also note that triangles QRS and PQB are similar.

$$\frac{QR}{RS} = \frac{PQ}{QB}$$
$$\frac{y}{8.5} = \frac{x}{\sqrt{8.5(2x - 8.5)}}$$

(a)
$$\frac{y^2}{8.5^2} = \frac{x^2}{8.5(2x - 8.5)}$$
$$y^2 = \frac{8.5x^2}{2x - 8.5}$$
$$L^2 = x^2 + y^2$$
$$L^2 = x^2 + \frac{8.5x^2}{2x - 8.5}$$
$$L^2 = \frac{x^2(2x - 8.5) + 8.5x^2}{2x - 8.5}$$
$$L^2 = \frac{2x^3}{2x - 8.5}$$

(b) Note that $x > 4.25$, and let $f(x) = L^2 = \frac{2x^3}{2x - 8.5}$. Since $y \le 11$, the approximate domain of f is $5.20 \le x \le 8.5$. Then

$$f'(x) = \frac{(2x - 8.5)(6x^2) - (2x^3)(2)}{(2x - 8.5)^2}$$
$$= \frac{x^2(8x - 51)}{(2x - 8.5)^2}$$

For $x > 5.20$, the critical point occurs at $x = \frac{51}{8} = 6.375$ in., and this corresponds to a minimum value of $f(x)$ because $f'(x) < 0$ for $5.20 < x < 6.375$ and $f'(x) > 0$ for $x > 6.375$. Therefore, the value of x that minimizes L^2 is $x = 6.375$ in.

(c) The minimum value of L is

$$\sqrt{\frac{2(6.375)^3}{2(6.375) - 8.5}} \approx 11.04 \text{ in.}$$

49. Since $R = M^2\left(\frac{C}{2} - \frac{M}{3}\right) = \frac{C}{2}M^2 - \frac{1}{3}M^3$, we have $\frac{dR}{dM} = CM - M^2$. Let $f(M) = CM - M^2$. Then $f'(M) = C - 2M$, and the critical point for f occurs at $M = \frac{C}{2}$. This value corresponds to a maximum because $f'(M) > 0$ for $M < \frac{C}{2}$ and $f'(M) < 0$ for $M > \frac{C}{2}$. The value of M that maximizes $\frac{dR}{dM}$ is $M = \frac{C}{2}$.

50. The profit is given by

$$\begin{aligned} P(x) &= (n)(x - c) \\ &= a + b(100 - x)(x - c) \\ &= -bx^2 + (100 + c)bx + (a - 100bc). \end{aligned}$$

Then $P'(x) = -2bx + (100 + c)b$
$= b(100 + c - 2x)$.

The critical point occurs at $x = \frac{100 + c}{2} = 50 + \frac{c}{2}$, and this value corresponds to the maximum profit because $P'(x) > 0$ for $x < 50 + \frac{c}{2}$ and $P'(x) < 0$ for $x > 50 + \frac{c}{2}$.

A selling price of $50 + \frac{c}{2}$ will bring the maximum profit.

51. True. This is guaranteed by the Extreme Value Theorem (Section 5.1).

52. False; for example, consider $f(x) = x^3$ at $c = 0$.

53. D; $f(x) = x^2(60 - x)$

$$\begin{aligned} f'(x) &= x^2(-1) + (60 - x)(2x) \\ &= -x^2 + 120x - 2x^2 \\ &= -3x^2 + 120x \\ &= -3x(x - 40) \end{aligned}$$

$x = 0$ or $x = 40$

$60 - x = 60$ $\quad$ $60 - x = 20$

$x^2(60 - x) = 0$

$$\begin{aligned} (40)^2(20) &= (1600)(20) \\ &= 32{,}000 \end{aligned}$$

54. B; since $f'(x)$ is negative, $f(x)$ is always decreasing, so $f(25) = 3$.

55. B; $A = \frac{1}{2}bh$

$$\begin{aligned} b^2 + h^2 &= 100 \\ b &= \sqrt{100 - h^2} \\ A &= \frac{h}{2}\sqrt{100 - h^2} \\ A' &= \frac{\sqrt{100 - h^2}}{2} - \frac{h^2}{2\sqrt{100 - h^2}} \end{aligned}$$

$A' = 0$ when $h = \sqrt{50}$

$b = \sqrt{100 - \sqrt{50}^2} = \sqrt{50}$

$A_{\max} = \frac{1}{2}\sqrt{50}\sqrt{50} = 25$

56. E; length $= 2x$

height $= 30 - x^2 - 4x^2 = 30 - 5x^2$

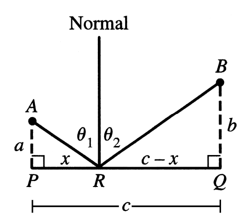
area $= A = 2x(30 - 5x^2) = 60x - 10x^3$

$\frac{dA}{dx}(60x - 10x^3) = 60 - 30x^2$

$x = \sqrt{2}$

$2\sqrt{2}\left(30 - 5(\sqrt{2})^2\right) = 40\sqrt{2}$.

57.



Let P be the foot of the perpendicular from A to the mirror, and Q be the foot of the perpendicular from B to the mirror. Suppose the light strikes the mirror at point R on the way from A to B. Let:

a = distance from A to P
b = distance from B to Q
c = distance from P to Q
x = distance from P to R

To minimize the time is to minimize the total distance the light travels going from A to B. The total distance is

$D(x) = \sqrt{x^2 + a^2} + \sqrt{(c - x)^2 + b^2}$

Then

$$D'(x) = \frac{1}{2\sqrt{x^2+a^2}}(2x) + \frac{1}{2\sqrt{(c-x)^2+b^2}}[-2(c-x)]$$
$$= \frac{x}{\sqrt{x^2+a^2}} - \frac{c-x}{\sqrt{(c-x)^2+b^2}}$$

Solving $D'(x) = 0$ gives the equation $\frac{x}{\sqrt{x^2+a^2}} = \frac{c-x}{\sqrt{(c-x)^2+b^2}}$ which we will refer to as Equation 1.

Squaring both sides, we have:

$$\frac{x^2}{x^2+a^2} = \frac{(c-x)^2}{(c-x)^2+b^2}$$
$$x^2\left[(c-x)^2+b^2\right] = (c-x)^2(x^2+a^2)$$
$$x^2(c-x)^2 + x^2b^2 = (c-x)^2x^2 + (c-x)^2a^2$$
$$x^2b^2 = (c-x)^2a^2$$
$$x^2b^2 = \left[c^2-2xc+x^2\right]a^2$$
$$0 = (a^2-b^2)x^2 - 2a^2cx + a^2c^2$$
$$0 = [(a+b)x-ac][(a-b)x-ac]$$
$$x = \frac{ac}{a+b} \text{ or } x = \frac{ac}{a-b}$$

Note that the value $x = \frac{ac}{a-b}$ is an extraneous solution because $c - x = c - \frac{ac}{a-b} = \frac{-cb}{a-b}$, so x and $c - x$ could not both be positive. The only critical point occurs at $x = \frac{ac}{a+b}$.

To verify that critical point represents the minimum distance, notice that D is differentiable for all x in $[0, c]$ with a single critical point in the interior of the interval. Since $D'(0) = \frac{-c}{\sqrt{c^2+b^2}} < 0$, D must be decreasing from 0 to the critical point, and since $D'(c) = \frac{c}{\sqrt{c^2+b^2}} > 0$, D must be increasing from the critical point to c.

We now know that $D(x)$ is minimized when Equation 1 is true, or, equivalently, $\frac{PR}{AR} = \frac{QR}{BR}$. This means that the two right triangles APR and BQR are similar, which in turn implies that the two angles must be equal.

58. $\frac{dv}{dx} = ka - 2kx$

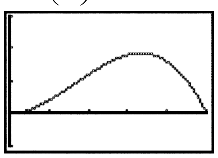
The critical point occurs at $x = \frac{ka}{2k} = \frac{a}{2}$, which represents a maximum value because $\frac{d^2v}{dx^2} = -2k$, which is negative for all x. The maximum value of v is $kax - kx^2 = ka\left(\frac{a}{2}\right) - k\left(\frac{a}{2}\right)^2 = \frac{ka^2}{4}$.

59. **(a)** $v = cr_0r^2 - cr^3$

$$\frac{dv}{dr} = 2cr_0r - 3cr^2 = cr(2r_0 - 3r)$$

The critical point occurs at $r = \frac{2r_0}{3}$. (Note that $r = 0$ is not in the domain of v.) The critical point represents a maximum because $\frac{d^2v}{dr^2} = 2cr_0 - 6cr = 2c(r_0 - 3r)$, which is negative when $r = \frac{2r_0}{3}$.

(b) We graph $v = (0.5 - r)r^2$, and observe that the maximum indeed occurs at $v = \left(\frac{2}{3}\right)0.5 = \frac{1}{3}$.



[0, 0.5] by [−0.01, 0.03]

60. **(a)** Since $A'(q) = -kmq^{-2} + \frac{h}{2}$, the critical point occurs when $\frac{km}{q^2} = \frac{h}{2}$, or $q = \sqrt{\frac{2km}{h}}$. This corresponds to the minimum value of $A(q)$ because $A''(q) = 2kmq^{-3}$, which is positive for $q > 0$.

(b) The new formula for average weekly cost is

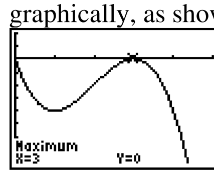
$$B(q) = \frac{(k+bq)m}{q} + cm + \frac{hq}{2} = \frac{km}{q} + bm + cm + \frac{hq}{2} = A(q) + bm$$

Since $B(q)$ differs from $A(q)$ by a constant, the minimum value of $B(q)$ will occur at the same q-value as the minimum value of $A(q)$. The most economical quantity is again $\sqrt{\frac{2km}{h}}$.

61. The profit is given by

$$p(x) = r(x) - c(x) = 6x - (x^3 - 6x^2 + 15x) = -x^3 + 6x^2 - 9x, \text{ for } x \ge 0.$$

Then $p'(x) = -3x^2 + 12x - 9 = -3(x-1)(x-3)$, so the critical points occur at $x = 1$ and $x = 3$. Since $p'(x) < 0$ for $0 \le x < 1$, $p'(x) > 0$ for $1 < x < 3$, and $p'(x) < 0$ for $x > 3$, the relative maxima occur at the endpoint $x = 0$ and at the critical point $x = 3$. Since $p(0) = p(3) = 0$, this means that for $x \ge 0$, the function $p(x)$ has its absolute maximum value at the points (0, 0) and (3, 0). This result can also be obtained graphically, as shown.



[0, 5] by [−8, 2]

62. The average cost is given by $a(x) = \frac{c(x)}{x} = x^2 - 20x + 20{,}000$. Therefore, $a'(x) = 2x - 20$ and the critical value is $x = 10$, which represents the minimum because $a''(x) = 2$, which is positive for all x. The average cost is minimized at a production level of 10 items.

63. **(a)** According to the graph, $y'(0) = 0$.

(b) According to the graph, $y'(-L) = 0$.

(c) $y(0) = 0$, so $d = 0$.

Now $y'(x) = 3ax^2 + 2bx + c$, so $y'(0) = 0$ implies that $c = 0$. Therefore, $y(x) = ax^3 + bx^2$ and $y'(x) = 3ax^2 + 2bx$. Then $y(-L) = -aL^3 + bL^2 = H$ and $y'(-L) = 3aL^2 - 2bL = 0$, so we have two linear equations in the two unknowns a and b. The second equation gives $b = \frac{3aL}{2}$. Substituting into the first equation, we have $-aL^3 + \frac{3aL^3}{2} = H$, or $\frac{aL^3}{2} = H$, so $a = 2\frac{H}{L^3}$. Therefore,

$b=3\dfrac{H}{L^2}$ and the equation for y is $y(x)=2\dfrac{H}{L^3}x^3+3\dfrac{H}{L^2}x^2$, or $y(x)=H\left[2\left(\dfrac{x}{L}\right)^3+3\left(\dfrac{x}{L}\right)^2\right]$.

64. **(a)** The base radius of the cone is $r=\dfrac{2\pi a-x}{2\pi}$ and so the height is $h=\sqrt{a^2-r^2}=\sqrt{a^2-\left(\dfrac{2\pi a-x}{2\pi}\right)^2}$.

Therefore, $V(x)=\dfrac{\pi}{3}r^2h=\dfrac{\pi}{3}\left(\dfrac{2\pi a-x}{2\pi}\right)^2\sqrt{a^2-\left(\dfrac{2\pi a-x}{2\pi}\right)^2}$.

(b) To simplify the calculations, we shall consider the volume as a function of *r:*

volume $=f(r)=\dfrac{\pi}{3}r^2\sqrt{a^2-r^2}$, where $0<r<a$.

$$\begin{aligned}f'(r)&=\frac{\pi}{3}\frac{d}{dr}(r^2\sqrt{a^2-r^2})\\&=\frac{\pi}{3}\left[r^2\cdot\frac{1}{2\sqrt{a^2-r^2}}\cdot(-2r)+(\sqrt{a^2-r^2})(2r)\right]\\&=\frac{\pi}{3}\left[\frac{-r^3+2r(a^2-r^2)}{\sqrt{a^2-r^2}}\right]\\&=\frac{\pi}{3}\left[\frac{(2a^2r-3r^3)}{\sqrt{a^2-r^2}}\right]\\&=\frac{\pi r(2a^2-3r^2)}{3\sqrt{a^2-r^2}}\end{aligned}$$

The critical point occurs when $r^2=\dfrac{2a^2}{3}$, which gives $r=a\sqrt{\dfrac{2}{3}}=\dfrac{a\sqrt{6}}{3}$. Then

$h=\sqrt{a^2-r^2}=\sqrt{a^2-\dfrac{2a^2}{3}}=\sqrt{\dfrac{a^2}{3}}=\dfrac{a\sqrt{3}}{3}$. Using $r=\dfrac{a\sqrt{6}}{3}$ and $h=\dfrac{a\sqrt{3}}{3}$, we may now find the values of *r* and *h* for the given values of *a*

when $a=4: r=\dfrac{4\sqrt{6}}{3}, h=\dfrac{4\sqrt{3}}{3}$; when $a=5: r=\dfrac{5\sqrt{6}}{3}, h=\dfrac{5\sqrt{3}}{3}$;

when $a=6: r=2\sqrt{6}, h=2\sqrt{3}$; when $a=8: r=\dfrac{8\sqrt{6}}{3}, h=\dfrac{8\sqrt{3}}{3}$

(c) Since $r=\dfrac{a\sqrt{6}}{3}$ and $h=\dfrac{a\sqrt{3}}{3}$, the relationship is $\dfrac{r}{h}=\sqrt{2}$.

65. **(a)** Let x_0 represent the fixed value of *x* at point *P*, so that *P* has coordinates (x_0, a) and let $m=f'(x_0)$ be the slope of line *RT*. Then the equation of line *RT* is $y=m(x-x_0)+a$. The *y*-intercept of this line is $m(0-x_0)+a=a-mx_0$, and the *x*-intercept is the solution of $m(x-x_0)+a=0$, or $x=\dfrac{mx_0-a}{m}$. Let *O* designate the origin. Then

(Area of triangle RST) = 2 (Area of triangle ORT)

$$= 2\cdot\frac{1}{2}\ (x\text{-intercept of line } RT)\ (y\text{-intercept of line } RT)$$

$$= 2\cdot\frac{1}{2}\left(\frac{mx_0 - a}{m}\right)(a - mx_0)$$

$$= -m\left(\frac{mx_0 - a}{m}\right)\left(\frac{mx_0 - a}{m}\right)$$

$$= -m\left(\frac{mx_0 - a}{m}\right)^2$$

$$= -m\left(x_0 - \frac{a}{m}\right)^2$$

Substituting x for x_0, $f'(x)$ for m, and $f(x)$ for a, we have $A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2$.

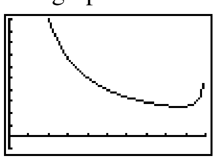
(b) The domain is the open interval (0, 10).

To graph, let $y_1 = f(x) = 5 + 5\sqrt{1 - \frac{x^2}{100}}$,

$y_2 = f'(x) = \text{NDER}(y_1)$, and

$y_3 = A(x) = -y_2\left(x - \frac{y_1}{y^2}\right)^2$.

The graph of the area function $y_3 = A(x)$ is shown below.



[0, 10] by [–100, 1000]

The vertical asymptotes at $x = 0$ and $x = 10$ correspond to horizontal or vertical tangent lines, which do not form triangles.

(c) Using our expression for the y-intercept of the tangent line, the height of the triangle is

$$a - mx = f(x) - f'(x)\cdot x$$

$$= 5 + \frac{1}{2}\sqrt{100 - x^2} - \frac{-x}{2\sqrt{100 - x^2}}x$$

$$= 5 + \frac{1}{2}\sqrt{100 - x^2} + \frac{x^2}{2\sqrt{100 - x^2}}$$

We may use graphing methods or the analytic method in part (d) to find that the minimum value of $A(x)$ occurs at $x \approx 8.66$. Substituting this value into the expression above, the height of the triangle is 15. This is 3 times the y-coordinate of the center of the ellipse.

(d) Part (a) remains unchanged. The domain is (0, C). To graph, note that

$$f(x) = B + B\sqrt{1 - \frac{x^2}{C^2}} = B + \frac{B}{C}\sqrt{C^2 - x^2} \text{ and } f'(x) = \frac{B}{C}\frac{1}{2\sqrt{C^2 - x^2}}(-2x) = \frac{-Bx}{C\sqrt{C^2 - x^2}}.$$

Therefore, we have

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2$$

$$= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{B + \frac{B}{C}\sqrt{C^2 - x^2}}{\frac{-Bx}{C\sqrt{C^2 - x^2}}}\right]^2$$

$$= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{(BC + B\sqrt{C^2 - x^2})\sqrt{C^2 - x^2}}{-Bx}\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + (BC + B\sqrt{C^2 - x^2})(\sqrt{C^2 - x^2})\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + BC\sqrt{C^2 - x^2} + B(C^2 - x^2)\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[BC(C + \sqrt{C^2 - x^2})\right]^2$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})^2}{x\sqrt{C^2 - x^2}}$$

$$A'(x) = \frac{BC\left(x\sqrt{C^2 - x^2}\right)(2)\left(C + \sqrt{C^2 - x^2}\right)\left(\frac{-x}{\sqrt{C^2 - x^2}}\right) - BC \cdot \left(C + \sqrt{C^2 - x^2}\right)^2\left(x\frac{-x}{\sqrt{C^2 - x^2}} + \sqrt{C^2 - x^2}(1)\right)}{x^2(C^2 - x^2)}$$

$$= \frac{BC\left(C + \sqrt{C^2 - x^2}\right)}{x^2(C^2 - x^2)}\left[-2x^2 - \left(C + \sqrt{C^2 - x^2}\right)\left(\frac{-x^2}{\sqrt{C^2 - x^2}} + \sqrt{C^2 - x^2}\right)\right]$$

$$= \frac{BC\left(C + \sqrt{C^2 - x^2}\right)}{x^2\sqrt{C^2 - x^2}}\left[-2x^2 + \frac{Cx^2}{\sqrt{C^2 - x^2}} - C\sqrt{C^2 - x^2} + x^2 - (C^2 - x^2)\right]$$

$$= \frac{BC\left(C + \sqrt{C^2 - x^2}\right)}{x^2(C^2 - x^2)}\left(\frac{Cx^2}{\sqrt{C^2 - x^2}} - C\sqrt{C^2 - x^2} - C^2\right)$$

$$= \frac{BC\left(C + \sqrt{C^2 - x^2}\right)}{x^2(C^2 - x^2)^{3/2}}\left[Cx^2 - C(C^2 - x^2) - C^2\sqrt{C^2 - x^2}\right]$$

$$= \frac{BC^2\left(C + \sqrt{C^2 - x^2}\right)}{x^2(C^2 - x^2)^{3/2}}\left(2x^2 - C^2 - C\sqrt{C^2 - x^2}\right)$$

To find the critical points for $0 < x < C$, we solve:

$$2x^2 - C^2 = C\sqrt{C^2 - x^2}$$

$$4x^4 - 4C^2x^2 + C^4 = C^4 - C^2x^2$$

$$4x^4 - 3C^2x^2 = 0$$

$$x^2(4x^2 - 3C^2) = 0$$

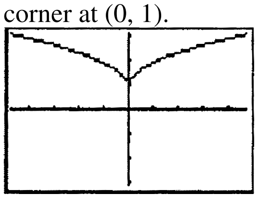
The minimum value of $A(x)$ for $0 < x < C$ occurs at the critical point $x = \frac{C\sqrt{3}}{2}$, or $x^2 = \frac{3C^2}{4}$. The corresponding triangle height is

$a-mx$
$= f(x)-f'(x)\cdot x$
$= B+\frac{B}{C}\sqrt{C^2-x^2}+\frac{Bx^2}{C\sqrt{C^2-x^2}}$
$= B+\frac{B}{C}\sqrt{C^2-\frac{3C^2}{4}}+\frac{B\left(\frac{3C^2}{4}\right)}{C\sqrt{C^2-\frac{3C^2}{4}}}$
$= B+\frac{B}{C}\left(\frac{C}{2}\right)+\frac{\frac{3BC^2}{4}}{\frac{C^2}{2}}$
$= B+\frac{B}{2}+\frac{3B}{2}$
$= 3B$
This shows that the triangle has minimum area when its height is $3B$.

Section 5.5 Linearization, Sensitivity, and Differentials (pp. 238–251)

Exploration 1 Appreciating Local Linearity

1. The graph appears to have either a cusp or a corner at (0, 1).



$y=\left(x^2+0.0001\right)^{1/4}+0.9$

2. $f'(x)=\frac{1}{4}(x^2+0.0001)^{-3/4}(2x)$

$=\frac{x}{\sqrt[4]{\left(x^2+0.0001\right)^3}}.$

Since $f'(0)=0,$ the tangent line at (0, 1) has equation $y = 1$.

3. The "corner" becomes smooth and the graph straightens out.

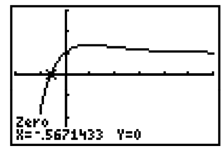
4. As with any differentiable curve, the graph comes to resemble the tangent line.

Quick Review 5.5

1. $\frac{dy}{dx}=\cos(x^2+1)\cdot\frac{d}{dx}(x^2+1)$
$=2x\cos(x^2+1)$

2. $\frac{dy}{dx}=\frac{(x+1)(1-\sin x)-(x+\cos x)(1)}{(x+1)^2}$
$=\frac{x-x\sin x+1-\sin x-x-\cos x}{(x+1)^2}$
$=\frac{1-\cos x-(x+1)\sin x}{(x+1)^2}$

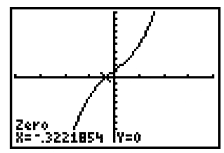
3.



[−2, 6] by [−3, 3]

$x\approx -0.567$

4.



[−4, 4] by [−10, 10]

$x\approx -0.322$

5. $f'(x)=(x)(-e^{-x})+(e^{-x})(1)=e^{-x}-xe^{-x}$
$f'(0) = 1$
The lines passes through (0, 1) and has slope 1. Its equation is $y = x + 1$.

6. $f'(x)=(x)(-e^{-x})+(e^{-x})(1)=e^{-x}-xe^{-x}$
$f'(-1)=e^1-(-e^1)=2e$
The lines passes through $(-1, -e + 1)$ and has slope $2e$.
Its equation is $y = 2e(x + 1) + (-e + 1)$, or $y = 2ex + e + 1$.

7. **(a)** $x+1=0$
$x=-1$

(b) $2ex+e+1=0$
$2ex=-(e+1)$
$x=-\frac{e+1}{2e}$
≈ -0.684

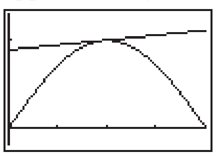
8. $f'(x)=3x^2-4$
$f'(1)=3(1)^2-4=-1$
Since $f(1)=-2$ and $f'(1)=-1,$ the graph of $g(x)$ passes through $(1, -2)$ and has slope -1.
Its equation is $g(x) = -1(x - 1) + (-2)$, or $g(x) = -x - 1$.

x	$f(x)$	$g(x)$
0.7	−1.457	−1.7
0.8	−1.688	−1.8
0.9	−1.871	−1.9
1.0	−2	−2
1.1	−2.069	−2.1
1.2	−2.072	−2.2
1.3	−2.003	−2.3

9. $f'(x) = \cos x$

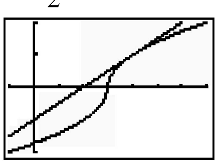
$f'(1.5) = \cos 1.5$

Since $f(1.5) = \sin 1.5$ and $f'(1.5) = \cos 1.5$, the tangent line passes through (1.5, sin 1.5) and has slope cos 1.5. Its equation is $y = (\cos 1.5)(x - 1.5) + \sin 1.5$, or approximately $y = 0.071x + 0.891$



$[0, \pi]$ by $[-0.2, 1.3]$

10. For $x > 3$, $f'(x) = \frac{1}{2\sqrt{x-3}}$, and so $f'(4) = \frac{1}{2}$. Since $f(4) = 1$ and $f'(4) = \frac{1}{2}$, the tangent line passes through (4, 1) and has slope $\frac{1}{2}$. Its equation is $y = \frac{1}{2}(x-4)+1$, or $y = \frac{1}{2}x - 1$.



$[-1, 7]$ by $[-2, 2]$

Section 5.5 Exercises

1. **(a)** $f'(x) = 3x^2 - 2$

We have $f(2) = 7$ and $f'(2) = 10$.

$$L(x) = f(2) + f'(2)(x-2) = 7 + 10(x-2) = 10x - 13$$

(b) Since $f(2.1) = 8.061$ and $L(2.1) = 8$, the approximation differs from the true value in absolute value by less than 10^{-1}.

2. **(a)** $f'(x) = \frac{1}{2\sqrt{x^2+9}}(2x) = \frac{x}{\sqrt{x^2+9}}$

We have $f(-4) = 5$ and $f'(-4) = -\frac{4}{5}$.

$$L(x) = f(-4) + f'(-4)(x-(-4)) = 5 - \frac{4}{5}(x+4) = -\frac{4}{5}x + \frac{9}{5}$$

(b) Since $f(-3.9) \approx 4.9204$ and $L(-3.9) = 4.92$, the approximation differs from the true value by less than 10^{-3}.

3. **(a)** $f'(x) = 1 - x^{-2}$

We have $f(1) = 2$ and $f'(1) = 0$.

$$L(x) = f(1) + f'(1)(x-1) = 2 + 0(x-1) = 2$$

(b) Since $f(1.1) = 2.009$ and $L(1.1) = 2$, the approximation differs from the true value by less than 10^{-2}.

4. **(a)** $f'(x) = \frac{1}{x+1}$

We have $f(0) = 0$ and $f'(0) = 1$.

$$L(x) = f(0) + f'(0)(x-0) = 0 + 1x = x$$

(b) Since $f(0.1) \approx 0.0953$ and $L(0.1) = 0.1$ the approximation differs from the true value by less than 10^{-2}.

5. **(a)** $f'(x)=\sec^2 x$

We have $f(\pi)=0$ and $f'(\pi)=1$.

$$\begin{aligned}L(x)&=f(\pi)+f'(\pi)(x-\pi)\\&=0+1(x-\pi)\\&=x-\pi\end{aligned}$$

(b) Since $f(\pi+0.1)\approx 0.10033$ and $L(\pi+0.1)=0.1$, the approximation differs from the true value in absolute value by less than 10^{-3}.

6. **(a)** $f'(x)=-\dfrac{1}{\sqrt{1-x^2}}$

We have $f(0)=\dfrac{\pi}{2}$ and $f'(0)=-1$.

$$\begin{aligned}L(x)&=f(0)+f'(0)(x-0)\\&=\frac{\pi}{2}+(-1)(x-0)\\&=-x+\frac{\pi}{2}\end{aligned}$$

(b) Since $f(0.1)\approx 1.47063$ and $L(0.1)\approx 1.47080$, the approximation differs from the true value in absolute value by less than 10^{-3}.

7. $f'(x)=k(1+x)^{k-1}$

We have $f(0)=1$ and $f'(0)=k$.

$$\begin{aligned}L(x)&=f(0)+f'(0)(x-0)\\&=1+k(x-0)\\&=1+kx\end{aligned}$$

8. **(a)**
$$\begin{aligned}(1.002)^{100}&=(1+0.002)^{100}\\&\approx 1+(100)(0.002)\\&=1.2;\end{aligned}$$
$$\left|1.002^{100}-1.2\right|\approx 0.021<10^{-1}$$

(b)
$$\begin{aligned}\sqrt[3]{1.009}&=(1+0.009)^{1/3}\\&\approx 1+\frac{1}{3}(0.009)\\&=1.003;\end{aligned}$$
$$\left|\sqrt[3]{1.009}-1.003\right|\approx 9\times10^{-6}<10^{-5}$$

9. **(a)**
$$\begin{aligned}f(x)&=(1-x)^6\\&=[1+(-x)]^6\\&\approx 1+6(-x)\\&=1-6x\end{aligned}$$

(b)
$$\begin{aligned}f(x)&=\frac{2}{1-x}\\&=2[1+(-x)]^{-1}\\&\approx 2[1+(-1)(-x)]\\&=2+2x\end{aligned}$$

(c) $f(x)=(1+x)^{-1/2}\approx 1+\left(-\dfrac{1}{2}\right)x=1-\dfrac{x}{2}$

10. **(a)**
$$\begin{aligned}f(x)&=(4+3x)^{1/3}\\&=4^{1/3}\left(1+\frac{3x}{4}\right)^{1/3}\\&\approx 4^{1/3}\left(1+\frac{1}{3}\left(\frac{3x}{4}\right)\right)\\&=4^{1/3}\left(1+\frac{x}{4}\right)\end{aligned}$$

(b)
$$\begin{aligned}f(x)&=\sqrt{2+x^2}\\&=\sqrt{2}\left(1+\frac{x^2}{2}\right)^{1/2}\\&\approx\sqrt{2}\left(1+\frac{1}{2}\left(\frac{x^2}{2}\right)\right)\\&=\sqrt{2}\left(1+\frac{x^2}{4}\right)\end{aligned}$$

(c)
$$\begin{aligned}f(x)&=\left(1-\frac{1}{2+x}\right)^{2/3}\\&=\left[1+\left(-\frac{1}{2+x}\right)\right]^{2/3}\\&\approx 1+\frac{2}{3}\left(-\frac{1}{2+x}\right)\\&=1-\frac{2}{6+3x}\end{aligned}$$

11. $x=100$

$$f'(100)=\frac{1}{2}(100)^{-1/2}=0.05$$
$$f(100)\approx 10+0.05(101-100)=10.05$$

12. $x=27$

$$f'(27)=\frac{1}{3}(27)^{-2/3}=\frac{1}{27}$$
$$f(27)\approx 3+\left(\frac{1}{27}\right)(26-27)$$
$$y=3-\frac{1}{27}\approx 2.\overline{962}$$

13. $x = 1000$

$f'(1000) = \frac{1}{3}(1000)^{-2/3} = \frac{1}{300}$

$y = 10 + \left(\frac{1}{300}\right)(x - 1000)$

$y = 10 - \frac{1}{150} = 9.99\overline{3}$

14. $x = 81$

$f'(81) = \frac{1}{2}(81)^{-1/2} = \frac{1}{18}$

$y = 9 + \frac{1}{18}(80 - 81)$

$y = 9 - \frac{1}{18} = 8.9\overline{4}$

15. **(a)** Since $\frac{dy}{dx} = 3x^2 - 3$, $dy = (3x^2 - 3)\,dx$.

(b) At the given values,
$dy = (3 \cdot 2^2 - 3)(0.05) = 9(0.05) = 0.45.$

16. **(a)** Since

$$\frac{dy}{dx} = \frac{(1+x^2)(2) - (2x)(2x)}{(1+x^2)^2} = \frac{2 - 2x^2}{(1+x^2)^2},$$

$$dy = \frac{2 - 2x^2}{(1+x^2)^2}\,dx.$$

(b) At the given values,

$$dy = \frac{2 - 2(-2)^2}{[1+(-2)^2]^2}(0.1) = \frac{2-8}{5^2}(0.1) = -0.024.$$

17. **(a)** Since

$$\frac{dy}{dx} = (x^2)\left(\frac{1}{x}\right) + (\ln x)(2x) = 2x\ln x + x,$$

$dy = (2x \ln x + x)dx.$

(b) At the given values,
$dy = [2(1)\ln(1) + 1](0.01) = 1(0.01) = 0.01$

18. **(a)** Since

$$\frac{dy}{dx} = (x)\left(\frac{1}{2\sqrt{1-x^2}}\right)(-2x) + (\sqrt{1-x^2})(1)$$
$$= \frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$$
$$= \frac{-x^2 + (1-x^2)}{\sqrt{1-x^2}}$$
$$= \frac{1-2x^2}{\sqrt{1-x^2}},$$

$$dy = \frac{1-2x^2}{\sqrt{1-x^2}}\,dx.$$

(b) At the given values,

$$dy = \frac{1-2(0)^2}{\sqrt{1-(0)^2}}(-0.2) = -0.2.$$

19. **(a)** Since $\frac{dy}{dx} = e^{\sin x}\cos x$,

$dy = (\cos x)e^{\sin x}dx.$

(b) At the given values,

$$dy = (\cos\pi)(e^{\sin\pi})(-0.1) = (-1)(1)(-0.1) = 0.1.$$

20. **(a)** Since

$$\frac{dy}{dx} = -3\csc\left(1 - \frac{x}{3}\right)\cot\left(1 - \frac{x}{3}\right)\left(-\frac{1}{3}\right)$$
$$= \csc\left(1 - \frac{x}{3}\right)\cot\left(1 - \frac{x}{3}\right),$$

$$dy = \csc\left(1 - \frac{x}{3}\right)\cot\left(1 - \frac{x}{3}\right)dx.$$

(b) At the given values,

$$dy = \csc\left(1 - \frac{1}{3}\right)\cot\left(1 - \frac{1}{3}\right)(0.1)$$
$$= 0.1\csc\frac{2}{3}\cot\frac{2}{3}$$
$$\approx 0.205525$$

21. **(a)** $y + xy - x = 0$

$y(1+x) = x$

$y = \frac{x}{x+1}$

Since $\frac{dy}{dx} = \frac{(x+1)(1)-(x)(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$,

$dy = \frac{dx}{(x+1)^2}$.

(b) At the given values, $dy = \frac{0.01}{(0+1)^2} = 0.01$.

22. **(a)** $2y = x^2 - xy$

$2dy = 2xdx - xdy - ydx$

$dy(2+x) = (2x-y)dx$

$dy = \left(\frac{2x-y}{2+x}\right)dx$

(b) At the given values, and $y = 1$ from the original equation,

$dy = \left(\frac{2(2)-1}{2+2}\right)(-0.05) = -0.0375$

23. $\frac{dy}{dx} = \sqrt{1-x^2}$

$dy = \left(-\frac{2x}{2\sqrt{1-x^2}}\right)dx$

$dy = -\frac{x}{\sqrt{1-x^2}}dx$

24. $\frac{dy}{dx} = e^{5x} + x^5$

$dy = (5e^{5x} + 5x^4)dx$

25. $\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

$u = 4x$

$\frac{du}{dx} = 4$

$dy = \left(\frac{4}{1+16x^2}\right)dx$

26. $\frac{d}{dx}a^x = (\ln a)a^x$

$dy = (8^x \ln 8 + 8x^7)dx$

27. **(a)** $\Delta f = f(0.1) - f(0) = 0.21 - 0 = 0.21$

(b) Since $f'(x) = 2x + 2$, $f'(0) = 2$.
Therefore, $f'(0)\Delta x = 2(0.1) = 0.2$.

(c) $|\Delta f - df| = |0.21 - 0.2| = 0.01$

28. **(a)** $\Delta f = f(1.1) - f(1) = 0.231 - 0 = 0.231$

(b) Since $f'(x) = 3x^2 - 1$, $f'(1) = 2$.
Therefore, $f'(1)\Delta x = 2(0.1) = 0.2$.

(c) $|\Delta f - f'(1)\Delta x| = |0.231 - 0.2| = 0.031$

29. **(a)** $\Delta f = f(0.55) - f(0.5) = \frac{20}{11} - 2 = -\frac{2}{11}$

(b) Since $f'(x) = -x^{-2}$, $f'(0.5) = -4$.
Therefore,

$f'(0.5)\Delta x = -4(0.05) = -0.2 = -\frac{1}{5}$

(c) $|\Delta f - f'(0.5)\Delta x| = \left|-\frac{2}{11} + \frac{1}{5}\right| = \frac{1}{55}$

30. **(a)** $\Delta f = f(1.01) - f(1)$

$= 1.04060401 - 1$

$= 0.04060401$

(b) Since $f'(x) = 4x^3$, $f'(1) = 4$.
Therefore, $f'(1)\Delta x = 4(0.01) = 0.04$.

(c) $|\Delta f - f'(1)\Delta x| = |0.04060401 - 0.04|$

$= 0.00060401$

31. Note that $V = \frac{4}{3}\pi r^3$, $\Delta V = 4\pi r^2 \Delta r$. When r changes from a to $a + \Delta r$, the change in volume is approximately $4\pi a^2 \Delta r$. When $a = 10$ and $\Delta r = 0.05$,

$\Delta V \approx 4\pi(10)^2(0.05) = 20\pi$ cm^3.

32. Note that $S = 4\pi r^2$, so $\Delta S = 8\pi r\,\Delta r$. When r changes from a to $a + \Delta r$, the change in surface area is approximately $8\pi a\,\Delta r$. When $a = 10$ and $\Delta r = 0.05$,

$\Delta S = 8\pi(10)(0.05) = 4\pi$ cm^2.

33. Note that $V = x^3$, so $\Delta V = 3x^2\,\Delta x$. When x changes from a to $a + \Delta x$, the change in volume is approximately $3a^2\ \Delta x$. When $a = 10$ and $\Delta x = 0.05$,
$\Delta V \approx 3(10)^2(0.05) = 15\text{ cm}^3$.

34. Note that $S = 6x^2$, so $\Delta S = 12x\ \Delta x$. When x changes from a to $a + \Delta x$, the change in surface area is approximately $12a\ \Delta x$. When $a = 10$ and $\Delta x = 0.05$,
$\Delta S \approx 12(10)(0.05) = 6\text{ cm}^2$.

35. Note that $V = \pi r^2 h$, so $\Delta V = 2\pi rh\ \Delta r$. When r changes from a to $a + \Delta r$, the change in volume is approximately $2\pi ah\ \Delta r$. When $a = 10$ and $\Delta r = 0.05$,
$\Delta V \approx 2\pi(10)h(0.05) = \pi h\text{ cm}^3$.

36. Note that $S = 2\pi rh$, so $\Delta S = 2\pi r\,\Delta h$. When h changes from a to $a + \Delta h$, the change in lateral surface area is approximately $2\pi r\ \Delta h$. When $a = 10$ and $\Delta h = 0.05$,
$\Delta S \approx 2\pi r(0.05) = 0.1\pi r\text{ cm}^2$.

37. $A = \pi r^2$
$\Delta A = 2\pi r\,\Delta r$
$\Delta A \approx 2\pi(10)(0.1) \approx 6.3\text{ in}^2$

38. $V = \frac{4}{3}\pi r^3$
$\Delta V = 4\pi r^2 \Delta r$
$\Delta V \approx 4\pi(8)^2(0.3) \approx 241\text{ in}^3$

39. $V = s^3$
$\Delta V = 3s^2 \Delta s$
$\Delta V \approx 3(15)^2(0.2) = 135\text{ cm}^3$

40. $A = \frac{\sqrt{3}}{4}s^2$
$\Delta A = \frac{\sqrt{3}}{2}s\,\Delta s$
$\Delta A \approx \frac{\sqrt{3}}{2}(20)(0.5) = 8.7\text{ cm}^2$

41. **(a)** Note that $f'(0) = \cos 0 = 1$.
$L(x) = f(0) + f'(0)(x - 0) = 1 + 1x = x + 1$

(b) $f(0.1) \approx L(0.1) = 1.1$

(c) The actual value is less than 1.1. This is because the derivative is decreasing over the interval [0, 0.1], which means that the graph of $f(x)$ is concave down and lies below its linearization in this interval.

42. **(a)** Note that $A = \pi r^2$ so $\Delta A = 2\pi r\,\Delta r$. When r changes from a to $a + \Delta r$, the change in area is approximately $2\pi a\ \Delta r$. Substituting 2 for a and 0.02 for Δr, the change in area is approximately
$2\pi(2)(0.02) = 0.08\pi \approx 0.2513$

(b) $\frac{\Delta A}{A} = \frac{0.08\pi}{4\pi} = 0.02 = 2\%$

43. Let A = cross section area, C = circumference, and D = diameter. Then $D = \frac{C}{\pi}$, $\Delta D = \frac{1}{\pi}\Delta C$.
Also, $A = \pi\left(\frac{D}{2}\right)^2 = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{4\pi}$, so
$\Delta A = \frac{C}{2\pi}\Delta C$. When C increases from 10π in. to $10\pi + 2$ in. the diameter increases by
$\Delta D \approx \frac{1}{\pi}(2) = \frac{2}{\pi} \approx 0.6366$ in. and the area increases by approximately
$\Delta A \approx \frac{10\pi}{2\pi}(2) = 10\text{ in}^2$.

44. Let x = edge length and V = volume. Then $V = x^3$, and so $\Delta V = 3x^2\ \Delta x$. With $x = 10$ cm and $\Delta x = 0.01x = 0.1$ cm, we have
$V = 10^3 = 1000\text{ cm}^3$ and
$\Delta V \approx 3(10)^2(0.1) = 30\text{ cm}^3$, so the percentage error in the volume measurement is approximately $\frac{\Delta V}{V} = \frac{30}{1000} = 0.03 = 3\%$.

45. Let x = side length and A = area. Then $A = x^2$ so $\Delta A = 2x\,\Delta x$. We want $|\Delta A| \le 0.02A$, which gives $|2x\,\Delta x| \le 0.02x^2$, or
$|\Delta dx| \le 0.01x$. The side length should be measured with an error of no more than 1%.

46. **(a)** Note that $V = \pi r^2 h = 10\pi r^2 = 2.5\pi D^2$, where D is the interior diameter of the tank. Then $\Delta V = 5\pi D\,\Delta D$. We want $|\Delta V| \le 0.01V$, which gives

$|5\pi D\,\Delta D| \le 0.01(2.5\pi D^2)$, or

$|\Delta D| \le 0.005D$. The interior diameter should be measured with an error of no more than 0.5%.

(b) Now we let D represent the *exterior* diameter of the tank, and we assume that the paint coverage rate (number of square feet covered per gallon of paint) is known precisely. Then, to determine the amount of paint within 5%, we need to calculate the lateral surface area S with an error of no more than 5%. Note that $S = 2\pi rh = 10\pi D$, so $\Delta S = 10\pi\,\Delta D$. We want $|\Delta S| \le 0.05S$, which gives

$|10\pi\,\Delta D| \le 0.05(10\pi D)$, or

$\Delta D \le 0.5D$. The exterior diameter should be measured with an error of no more than 5%.

47. Note that $V = \pi r^2 h$, where h is constant. Then $\Delta V = 2\pi rh\Delta r$. The percent change is given by
$$\frac{\Delta V}{V} = \frac{2\pi rh\Delta r}{\pi r^2 h} = 2\frac{\Delta r}{r} = 2\frac{0.1\% r}{r} = 0.2\%.$$

48. Note that $V = \pi h^3$, so $\Delta V = 3\pi h^2\,\Delta h$. We want $|\Delta V| \le 0.01V$, which gives

$\left|3\pi h^2 \Delta h\right| \le 0.01(\pi h^3)$, or $|\Delta h| \le \frac{0.01h}{3}$. The height should be measured with an error of no more than $\frac{1}{3}\%$.

49. If $\Delta C = 2\pi\,\Delta r$ and $\Delta C = \frac{1}{8}$ inch, then

$\Delta r = \frac{1}{16\pi}$ inch. Since $V = \frac{4}{3}\pi r^3$, we have

$$\Delta V = 4\pi r^2 \Delta r = 4\pi r^2\left(\frac{1}{16\pi}\right) = \frac{r^2}{4}.$$

The volume error in each case is simply $\frac{r^2}{4}\text{in}^3$.

Sphere Type	True Radius	Tape error	Radius Error	Volume Error
Orange	2 in.	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in.	$1\,\text{in}^3$
Melon	4 in.	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in.	$4\,\text{in}^3$
Beach Ball	7 in.	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in..	$12.25\ \text{in}^3$

50. If $\Delta C = 2\pi\,\Delta r$ and $\Delta C = \frac{1}{8}$ inch, then

$\Delta r = \frac{1}{16\pi}$ inch. Since $A = 4\pi r^2$, we have

$$\Delta A = 8\pi r\Delta r = 8\pi r\left(\frac{1}{16\pi}\right) = \frac{r}{2}.$$

The surface area error in each case is simply $\frac{r}{2}\text{ in}^2$.

Sphere Type	True Radius	Tape Error	Radius Error	Volume Error
Orange	2 in.	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in.	$1\,\text{in}^2$
Melon	4 in.	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in.	$2\,\text{in}^2$
Beach Ball	7 in	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in.	$3.5\,\text{in}^2$

51. We have $W = a + \frac{b}{g}$, so $\Delta W = -bg^{-2}\Delta g$.

Then
$$\frac{dW_{\text{moon}}}{dW_{\text{earth}}} = \frac{-b(5.2)^{-2}\Delta g}{-b(32)^{-2}\Delta g} = \frac{32^2}{5.2^2} \approx 37.87.$$ The ratio is about 37.87 to 1.

52. **(a)** Note that $T = 2\pi L^{1/2} g^{-1/2}$, so
$$\Delta T = -\pi L^{1/2} g^{-3/2}\Delta g.$$

(b) Note that ΔT and Δg have opposite signs. Thus, if g increases, T decreases and the clock speeds up.

(c)
$$-\pi L^{1/2} g^{-3/2}\Delta g = \Delta T$$
$$-\pi(100)^{1/2}(980)^{-3/2}\Delta g = 0.001$$
$$\Delta g \approx -0.9765$$

Since $\Delta g \approx -0.9765$,

$g \approx 980 - 0.9765 = 979.0235.$

53. Let $f(x) = x^3 + x - 1$. Then $f'(x) = 3x^2 + 1$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}.$$

Note that f is cubic and f' is always positive, so there is exactly one solution. We choose $x_1 = 0$.

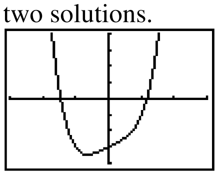
$x_1 = 0$
$x_2 = 1$
$x_3 = 0.75$
$x_4 \approx 0.6860465$
$x_5 \approx 0.6823396$
$x_6 \approx 0.6823278$
$x_7 \approx 0.6823278$

Solution: $x \approx 0.682328$.

54. Let $f(x) = x^4 + x - 3$. Then $f'(x) = 4x^3 + 1$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 + x_n - 3}{4x_n^3 + 1}$$

The graph of $y = f(x)$ shows that $f(x) = 0$ has two solutions.



[−3, 3] by [−4, 4]

$x_1 = -1.5$	$x_1 = 1.2$
$x_2 = -1.455$	$x_2 \approx 1.6541962$
$x_3 \approx -1.4526332$	$x_3 \approx 1.1640373$
$x_4 \approx -1.4526269$	$x_4 \approx 1.1640351$
$x_5 \approx -1.4526269$	$x_5 \approx 1.1640351$

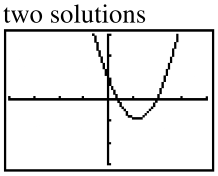
Solution: $x \approx -1.452627, 1.164035$

55. Let $f(x) = x^2 - 2x + 1 - \sin x$.

Then $f'(x) = 2x - 2 - \cos x$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2x_n + 1 - \sin x_n}{2x_n - 2 - \cos x_n}$$

The graph of $y = f(x)$ shows that $f(x) = 0$ has two solutions



[−4, 4] by [−3, 3]

$x_1 = 0.3$	$x_1 = 2$
$x_2 \approx 0.3825699$	$x_2 \approx 1.9624598$
$x_3 \approx 0.3862295$	$x_3 \approx 1.9615695$
$x_4 \approx 0.3862369$	$x_4 \approx 1.9615690$
$x_5 \approx 0.3862369$	$x_5 \approx 1.9615690$

Solutions: $x \approx 0.386237, 1.961569$

56. Let $f(x) = x^4 - 2$. Then $f'(x) = 4x^3$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 2}{4x_n^3}.$$

Note that $f(x) = 0$ clearly has two solutions, namely $x = \pm\sqrt[4]{2}$. We use Newton's method to find the decimal equivalents.

$x_1 = 1.5$
$x_2 \approx 1.2731481$
$x_3 \approx 1.1971498$
$x_4 \approx 1.1892858$
$x_5 \approx 1.1892071$
$x_6 \approx 1.1892071$

Solutions: $x \approx \pm 1.189207$

57. True; a look at the graph reveals the problem. The graph decreases after $x = 1$ toward a horizontal asymptote of $y = 0$, so the x-intercepts of the tangent lines keep getting bigger without approaching a zero.

58. False; by the product rule, $d(uv) = u\,dv + v\,du$.

59. B; $f(x) = e^x$

$f'(x) = e^x$

$L(x) = e^1 + e^1(x - 1)$

$L(x) = ex$

60. D; $y = \tan x$

$dy = (\sec^2 x)dx = (\sec^2 \pi)0.5 = 0.5$

61. D; $f(x) = x - x^3 + 2$

$f'(x) = 1 - 3x^2$

$$x_{n+1} = x_n - \frac{x_n - x_n^3 + 2}{1 - 3x_n^2}$$

$$x_2 = 1 - \frac{1 - (1)^3 + 2}{1 - 3(1)^2} = 2$$

$$x_3 = 2 - \frac{2 - (2)^3 + 2}{1 - 3(2)^2} = \frac{18}{11}$$

62. A; $f(x)=\sqrt[3]{x};\ x=64$

$$f'(64)=\frac{1}{3}(64)^{-2/3}=\frac{1}{48}$$

$$\sqrt[3]{66}\approx 4+\frac{1}{48}(66-64)=\frac{97}{24}$$

The percentage error is

$$\frac{\sqrt[3]{66}-97/24}{\sqrt[3]{66}}\approx 0.01\%.$$

63. If $f'(x)\neq 0$, we have

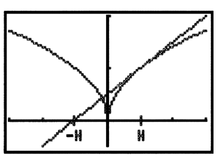
$x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}=x_1-\dfrac{0}{f'(x_1)}=x_1$. Therefore, $x_2=x_1$, and all later approximations are also equal to x_1.

64. If $x_1=h$, then $f'(x_1)=\dfrac{1}{2h^{1/2}}$ and

$x_2=h-\dfrac{h^{1/2}}{\frac{1}{2h^{1/2}}}=h-2h=-h$. If $x_1=-h$, then

$f'(x_1)=-\dfrac{1}{2\sqrt{h}}$ and

$$x_2=-h-\frac{h^{1/2}}{-\frac{1}{2h^{1/2}}}=-h+2h=h$$



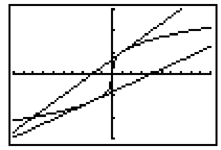
[−3, 3] by [−0.5, 2]

65. Note that $f'(x)=\dfrac{1}{3}x^{-2/3}$ and so

$$\begin{aligned}x_{n+1}&=x_n-\frac{f(x_n)}{f'(x_n)}\\&=x_n-\frac{x_n^{1/3}}{\frac{x_n^{-2/3}}{3}}\\&=x_n-3x_n\\&=-2x_n.\end{aligned}$$

For $x_1=1$, we have $x_2=-2,\ x_3=4,\ x_4=-8$, and $x_5=16;\ |x_n|=2^{n-1}$.

The approximations alternate in sign and rapidly get farther and farther away from the zero at the origin.


[−10, 10] by [−3, 3]

66. (a) i. $Q(a)=f(a)$ implies that $b_0=f(a)$.

ii. Since $Q'(x)=b_1+2b_2(x-a)$, $Q'(a)=f'(a)$ implies that $b_1=f'(a)$.

iii. Since $Q''(x)=2b_2$, $Q''(a)=f''(a)$ implies that $b_2=\dfrac{f''(a)}{2}$

In summary,

$b_0=f(a)$, $b_1=f'(a)$, and $b_2=\dfrac{f''(a)}{2}$.

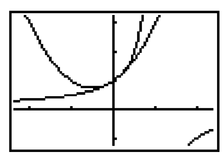
(b) $f(x)=(1-x)^{-1}$

$f'(x)=-1(1-x)^{-2}(-1)=(1-x)^{-2}$

$f''(x)=-2(1-x)^{-3}(-1)=2(1-x)^{-3}$

Since $f(0)=1, f'(0)=1$, and $f''(0)=2$, the coefficients are $b_0=1, b_1=1$, and $b_2=\dfrac{2}{2}=1$. The quadratic approximation is $Q(x)=1+x+x^2$.

(c)


[−2.35, 2.35] by [−1.25, 3.25]

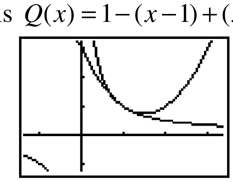
As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(d) $g(x)=x^{-1}$

$g'(x)=-x^{-2}$

$g''(x)=2x^{-3}$

Since $g(1)=1, g'(1)=-1$, and $g''(1)=2$, the coefficients are $b_o=1, b_1=-1$, and $b_2=\dfrac{2}{2}=1$. The quadratic approximation is $Q(x)=1-(x-1)+(x-1)^2$.


[−1.35, 3.35] by [−1.25, 3.25]

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical

(e) $h(x) = (1+x)^{1/2}$

$h'(x) = \frac{1}{2}(1+x)^{-1/2}$

$h''(x) = -\frac{1}{4}(1+x)^{-3/2}$

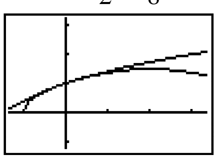
Since

$h(0) = 1, h'(0) = \frac{1}{2},$ and $h''(0) = -\frac{1}{4}$, the coefficients are $b_0 = 1,\ b_1 = \frac{1}{2},$ and

$b_2 = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}.$

The quadratic approximation is

$Q(x) = 1 + \frac{x}{2} - \frac{x^2}{8}.$



[–1.35, 3.35] by [–1.25, 3.25]

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(f) The linearization of any differentiable function $u(x)$ at $x = a$ is $L(x) = u(a) + u'(a)(x-a) = b_0 + b_1(x-a)$, where b_0 and b_1 are the coefficients of the constant and linear terms of the quadratic approximation. Thus, the linearization for $f(x)$ at $x = 0$ is $1 + x$; the linearization for $g(x)$ at $x = 1$ is $1 - (x-1)$ or $2 - x$; and the linearization for $h(x)$ at $x = 0$ is $1 + \frac{x}{2}$.

67. Finding a zero of sin x by Newton's method would use the recursive formula $x_{n+1} = x_n - \frac{\sin(x_n)}{\cos(x_n)} = x_n - \tan x_n$, and that is exactly what the calculator would be doing. Any zero of sin x would be a multiple of π.

68. Just multiply the corresponding derivative formulas by dx.

(a) Since $\frac{d}{dx}(c) = 0,\ d(c) = 0.$

(b) Since $\frac{d}{dx}(cu) = c\frac{du}{dx},\ d(cu) = c\ du.$

(c) Since $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$,
$d(u + v) = du + dv$.

(d) Since $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$,
$d(u \cdot v) = u\ dv + v\ du.$

(e) Since $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$,
$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$

(f) Since $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$,
$d(u^n) = nu^{n-1}du.$

69. $g(a) = c$, so if $E(a) = 0$, then $g(a) = f(a)$ and $c = f(a)$. Then
$E(x) = f(x) - g(x) = f(x) - f(a) - m(x - a)$.

Thus, $\frac{E(x)}{x-a} = \frac{f(x) - f(a)}{x-a} - m.$

$\lim_{x \to a} \frac{f(x) - f(a)}{x-a} = f'(a)$, so

$\lim_{x \to a} \frac{E(x)}{x-a} = f'(a) - m.$

Therefore, if the limit of $\frac{E(x)}{x-a}$ is zero, then $m = f'(a)$ and $g(x) = L(x)$.

70. $f'(x) = \frac{1}{2\sqrt{x+1}} + \cos x$

We have $f(0) = 1$ and $f'(0) = \frac{3}{2}$

$L(x) = f(0) + f'(0)(x-0)$
$= 1 + \frac{3}{2}x$

The linearization is the sum of the two individual linearizations, which are x for sin x and $1 + \frac{1}{2}x$ for $\sqrt{x+1}$.

71. The equation for the tangent is $y - f(x_n) = f'(x_n)(x - x_n)$. Set $y = 0$ and solve for x.

$$0 - f(x_n) = f'(x_n)(x - x_n)$$
$$-f(x_n) = f'(x_n) \cdot x - f'(x_n) \cdot x_n$$
$$f'(x_n) \cdot x = f'(x_n) \cdot x_n - f(x_n)$$
$$x = x_n - \frac{f(x_n)}{f'(x_n)} \text{ (If } f'(x_n) \neq 0)$$

The value of x is the next approximation x_{n+1}.

72. **(a)** $f''(x) = -\sin x$ and $|-\sin(c)| \le 1$.

$$\left|\Delta y - f'\left(\frac{\pi}{4}\right)\Delta x\right| = \left|\Delta y - \cos\left(\frac{\pi}{4}\right)\Delta x\right| = \left|\Delta y - \frac{\sqrt{2}}{2}\Delta x\right| = \frac{1}{2}|-\sin(c)|(\Delta x)^2 \le \frac{1}{2}(\Delta x)^2$$

(b) $f''(x) = 2$ for all x, i.e., $f''(c) = 2$.

$$|\Delta y - f'(1)\Delta x| = |\Delta y - 2(1)\Delta x| = \frac{1}{2}|2|(\Delta x)^2 = (\Delta x)^2$$

This is the exact value of the difference since $f''(x)$ is a constant.

(c) $f''(x) = e^x$ and within 0.1 unit of $x = 1$, $|f''(x)| \le e^{1.1}$.

$$|\Delta y - f(a)\Delta x| = |\Delta y - e\Delta x| = \frac{1}{2}|e^c|(\Delta x)^2 \le \frac{e^{1.1}}{2}(\Delta x)^2$$

73. **(a)**
$$g(a) = (f(a) - f(a)) - f'(a)(a - a) = 0 - f'(a) \cdot 0 = 0$$
$$g'(x) = (f'(x) - 0) - f'(a)(1 - 0) = f'(x) - f'(a)$$
so $g'(a) = f'(a) - f'(a) = 0$
$$g''(x) = f''(x) - 0 = f''(x)$$

(b) Let A be the minimum value of $\frac{1}{2}g''(t)$ and B be the maximum value of $\frac{1}{2}g''(t)$ for t in the interval $[a, x]$. Since $g''(x) = f''(x)$, $\frac{1}{2}g''(t)$ is continuous. Then by the Intermediate Value Theorem, $\frac{1}{2}g''(t)$ takes on every value between A and B. That is, for every number r between A and B there is some value c in $[a, x]$ for which $r = \frac{1}{2}g''(c)$.

(c) Since A is the minimum value of $\frac{1}{2}g''(t)$ for t in the interval $[a, x]$, $A \le \frac{1}{2}g''(t)$ or $2A \le g''(t)$, so $g'''(t) - 2A \ge 0$.

Similarly, $B \ge \frac{1}{2}g''(t)$ or $2B \ge g''(t)$, so $g''(t) - 2B \le 0$.

(d) Since $g'(a) = 0$, $g'(a) - 2A(a - a) = g'(a) - 2B(a - a) = 0$. Let $G(t) = g'(t) - 2A(t - a)$, then $G(0) = 0$ and $G'(t) = g''(t) - 2A$, so by Corollary 1 on page 204, $G(t)$ is increasing on $[a, x]$, so $G(t) = g'(t) = 2A(t - a) \ge 0$ for all t in $[a, x]$.
Let $H(t) = g'(t) - 2B(t - a)$, so $H'(t) = g''(t) - 2B$. By Corollary 1 on page 204, $H(t)$ is decreasing on $[a, x]$, so $H(t) = g'(t) - 2B(t - a) \le 0$ for all t in $[a, x]$.

(e) Similar to part (d), now let $G(t) = g(t) - A(t - a)^2$. Then $G(a) = g(a) - A(a - a)^2 = 0$ since $g(a) = 0$. Here $G'(t) = g'(t) - 2A(t - a)$, thus $G'(t) \ge 0$ for all t in $[a, x]$. By Corollary 1 on page 204, then $G(t)$ is increasing on $[a, x]$ and since $G(a) = 0$, $G(t) = g(t) - A(t - a)^2 \ge 0$ for all t in $[a, x]$. In a similar manner, $g(t) - B(t - a)^2 \le 0$ for all t in $[a, x]$.

(f) Since $g(t) - A(t-a)^2 \geq 0$ for all t in $[a, x]$, then specifically

$$g(x) - A(x-a)^2 \geq 0 \text{ or } \frac{g(x)}{(x-a)^2} \geq A.$$

Similarly, since $g(t) - B(t-a)^2 \leq 0$ for all t in $[a, x]$, then $g(x) - B(x-a)^2 \leq 0$ or

$$\frac{g(x)}{(x-a)^2} \leq B.$$

Thus, $A \leq \frac{g(x)}{(x-a)^2} \leq B$, and by part (b), there is some value of c in (a, x) for which

$$\frac{g(x)}{(x-a)^2} = \frac{1}{2}g''(c) = \frac{1}{2}f''(c).$$

Alternatively, there is some value of c in (a, x) for which

$$(f(x) - f(a)) - f'(a)(x-a) = g(x) = \frac{1}{2}f''(c)(x-a)^2.$$

Hence, $|\Delta y - f'(a)\Delta x| = \frac{1}{2}|f''(c)|(\Delta x)^2.$

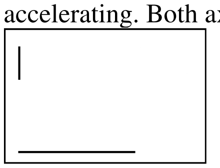
Section 5.6 Related Rates (pp. 252–261)

Exploration 1 The Sliding Ladder

1. Here the x-axis represents the ground and the y-axis represents the wall. The curve (x_1, y_1) gives the position of the bottom of the ladder (distance from the wall) at any time t in $0 \leq t \leq 5$. The curve (x_2, y_2) gives the position of the top of the ladder at any time in $0 \leq t \leq 5$.

2. $0 \leq t \leq 5$

4. This is a snapshot at $t \approx 3.1$. The top of the ladder is moving down the y-axis and the bottom of the ladder is moving to the right on the x-axis. The end of the ladder is accelerating. Both axes are hidden from view.



[–1, 15] by [–1, 15]

6. $\frac{dy}{dt} = \frac{-4T}{\sqrt{10^2 - (2T)^2}}$

7. $y'(3) = -1.5$ ft/sec^2. The negative number means the y-side of the right triangle is decreasing in length.

8. Since $\lim_{t \to 5^-} y'(t) = -\infty$, the speed of the top of the ladder is infinite as it hits the ground.

Quick Review 5.6

1. $D = \sqrt{(7-0)^2 + (0-5)^2} = \sqrt{49+25} = \sqrt{74}$

2. $D = \sqrt{(b-0)^2 + (0-a)^2} = \sqrt{a^2 + b^2}$

3. Use implicit differentiation.

$$\frac{d}{dx}(2xy + y^2) = \frac{d}{dx}(x+y)$$

$$2x\frac{dy}{dx} + 2y(1) + 2y\frac{dy}{dx} = (1) + \frac{dy}{dx}$$

$$(2x + 2y - 1)\frac{dy}{dx} = 1 - 2y$$

$$\frac{dy}{dx} = \frac{1-2y}{2x+2y-1}$$

4. Use implicit differentiation.

$$\frac{d}{dx}(x\sin y) = \frac{d}{dx}(1 - xy)$$

$$(x)(\cos y)\frac{dy}{dx} + (\sin y)(1) = -x\frac{dy}{dx} - y(1)$$

$$(x + x\cos y)\frac{dy}{dx} = -y - \sin y$$

$$\frac{dy}{dx} = \frac{-y - \sin y}{x + x\cos y}$$

$$\frac{dy}{dx} = -\frac{y + \sin y}{x + x\cos y}$$

5. Use implicit differentiation.

$$\frac{d}{dx}x^2 = \frac{d}{dx}\tan y$$

$$2x = \sec^2 y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x}{\sec^2 y}$$

$$\frac{dy}{dx} = 2x\cos^2 y$$

6. Use implicit differentiation.

$$\frac{d}{dx}\ln(x+y) = \frac{d}{dx}(2x)$$
$$\frac{1}{x+y}\left(1+\frac{dy}{dx}\right) = 2$$
$$1+\frac{dy}{dx} = 2(x+y)$$
$$\frac{dy}{dx} = 2x+2y-1$$

7. Using $A(-2, 1)$ we create the parametric equations $x = -2 + at$ and $y = 1 + bt$, which determine a line passing through A *at* $t = 0$. We determine a and b so that the line passes through $B(4, -3)$ *at* $t = 1$. Since $4 = -2 + a$, we have $a = 6$, and since $-3 = 1 + b$, we have $b = -4$. Thus, one parametrization for the line segment is $x = -2 + 6t$, $y = 1 - 4t$, $0 \le t \le 1$. (Other answers are possible.)

8. Using $A(0, -4)$, we create the parametric equations $x = 0 + at$ and $y = -4 + bt$, which determine a line passing through A at $t = 0$. We now determine a and b so that the line passes through $B(5, 0)$ at $t = 1$. Since $5 = 0 + a$, we have $a = 5$, and since $0 = -4 + b$, we have $b = 4$. Thus, one parametrization for the line segment is $x = 5t$, $y = -4 + 4t$, $0 \le t \le 1$. (Other answers are possible.)

9. One possible answer: $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$

10. One possible answer: $\frac{3\pi}{2} \le t \le 2\pi$

Section 5.6 Exercises

1. Since $\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$, we have $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$.

2. Since $\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}$, we have $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$.

3. **(a)** Since $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$, we have

$$\frac{dV}{dt} = \pi r^2\frac{dh}{dt}.$$

(b) Since $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$, we have

$$\frac{dV}{dt} = 2\pi rh\frac{dr}{dt}.$$

(c)
$$\frac{dV}{dt} = \frac{d}{dt}\pi r^2 h = \pi\frac{d}{dt}(r^2 h)$$
$$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + h(2r)\frac{dr}{dt}\right)$$
$$\frac{dV}{dt} = \pi r^2\frac{dh}{dt} + 2\pi rh\frac{dr}{dt}$$

4. **(a)**
$$\frac{dP}{dt} = \frac{d}{dt}(RI^2)$$
$$\frac{dP}{dt} = R\frac{d}{dt}I^2 + I^2\frac{dR}{dt}$$
$$\frac{dP}{dt} = R\left(2I\frac{dI}{dt}\right) + I^2\frac{dR}{dt}$$
$$\frac{dP}{dt} = 2RI\frac{dI}{dt} + I^2\frac{dR}{dt}$$

(b) If P is constant, we have $\frac{dP}{dt} = 0$, which means $2RI\frac{dI}{dt} + I^2\frac{dR}{dt} = 0$, or

$$\frac{dR}{dt} = -\frac{2R}{I}\frac{dI}{dt} = -\frac{2P}{I^3}\frac{dI}{dt}.$$

5.
$$\frac{ds}{dt} = \frac{d}{dt}\sqrt{x^2+y^2+z^2}$$
$$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2+y^2+z^2}}\frac{d}{dt}(x^2+y^2+z^2)$$
$$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2+y^2+z^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}\right)$$
$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}}{\sqrt{x^2+y^2+z^2}}$$

6. $\frac{dA}{dt} = \frac{d}{dt}\left(\frac{1}{2}ab\sin\theta\right)$

$\frac{dA}{dt} = \frac{1}{2}\left(\frac{da}{dt}\cdot b\cdot\sin\theta + a\cdot\frac{db}{dt}\cdot\sin\theta + ab\cdot\frac{d}{dt}\sin\theta\right)$

$\frac{dA}{dt} = \frac{1}{2}\left(b\sin\theta\frac{da}{dt} + a\sin\theta\frac{db}{dt} + ab\cos\theta\frac{d\theta}{dt}\right)$

7. **(a)** Since V is increasing at the rate of 1 volt/sec, $\frac{dV}{dt} = 1$ volt/sec.

(b) Since I is decreasing at the rate of $\frac{1}{3}$ amp/sec, $\frac{dI}{dt} = -\frac{1}{3}$ amp/sec.

(c) Differentiating both sides of $V = IR$, we have $\frac{dV}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}$.

(d) Note that $V = IR$ gives $12 = 2R$, so $R = 6$ ohms. Now substitute the known values into the equation in (c).

$1 = 2\frac{dR}{dt} + 6\left(-\frac{1}{3}\right)$

$3 = 2\frac{dR}{dt}$

$\frac{dR}{dt} = \frac{3}{2}$ ohms/sec

R is changing at the rate of $\frac{3}{2}$ ohms/sec. Since this value is positive, R is increasing.

8. Step 1:
r = radius of plate
A = area of plate
Step 2:
At the instant in question, $\frac{dr}{dt} = 0.01$ cm/sec, $r = 50$ cm.
Step 3:
We want to find $\frac{dA}{dt}$.
Step 4:
$A = \pi r^2$
Step 5:
$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$
Step 6:
$\frac{dA}{dt} = 2\pi(50)(0.01) = \pi\ \text{cm}^2/\text{sec}$
At the instant in question, the area is increasing at the rate of $\pi\ \text{cm}^2/\text{sec}$.

9. Step 1:
l = length of rectangle
w = width of rectangle
A = area of rectangle
P = perimeter of rectangle
D = length of a diagonal of the rectangle

Step 2:

At the instant in question, $\frac{dl}{dt} = -2$ cm/sec,

$\frac{dw}{dt} = 2$ cm/sec, $l = 12$ cm, and $w = 5$ cm.

Step 3:

We want to find $\frac{dA}{dt}, \frac{dP}{dt}$, and $\frac{dD}{dt}$.

Steps 4, 5, and 6:

(a) $A = lw$

$$\frac{dA}{dt} = l\frac{dw}{dt} + w\frac{dl}{dt}$$

$$\frac{dA}{dt} = (12)(2) + (5)(-2) = 14 \text{ cm}^2/\text{sec}$$

The rate of change of the area is $14 \text{ cm}^2/\text{sec}$.

(b) $P = 2l + 2w$

$$\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$

$$\frac{dP}{dt} = 2(-2) + 2(2) = 0 \text{ cm/sec}$$

The rate of change of the perimeter is 0 cm/sec.

(c) $D = \sqrt{l^2 + w^2}$

$$\frac{dD}{dt} = \frac{1}{2\sqrt{l^2 + w^2}}\left(2l\frac{dl}{dt} + 2w\frac{dw}{dt}\right) = \frac{l\frac{dl}{dt} + w\frac{dw}{dt}}{\sqrt{l^2 + w^2}}$$

$$\frac{dD}{dt} = \frac{(12)(-2) + (5)(2)}{\sqrt{12^2 + 5^2}} = -\frac{14}{13} \text{ cm/sec}$$

The rate of change of the length of the diameter is $-\frac{14}{13}$ cm/sec.

(d) The area is increasing, because its derivative is positive. The perimeter is not changing, because its derivative is zero. The diagonal length is decreasing, because its derivative is negative.

10. Step 1:

x, y, z = edge lengths of the box

V = volume of the box

S = surface area of the box

s = diagonal length of the box

Step 2:

At the instant in question,

$\frac{dx}{dt} = 1$ m/sec, $\frac{dy}{dt} = -2$ m/sec, $\frac{dz}{dt} = 1$ m/sec, $x = 4$m, $y = 3$m, and $z = 2$m.

Step 3:

We want to find $\frac{dV}{dt}, \frac{dS}{dt}$, and $\frac{ds}{dt}$.

Steps 4, 5, and 6:

(a) $V = xyz$

$$\frac{dV}{dt} = xy\frac{dz}{dt} + xz\frac{dy}{dt} + yz\frac{dx}{dt}$$

$$\frac{dV}{dt} = (4)(3)(1) + (4)(2)(-2) + (3)(2)(1) = 2 \text{ m}^3/\text{sec}$$

The rate of change of the volume is $2 \text{ m}^3/\text{sec}$.

(b) $S = 2(xy + xz + yz)$ π

$$\frac{dS}{dt} 2\left(x\frac{dy}{dt} + y\frac{dx}{dt} + x\frac{dz}{dt} + z\frac{dx}{dt} + y\frac{dz}{dt} + z\frac{dy}{dt}\right)$$

$$\frac{dS}{dt} = 2[(4)(-2) + (3)(1) + (4)(1) + (2)(1) + (3)(1) + (2)(-2)] = 0 \text{ m}^2/\text{sec}$$

The rate of change of the surface area is $0 \text{ m}^2/\text{sec}$.

(c) $s = \sqrt{x^2 + y^2 + z^2}$

$$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dy}\right) = \frac{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}}{\sqrt{x^2 + y^2 + z^2}}$$

The rate of change of the diagonal length is 0 m/sec.

11. Step 1:

r = radius of spherical balloon

S = surface area of spherical balloon

V = volume of spherical balloon

Step 2:

At the instant in question, $\frac{dV}{dt} = 100\pi \text{ ft}^3/\text{min}$ and $r = 5$ ft.

Step 3:

We want to find the values of $\frac{dr}{dt}$ and $\frac{dS}{dt}$.

Steps 4, 5, and 6:

(a) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$100\pi = 4\pi(5)^2 \frac{dr}{dt}$

$\frac{dr}{dt} = 1$ ft/min

The radius is increasing at the rate of 1 ft/min.

(b) $S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\frac{dS}{dt} = 8\pi(5)(1)$

$\frac{dS}{dt} = 40\pi \text{ ft}^2/\text{min}$

The surface area is increasing at the rate of $40\pi \text{ ft}^2/\text{min}$.

12. Step 1:
r = radius of spherical droplet
S = surface area of spherical droplet
V = volume of spherical droplet
Step 2:
No numerical information is given.
Step 3:
We want to show that $\frac{dr}{dt}$ is constant.
Step 4:
$S = 4\pi r^2, V = \frac{4}{3}\pi r^3, \frac{dV}{dt} = kS$ for some constant k
Steps 5 and 6:
Differentiating $V = \frac{4}{3}\pi r^3$, we have

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

Substituting kS for $\frac{dV}{dt}$ and S for $4\pi r^2$, we have $kS = S\frac{dr}{dt}$, or $\frac{dr}{dt} = k$.

13. Step 1:
s = (diagonal) distance from antenna to airplane
x = horizontal distance from antenna to airplane
Step 2:
At the instant in question, $s = 10$ mi and $\frac{ds}{dt} = 300$ mph.
Step 3:
We want to find $\frac{dx}{dt}$.
Step 4:
$x^2 + 49 = s^2$ or $x = \sqrt{s^2 - 49}$
Step 5:

$\frac{dx}{dt} = \frac{1}{2\sqrt{s^2-49}}\left(2s\frac{ds}{dt}\right) = \frac{s}{\sqrt{s^2-49}}\frac{ds}{dt}$

Step 6:

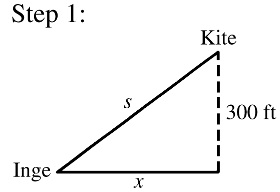
$\frac{dx}{dt} = \frac{10}{\sqrt{10^2-49}}(300)$

$= \frac{3000}{\sqrt{51}}$ mph

≈ 420.08 mph

The speed of the airplane is about 420.08 mph.

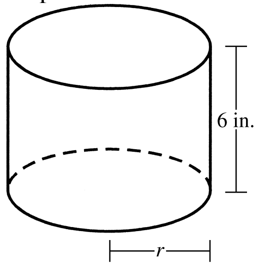
14. Step 1:



s = length of kite string
x = horizontal distance from Inge to kite
Step 2:
At the instant in question, $\frac{dx}{dt} = 25$ ft/sec and $s = 500$ ft
Step 3:
We want to find $\frac{ds}{dt}$.
Step 4:
$x^2 + 300^2 = s^2$
Step 5:
$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$ or $x\frac{dx}{dt} = s\frac{ds}{dt}$
Step 6:
At the instant in question, since $x^2 + 300^2 = s^2$, we have $x = \sqrt{s^2 - 300^2} = \sqrt{500^2 - 300^2} = 400$.

Thus $(400)(25) = (500)\frac{ds}{dt}$, so $\frac{ds}{dt}$, so $\frac{ds}{dt} = 20$ ft/sec. Inge must let the string out at the rate of 20 ft/sec.

15. Step 1:



The cylinder shown represents the shape of the hole.

r = radius of cylinder
V = volume of cylinder

Step 2:
At the instant in question,
$\frac{dr}{dt} = \frac{0.001\text{ in.}}{3\text{ min}} = \frac{1}{3000}$ in./min and (since the diameter is 3.800 in.), $r = 1.900$ in.

Step 3:
We want to find $\frac{dV}{dt}$.

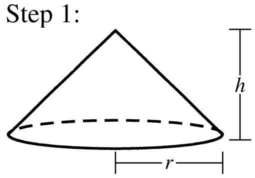
Step 4:
$V = \pi r^2(6) = 6\pi r^2$

Step 5:
$\frac{dV}{dt} = 12\pi r\frac{dr}{dt}$

Step 6:
$$\frac{dV}{dt} = 12\pi(1.900)\left(\frac{1}{3000}\right) = \frac{19\pi}{2500} = 0.0076\pi \approx 0.0239 \text{ in}^3\text{/min.}$$

The volume is increasing at the rate of approximately 0.0239 in^3/min.

16. Step 1:



r = base radius of cone
h = height of cone
V = volume of cone

Step 2:
At the instant in question, $h = 4$ m and $\frac{dV}{dt} = 10 \text{ m}^3\text{/min.}$

Step 3:
We want to find $\frac{dh}{dt}$ and $\frac{dr}{dt}$.

Step 4:
Since the height is $\frac{3}{8}$ of the base diameter, we have $h = \frac{3}{8}(2r)$ or $r = \frac{4}{3}h$.
We also have
$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{4}{3}h\right)^2 h = \frac{16\pi h^3}{27}$. We will use the equations $V = \frac{16\pi h^3}{27}$ and $r = \frac{4}{3}h$.

Step 5 and 6:

(a) $\frac{dV}{dt} = \frac{16\pi h^2}{9}\frac{dh}{dt}$

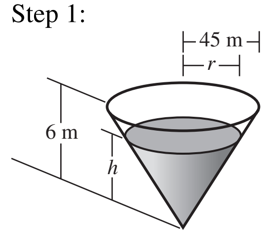
$10 = \frac{16\pi(4)^2}{9}\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{45}{128\pi}\text{ m/min} = \frac{1125}{32\pi}\text{ cm/min}$

The height is changing at the rate of $\frac{1125}{32\pi} \approx 11.19$ cm/min.

(b) Using the results from Step 4 and part (a), we have
$\frac{dr}{dt} = \frac{4}{3}\frac{dh}{dt} = \frac{4}{3}\left(\frac{1125}{32\pi}\right) = \frac{375}{8\pi}\text{ cm/min}$.
The radius is changing at the rate of $\frac{375}{8\pi} \approx 14.92$ cm/min.

17. Step 1:



r = radius of top surface of water
h = depth of water in reservoir
V = volume of water in reservoir

Step 2:
At the instant in question,
$\frac{dV}{dt} = -50 \text{ m}^3\text{/min}$ and $h = 5$ m.

Step 3:

We want to find $-\frac{dh}{dt}$ and $\frac{dr}{dt}$.

Step 4:

Note that $\frac{h}{r} = \frac{6}{45}$ by similar cones, so $r = 7.5h$.

Then $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(7.5h)^2 h = 18.75\pi h^3$

Steps 5 and 6:

(a) Since $V = 18.75\pi h^3, \frac{dV}{dt} = 56.25\pi h^2 \frac{dh}{dt}$.

Thus $-50 = 56.25\pi(5^2)\frac{dh}{dt}$, and

so $\frac{dh}{dt} = -\frac{8}{225\pi}$ m/min $= -\frac{32}{9\pi}$ cm/min.

The water level is falling by

$\frac{32}{9\pi} \approx 1.13$ cm/min.

(Since $\frac{dh}{dt} < 0$, the rate at which the water level is *falling* is positive.)

(b) Since $r = 7.5h$,

$\frac{dr}{dt} = 7.5\frac{dh}{dt} = -\frac{80}{3\pi}$ cm/min. The rate of change of the radius of the water's surface is $-\frac{80}{3\pi} \approx -8.49$ cm/min.

18. (a) Step 1:

y = depth of water in bowl

V = volume of water in bowl

Step 2:

At the instant in question,

$\frac{dV}{dt} = -6\ \text{m}^3/\text{min}$ and $y = 8$ m.

Step 3:

We want to find the value of $\frac{dy}{dt}$.

Step 4:

$V = \frac{\pi}{3}y^2(39 - y)$ or $V = 13\pi y^2 - \frac{\pi}{3}y^3$

Step 5:

$$\frac{dV}{dt} = (26\pi y - \pi y^2)\frac{dy}{dt}$$

Step 6:

$$-6 = \left[26\pi(8) - \pi(8^2)\right]\frac{dy}{dt}$$

$$-6 = 144\pi\frac{dy}{dt}$$

$$\frac{dy}{dt} = -\frac{1}{24\pi} \approx -0.01326 \text{ m/min}$$

or $-\frac{25}{6\pi} \approx -1.326$ cm/min

(b) Since $r^2 + (13 - y)^2 = 13^2$,

$r = \sqrt{169 - (13 - y)^2} = \sqrt{26y - y^2}$.

(c) Step 1:

y = depth of water

r = radius of water surface

V = volume of water in bowl

Step 2:

At the instant in question,

$\frac{dV}{dt} = -6\ \text{m}^3/\text{min}$, $y = 8$ m, and

therefore (from part (a))

$\frac{dy}{dt} = -\frac{1}{24\pi}$ m/min.

Step 3:

We want to find the value of $\frac{dr}{dt}$.

Step 4:

From part (b), $r = \sqrt{26y - y^2}$.

Step 5:

$$\frac{dr}{dt} = \frac{1}{2\sqrt{26y - y^2}}(26 - 2y)\frac{dy}{dt} = \frac{13 - y}{\sqrt{26y - y^2}}\frac{dy}{dt}$$

Step 6:

$$\frac{dr}{dt} = \frac{13 - 8}{\sqrt{26(8) - 8^2}}\left(\frac{1}{-24\pi}\right) = \frac{5}{12}\left(-\frac{1}{24\pi}\right) = -\frac{5}{288\pi} \approx -0.00553 \text{ m/min}$$

or $-\frac{125}{72\pi} \approx -0.553$ cm/min

19. Step 1:
x = distance from wall to base of ladder
y = height of top of ladder
A = area of triangle formed by the ladder, wall, and ground
θ = angle between the ladder and the ground
Step 2:
At the instant in question, $x = 12$ ft and $\frac{dx}{dt} = 5$ ft/sec.
Step 3:
We want to find $-\frac{dy}{dt}, \frac{dA}{dt},$ and $\frac{d\theta}{dt}$.
Steps 4, 5, and 6:

(a) $x^2 + y^2 = 169$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

To evaluate, note that, at the instant in question,
$y = \sqrt{169 - x^2} = \sqrt{169 - 12^2} = 5.$

Then $2(12)(5) + 2(5)\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -12$ ft/sec $\left(\text{or } -\frac{dy}{dt} = 12 \text{ ft/sec}\right)$

The top of the ladder is sliding down the wall at the rate of 12 ft/sec. (Note that the *downward* rate of motion is positive.)

(b) $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$

Using the results from step 2 and from part (a), we have

$$\frac{dA}{dt} = \frac{1}{2}[(12)(-12) + (5)(5)]$$
$$= -\frac{119}{2} \text{ ft}^2/\text{sec}$$

The area of the triangle is changing at the rate of $-59.5 \text{ ft}^2/\text{sec}$.

(c) $\tan\theta = \frac{y}{x}$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

Since $\tan\theta = \frac{5}{12}$, we have $\left(\text{for } 0 \le \theta < \frac{\pi}{2}\right)\cos\theta = \frac{12}{13}$ and so

$$\sec^2\theta = \frac{1}{\left(\frac{12}{13}\right)^2} = \frac{169}{144}.$$

Combining this result with the results from step 2 and from part (a), we have

$$\frac{169}{144}\frac{d\theta}{dt} = \frac{(12)(-12) - (5)(5)}{12^2},$$ so

$\frac{d\theta}{dt} = -1$ radian/sec. The angle is changing at the rate of -1 radian/sec.

20. Step 1:
h = height (or depth) of the water in the trough
V = volume of water in the trough
Step 2:
At the instant in question, $\frac{dV}{dt} = 2.5 \text{ ft}^3/\text{min}$ and $h = 2$ ft.
Step 3:
We want to find $\frac{dh}{dt}$.
Step 4:
The width of the top surface of the water is $\frac{4}{3}h$, so we have $V = \frac{1}{2}(h)\left(\frac{4}{3}h\right)(15)$, or $V = 10h^2$
Step 5:

$$\frac{dV}{dt} = 20h\frac{dh}{dt}$$

Step 6:

$$2.5 = 20(2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.0625 = \frac{1}{16} \text{ ft/min}$$

The water level is increasing at the rate of $\frac{1}{16}$ ft/min.

21. Step 1:
l = length of rope
x = horizontal distance from boat to dock
θ = angle between the rope and a vertical line
Step 2:
At the instant in question, $\frac{dl}{dt} = -2$ ft/sec and $l = 10$ ft.

Step 3:

We want to find the values of $-\frac{dx}{dt}$ and $\frac{d\theta}{dt}$.

Steps 4, 5, and 6:

(a) $x = \sqrt{l^2 - 36}$

$$\frac{dx}{dt} = \frac{l}{\sqrt{l^2 - 36}}\frac{dl}{dt}$$

$$\frac{dx}{dt} = \frac{10}{\sqrt{10^2 - 36}}(-2) = -2.5 \text{ ft/sec}$$

The boat is approaching the dock at the rate of 2.5 ft/sec.

(b) $\theta = \cos^{-1}\frac{6}{l}$

$$\frac{d\theta}{dt} = -\frac{1}{\sqrt{1-\left(\frac{6}{l}\right)^2}}\left(-\frac{6}{l^2}\right)\frac{dl}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{\sqrt{1-0.6^2}}\left(-\frac{6}{10^2}\right)(-2)$$

$$= -\frac{3}{20} \text{ radian/sec}$$

The rate of change of angle θ is $-\frac{3}{20}$ radian/sec.

22. Step 1:

x = distance from origin to bicycle

y = height of balloon (distance from origin to balloon)

s = distance from balloon to bicycle

Step 2:

We know that $\frac{dy}{dt}$ is a constant 1 ft/sec and $\frac{dx}{dt}$ is a constant 17 ft/sec. Three seconds before the instant in question, the values of x and y are $x = 0$ ft and $y = 65$ ft. Therefore, at the instant in question $x = 51$ ft and $y = 68$ ft.

Step 3:

We want to find the value of $\frac{ds}{dt}$ at the instant in question.

Step 4:

$s = \sqrt{x^2 + y^2}$

Step 5:

$$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2 + y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

$$= \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2 + y^2}}$$

Step 6:

$$\frac{ds}{dt} = \frac{(51)(17) + (68)(1)}{\sqrt{51^2 + 68^2}} = 11 \text{ ft/sec}$$

The distance between the balloon and the bicycle is increasing at the rate of 11 ft/sec.

23. $$\frac{dy}{dt} = \frac{dy}{dt}\frac{dx}{dt} = -10(1+x^2)^{-2}(2x)\frac{dx}{dt}$$

$$= -\frac{20x}{(1+x^2)^2}\frac{dx}{dt}$$

Since $\frac{dx}{dt} = 3$cm/sec, we have

$$\frac{dy}{dt} = -\frac{60x}{(1+x^2)^2} \text{ cm/sec.}$$

(a) $\frac{dy}{dt} = -\frac{60(-2)}{[1+(-2)^2]^2} = \frac{120}{5^2} = \frac{24}{5}$ cm/sec

(b) $\frac{dy}{dt} = -\frac{60(0)}{(1+0^2)^2} = 0$ cm/sec

(c) $\frac{dy}{dt} = -\frac{60(20)}{(1+20^2)^2} \approx -0.00746$ cm/sec

24. $$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = (3x^2 - 4)\frac{dx}{dt}$$

Since $\frac{dx}{dt} = -2$ cm/sec, we have

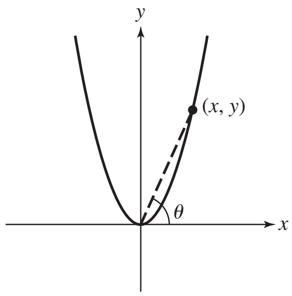
$$\frac{dy}{dt} = 8 - 6x^2 \text{ cm/sec.}$$

(a) $\frac{dy}{dt} = 8 - 6(-3)^2 = -46$ cm/sec

(b) $\frac{dy}{dt} = 8 - 6(1)^2 = 2$ cm/sec

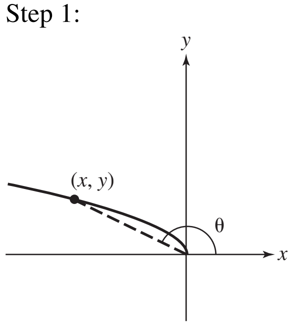
(c) $\frac{dy}{dt} = 8 - 6(4)^2 = -88$ cm/sec

25. Step 1:



$x = x$-coordinate of particle's location
$y = y$-coordinate of particle's location
θ = angle of inclination of line joining the particle to the origin.
Step 2:
At the instant in question,
$\frac{dx}{dt} = 10$ m/sec and $x = 3$ m.
Step 3:
We want to find $\frac{d\theta}{dt}$.
Step 4:
Since $y = x^2$, we have $\tan\,\theta = \frac{y}{x} = \frac{x^2}{x} = x$
and so, for $x > 0$, $\theta = \tan^{-1} x$.
Step 5:
$$\frac{d\theta}{dt} = \frac{1}{1+x^2}\frac{dx}{dt}$$
Step 6:
$$\frac{d\theta}{dt} = \frac{1}{1+3^2}(10) = 1 \text{ radian/sec}$$
The angle of inclination is increasing at the rate of 1 radian/sec.

26. Step 1:



$x = x$-coordinate of particle's location
$y = y$-coordinate of particle's location
θ = angle of inclination of line joining the particle to the origin
Step 2:
At the instant in question, $\frac{dx}{dt} = -8$ m/ sec and $x = -4$ m.
Step 3:
We want to find $\frac{d\theta}{dt}$,
Step 4:
Since $y = \sqrt{-x}$, we have
$\tan\theta = \frac{y}{x} = \frac{\sqrt{-x}}{x} - (-x)^{-1/2}$, and so, for $x < 0$,
$\theta = \pi + \tan^{-1}[-(-x)^{1/2}] = \pi - \tan^{-1}(-x)^{-1/2}$.
Step 5:
$$\frac{d\theta}{dt} = -\frac{1}{1+[(-x)^{-1/2}]^2}\left((-\frac{1}{2}(-x)^{-3/2}(-1)\right)\frac{dx}{dt}$$
$$= -\frac{1}{1-\left(\frac{1}{x}\right)}\frac{1}{2(-x)^{3/2}}\frac{dx}{dt}$$
$$= \frac{1}{2\sqrt{-x}(x-1)}\frac{dx}{dt}$$
Step 6:
$$\frac{d\theta}{dt} = \frac{1}{2\sqrt{4}(-4-1)}(-8) = \frac{2}{5} \text{ radian/sec}$$
The angle of inclination is increasing at the rate of $\frac{2}{5}$ radian/sec.

27. Step 1:
r = radius of balls plus ice
S = surface area of ball plus ice
V = volume of ball plus ice
Step 2:
At the instant in question,
$$\frac{dV}{dt} = -8 \text{ mL/min}$$
$$= -8 \text{ cm}^3\text{/min and } r$$
$$= \frac{1}{2}(20)$$
$$= 10 \text{ cm}.$$
Step 3:
We want to find $-\frac{dS}{dt}$.
Step 4:
We have $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. These equations can be combined by noting that
$r = \left(\frac{3V}{4\pi}\right)^{1/3}$, so $S = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$

Step 5:

$$\frac{dS}{dt} = 4\pi\left(\frac{2}{3}\right)\left(\frac{3V}{4\pi}\right)^{-1/3}\left(\frac{3}{4\pi}\right)\frac{dV}{dt}$$
$$= 2\left(\frac{3V}{4\pi}\right)^{-1/3}\frac{dV}{dt}$$

Step 6:

Note that $V = \frac{4}{3}\pi(10)^3 = \frac{4000\pi}{3}$.

$$\frac{dS}{dt} = 2\left(\frac{3}{4\pi}\bullet\frac{4000\pi}{3}\right)^{-1/3}(-8)$$
$$= \frac{-16}{\sqrt[3]{1000}}$$
$$= -1.6\,\text{cm}^2/\text{min}$$

Since $\frac{dS}{dt} < 0$, the rate of *decrease* is positive.

The surface area is decreasing at the rate of $1.6\,\text{cm}^2/\text{min}$.

28. Step 1:
x = x-coordinate of particle
y = y-coordinate of particle
D = distance from origin to particle
Step 2:
At the instant in question, $x = 5$ m, $y = 12$ m, $\frac{dx}{dt} = -1$ m/sec, and $\frac{dy}{dt} = -5$ m/sec.
Step 3:
We want to find $\frac{dD}{dt}$.
Step 4:

$$D = \sqrt{x^2 + y^2}$$

Step 5:

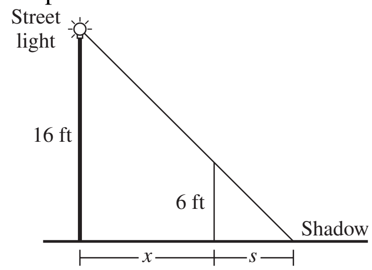
$$\frac{dD}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$
$$= \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}}$$

Step 6:

$$\frac{dD}{dt} = \frac{(5)(-1)+(12)(-5)}{\sqrt{5^2+12^2}} = -5\,\text{m/sec}$$

The particle's distance from the origin is changing at the rate of –5 m/sec.

29. Step 1:



x = distance from streetlight base to man
s = length of shadow
Step 2:
At the instant in question, $\frac{dx}{dt} = -5$ ft/sec and $x = 10$ ft.
Step 3:
We want to find $\frac{ds}{dt}$.
Step 4:
By similar triangles, $\frac{s}{6} = \frac{s+x}{16}$. This is equivalent to $16s = 6s + 6x$, or $s = \frac{3}{5}x$.
Step 5:

$$\frac{ds}{dt} = \frac{3}{5}\frac{dx}{dt}$$

Step 6:

$$\frac{ds}{dt} = \frac{3}{5}(-5) = -3\,\text{ft/sec}$$

The shadow length is changing at the rate of –3 ft/sec.

30. Step 1:
s = distance ball has fallen
x = distance from bottom of pole to shadow
Step 2:
At the instant in question, $s = 16\left(\frac{1}{2}\right)^2 = 4$ ft and $\frac{ds}{dt} = 32\left(\frac{1}{2}\right) = 16$ ft/sec.
Step 3:
We want to find $\frac{dx}{dt}$.
Step 4:
By similar triangles, $\frac{x-30}{50-s} = \frac{x}{50}$. This is equivalent to $50x - 1500 = 50x - sx$, or $sx = 1500$. We will use $x = 1500s^{-1}$.

Step 5 :
$\frac{dx}{dt} = -500s^{-2}\frac{ds}{dt}$
Step 6:
$\frac{dx}{dt} = -1500(4)^{-2}(16) = -1500\text{ ft/sec}$
The shadow is moving at a velocity of –1500 ft/sec.

31. Step 1:
x = position of car ($x = 0$ when car is right in front of you)
θ = camera angle. (We assume θ is negative until the car passess in front of you, and then positive.)
Step 2:
At the first instant in question, $x = 0$ ft and
$\frac{dx}{dt} = 264$ ft/sec.
A half second later, $x = \frac{1}{2}(264) = 132$ ft and
$\frac{dx}{dt} = 264$ ft/sec.
Step 3:
We want to find $\frac{d\theta}{dt}$ at each of the two instants.
Step 4:
$\theta = \tan^{-1}\left(\frac{x}{132}\right)$
Step 5:
$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{x}{132}\right)^2}\cdot\frac{1}{132}\frac{dx}{dt}$
Step 6:
When $x = 0$:
$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{0}{132}\right)^2}\left(\frac{1}{132}\right)(264) = 2$ radians/sec
When $x = 132$:
$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{132}{132}\right)^2}\left(\frac{1}{132}\right)(264) = 1$ radians/sec

32. Step 1:
p = x-coordinate of plane's position
x = x-coordinate of car's position
s = distance from plane to car (line-of-sight)
Step 2:
At the instant in question, $p = 0$,
$\frac{dp}{dt} = 120\text{ mph}$, $s = 5$ mi, and $\frac{ds}{dt} = -160\text{ mph}$.
Step 3:
We want to find $-\frac{dx}{dt}$.
Step 4:
$(x-p)^2 + 3^2 = s^2$
Step 5:
$2(x-p)\left(\frac{dx}{dt} - \frac{dp}{dt}\right) = 2s\frac{ds}{dt}$
Step 6:
Note that, at the instant in question,
$x = \sqrt{5^2 - 3^2} = 4\text{ mi}$.

$$2(4-0)\left(\frac{dx}{dt} - 120\right) = 2(5)(-160)$$
$$8\left(\frac{dx}{dt} - 120\right) = -1600$$
$$\frac{dx}{dt} - 120 = -200$$
$$\frac{dx}{dt} = -80\text{ mph}$$

The car's speed is 80 mph.

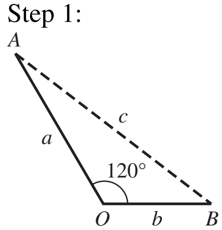
33. Step 1:
s = shadow length
θ = sun's angle of elevation
Step 2:
At the instant in question, $s = 60$ ft and
$\frac{d\theta}{dt} = 0.27°/\text{min} = 0.0015\pi$ radian/min.
Step 3:
We want to find $-\frac{ds}{dt}$.
Step 4:
$\tan\theta = \frac{80}{s}$ or $s = 80\cot\theta$
Step 5:
$\frac{ds}{dt} = -80\csc^2\theta\frac{d\theta}{dt}$
Step 6:
Note that, at the moment in question, since
$\tan\theta = \frac{80}{60}$ and $0 < \theta < \frac{\pi}{2}$, we have
$\sin\theta = \frac{4}{5}$ and so $\csc\theta = \frac{5}{4}$.

$$\frac{ds}{dt} = -80\left(\frac{5}{4}\right)^2(0.0015\pi)$$
$$= -0.1875\pi\ \frac{\text{ft}}{\text{min}}\cdot\frac{12\text{ in}}{1\text{ ft}}$$
$$= -2.25\pi\text{ in./min}$$
$$\approx -7.1\text{ in./min}$$

Since $\frac{ds}{dt} < 0,$ the rate at which the shadow length is *decreasing* is positive. The shadow length is decreasing at the rate of approximately 7.1 in./min.

34. Step 1:
$a =$ distance from origin to A
$b =$ distance from origin to B
$\theta =$ angle shown in problem statement
Step 2:
At the instant in question,
$\frac{da}{dt} = -2\text{m/ sec}, \frac{db}{dt} = 1 \text{ m/sec},$
$a = 10\text{m, and } b = 20\text{m.}$
Step 3:
We want to find $\frac{d\theta}{dt}.$
Step 4:
$\tan\theta = \frac{a}{b} \text{ or } \theta = \tan^{-1}\left(\frac{a}{b}\right)$
Step 5:
$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{a}{b}\right)^2}\frac{b\frac{da}{dt} - a\frac{db}{dt}}{b^2} = \frac{b\frac{da}{dt} - a\frac{db}{dt}}{a^2+b^2}$$
Step 6:
$$\frac{d\theta}{dt} = \frac{(20)(-2)-(10)(1)}{10^2+20^2}$$
$$= -0.1 \text{radian / sec}$$
$$\approx -5.73 \text{ degrees/sec}$$
To the nearest degree, the angle is changing at the rate of –6 degrees per second.

35. Step 1:



$a =$ distance from O to A
$b =$ distance from O to B
$c =$ distance from A to B
Step 2:
At the instant in question, $a = 5$ nautical miles, $b = 3$ nautical miles,
$\frac{da}{dt} = 14 \text{knots, and } \frac{db}{dt} = 21 \text{ knots.}$
Step 3:
We want to find $\frac{dc}{dt},$
Step 4:
Law of Cosines :
$$c^2 = a^2 + b^2 - 2ab\cos 120°$$
$$c^2 = a^2 + b^2 + ab$$
Step 5:
$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} + a\frac{db}{dt} + b\frac{da}{dt}$$
Step 6:
Note that, at the instant in question,
$$c = \sqrt{a^2+b^2+ab}$$
$$= \sqrt{(5)^2+(3)^2+(5)(3)}$$
$$= \sqrt{49}$$
$$= 7$$
$$2(7)\frac{dc}{dt} = 2(5)(14) + 2(3)(21) + (5)(21) + (3)(14)$$
$$14\frac{dc}{dt} = 413$$
$$\frac{dc}{dt} = 29.5 \text{ knots}$$
The ships are moving apart at a rate of 29.5 knots.

36. True. Since $\frac{dC}{dt} = 2\pi\frac{dr}{dt},$ a constant $\frac{dr}{dt}$ results in a constant $\frac{dC}{dt}.$

37. False. Since $\frac{dA}{dt} = 2\pi r\frac{dr}{dt},$ the value of $\frac{dA}{dt}$ depends on r.

38. A; $V = s^3$
$$\frac{dV}{dt} = 3s^2\frac{ds}{dt}$$
$$24 = 3s^2(2)$$
$$s = 2\text{in}$$

39. E; $sA = 6s^2$
$$\frac{dsA}{dt} = 12s\frac{ds}{dt}$$
$$12 = 12s\frac{ds}{dt}$$
$$\frac{ds}{dt} = \frac{1}{s}$$
$$V = s^3$$
$$\frac{dV}{dt} = 3s^2\frac{ds}{dt} = 3s^2\frac{1}{s}$$
$$24 = 3s$$
$$s = 8 \text{ in}$$

40. C; $x^2 + y^2 = 1$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} = -y\frac{dy}{dt}$$

$$\frac{x}{-y}\frac{dx}{dt} = \frac{dy}{dt}$$

$$\left(\frac{0.6}{-0.8}\right)3 = \frac{dy}{dt}$$

$$\frac{dy}{dt} = -2.25.$$

41. B; $v = \pi r^2 l$

$$\frac{dv}{dt} = 2\pi rl\frac{dr}{dt} + \pi r^2\frac{dl}{dt}$$

$$0 = 2\pi(1)(100)\frac{dr}{dt} + \pi(1)^2 2$$

$$\frac{dr}{dt} = \frac{-2\pi}{200\pi}$$

$$\frac{dr}{dt} = -.01 \text{ cm/s}$$

42. **(a)** Note that the level of the coffee in the cone is not needed until part (b).

Step 1:

V_1 = volume of coffee in pot

y = depth of coffee in pot

Step 2:

$$\frac{dV_1}{dt} = 10 \text{ in}^3/\text{min}$$

Step 3:

We want to find the value of $\frac{dy}{dt}$.

Step 4:

$V_1 = 9\pi y$

Step 5:

$$\frac{dV_1}{dt} = 9\pi\frac{dy}{dt}$$

Step 6:

$$10 = 9\pi\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{10}{9\pi} \approx 0.354 \text{ in./min}$$

The level in the pot is increasing at the rate of approximately 0.354 in./min.

(b) Step 1:

V_2 = volume of coffee in filter

r = radius of surface of coffee in filter

h = depth of coffee in filter

Step 2:

At the instant in question,

$\frac{dV_2}{dt} = -10 \text{ in}^3/\text{min}$ and $h = 5$ in.

Step 3:

We want to find $-\frac{dh}{dt}$.

Step 4:

Note that $\frac{r}{h} = \frac{3}{6}$, so $r = \frac{h}{2}$.

Then $V_2 = \frac{1}{3}\pi r^2 h = \frac{\pi h^3}{12}$.

Step 5:

$$\frac{dV_2}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$$

Step 6:

$$-10 = \frac{\pi(5)^2}{4}\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{8}{5\pi} \text{in./min}$$

Note that $\frac{dh}{dt} < 0$, so the rate at which the level is *falling* is positive. The level in the cone is falling at the rate of

$\frac{8}{5\pi} \approx 0.509$ in./min.

43. **(a)**
$$\frac{dc}{dt} = \frac{d}{dt}(x^3 - 6x^2 + 15x)$$
$$= (3x^2 - 12x + 15)\frac{dx}{dt}$$
$$= [3(2)^2 - 12(2) + 15](0.1)$$
$$= 0.3$$

$$\frac{dr}{dt} = \frac{d}{dt}(9x) = 9\frac{dx}{dt} = 9(0.1) = 0.9$$

$$\frac{dp}{dt} = \frac{dr}{dt} - \frac{dc}{dt} = 0.9 - 0.3 = 0.6$$

(b)
$$\frac{dc}{dt} = \frac{d}{dt}\left(x^3 - 6x^2 + \frac{45}{x}\right)$$
$$= \left(3x^2 - 12x - \frac{45}{x^2}\right)\frac{dx}{dt}$$
$$= \left[3(1.5)^2 - 12(1.5) - \frac{45}{1.5^2}\right](0.05)$$
$$= -1.5625$$

$$\frac{dr}{dt} = \frac{d}{dt}(70x) = 70\frac{dx}{dt} = 70(0.05) = 3.5$$

$$\frac{dp}{dt} = \frac{dr}{dt} - \frac{dc}{dt} = 3.5 - (-1.5625) = 5.0625$$

44. Step 1:
Q = rate of CO_2 exhalation (mL/min)
D = difference between CO_2 concentration in blood pumped to the lungs and CO_2 concentration in blood returning from the lungs (mL/L)
y = cardiac output
Step 2:
At the instant in question, $Q = 233$ mL/min, $D = 41$ mL/L, $\frac{dD}{dt} = -2$ (mL/L)/min, and $\frac{dQ}{dt} = 0$ mL/min^2.
Step 3:
We want to find the value of $\frac{dy}{dt}$.
Step 4:
$y = \frac{Q}{D}$
Step 5:
$\frac{dy}{dt} = \frac{D\frac{dQ}{dt} - Q\frac{dD}{dt}}{D^2}$
Step 6:
$$\frac{dy}{dt} = \frac{(41)(0)-(233)(-2)}{(41)^2} = \frac{466}{1681} \approx 0.277 \text{ L/min}^2$$
The cardiac output is increasing at the rate of approximately 0.277 L/min^2.

45. **(a)** The point being plotted would correspond to a point on the edge of the wheel as the wheel turns.

(b) One possible answer is $\theta = 16\pi t$, where t is in seconds. (An arbitrary constant may be added to this expression, and we have assumed counterclockwise motion.)

(c) In general, assuming counterclockwise motion:
$$\frac{dx}{dt} = -2\sin\theta\frac{d\theta}{dt} = -2(\sin\theta)(16\pi) = -32\pi\sin\theta$$
$$\frac{dy}{dt} = 2\cos\theta\frac{d\theta}{dt} = 2(\cos\theta)(16\pi) = 32\pi\cos\theta$$

At $\theta = \frac{\pi}{4}$:
$$\frac{dx}{dt} = -32\pi\ \sin\frac{\pi}{4} = -16\pi(\sqrt{2}) \approx -71.086 \text{ ft/sec}$$
$$\frac{dy}{dt} = 32\pi\ \cos\frac{\pi}{4} = 16\pi(\sqrt{2}) \approx 71.086 \text{ ft/sec}$$
At $\theta = \frac{\pi}{2}$:
$$\frac{dx}{dt} = -32\pi\sin\frac{\pi}{2} = -32\pi \approx -100.531 \text{ ft/sec}$$
$$\frac{dy}{dt} = 32\pi\cos\frac{\pi}{2} = 0 \text{ ft/sec}$$
At $\theta = \pi$:
$$\frac{dx}{dt} = -32\pi\ \sin\pi = 0 \text{ ft/sec}$$
$$\frac{dy}{dt} = 32\pi\ \cos\pi = -32\pi \approx -100.531 \text{ ft/sec}$$

46. **(a)** One possible answer: $y = 30\cos\theta$, $y = 40 + 30\sin\theta$

(b) Since the ferris wheel makes one revolution every 10 sec, we may let $\theta = 0.2\pi t$ and we may write $x = 30\cos 0.2\pi t$, $y = 40 + 30\sin 0.2\pi t$.
(This assumes that the ferris wheel revolves counterclockwise.)
In general:
$$\frac{dx}{dt} = -30(\sin 0.2\pi t)(0.2\pi) = -6\pi\sin 0.2\pi t$$
$$\frac{dy}{dt} = 30(\cos 0.2\pi t)(0.2\pi) = 6\pi\cos 0.2\pi t$$

At $t = 5$:
$$\frac{dx}{dt} = -6\pi\sin\pi = 0 \text{ ft/sec}$$
$$\frac{dy}{dt} = 6\pi\cos\pi = 6\pi(-1) \approx -18.850 \text{ ft/sec}$$

At $t = 8$:
$$\frac{dx}{dt} = -6\pi\sin 1.6\pi \approx 17.927 \text{ ft/sec}$$
$$\frac{dy}{dt} = 6\pi\cos 1.6\pi \approx 5.825 \text{ ft/sec}$$

47. (a) $\frac{dy}{dt} = \frac{d}{dt}(uv)$

$= u\frac{dv}{dt} + v\frac{du}{dt}$

$= u(0.05v) + v(0.04u)$

$= 0.09uv$

$= 0.09y$

Since $\frac{dy}{dt} = 0.09y$, the rate of growth of total production is 9% per year.

(b) $\frac{dy}{dt} = \frac{d}{dt}(uv)$

$= u\frac{dv}{dt} + v\frac{du}{dt}$

$= u(0.03v) + v(-0.02u)$

$= 0.01uv$

$= 0.01y$

The total production is increasing at the rate of 1% per year.

Quick Quiz Sections 5.4–5.6

1. B; $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

$f(x) = x^3 + 2x - 1$

$f'(x) = 3x^2 + 2$

$x_2 = 1 - \frac{(1)^3 + 2(1) - 1}{3(1)^2 + 2} = \frac{3}{5}$

$x_3 = \frac{3}{5} - \frac{\left(\frac{3}{5}\right)^3 + 2\left(\frac{3}{5}\right) - 1}{3\left(\frac{3}{5}\right)^2 + 2} = 0.465$

2. B; $z^2 = x^2 + y^2$

$z = \sqrt{4^2 + 3^2} = 5$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$5 = 4\left(3\frac{dy}{dt}\right) + 3\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{1}{3}$

$\frac{dx}{dt} = 3\frac{dy}{dt} = 3\left(\frac{1}{3}\right) = 1$

3. A; $x(t) = 70$

$y(t) = 60t$

$z(t) = ((60t)^2 + 70^2)^{1/2}$

$\frac{dz}{dt} = \frac{1}{2}(3600t^2 + 4900)^{-1/2}(7200t)$

$\frac{dz}{dt} = \frac{7200(4)}{2(3600(4)^2 + 4900)^{1/2}}$

$\frac{dz}{dt} = 57.6$

4. (a) $f(x) = \sqrt{x}$

$x = 25$

$f'(25) = \frac{1}{2}(25)^{-1/2} = \frac{1}{10}$

$L(26) = 5 + \frac{1}{10}(26 - 25) = 5.1$

(b) $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$, $f(x) = x^2 - 26 = 0$

$x_2 = 5 - \frac{(5)^2 - 26}{2(5)} = 5.1$

(c) $f(x) = \sqrt[3]{x}$

$x = 3$

$f'(27) = \frac{1}{3}(27)^{-2/3} = \frac{1}{27}$

$L(26) = 3 + \frac{1}{27}(26 - 27)$

$L(26) = 2.963$

Chapter 5 Review Exercises (pp. 262–266)

1. $y = x\sqrt{2 - x}$

$y' = x\left(\frac{1}{2\sqrt{2 - x}}\right)(-1) + (\sqrt{2 - x})(1)$

$= \frac{-x + 2(2 - x)}{2\sqrt{2 - x}}$

$= \frac{4 - 3x}{2\sqrt{2 - x}}$

The first derivative has a zero at $\frac{4}{3}$.

Critical point value: $x = \frac{4}{3}$ $\quad y = \frac{4\sqrt{6}}{9}$

Endpoint values: $x = -2$ $\quad y = -4$

$x = 2$ $\quad y = 0$

The global maximum value is $\frac{4\sqrt{6}}{9}$ at $x = \frac{4}{3}$, and the global minimum value is –4 at $x = -2$.

2. Since y is a cubic function with a positive leading coefficient, we have $\lim_{x\to-\infty} y = -\infty$ and $\lim_{x\to\infty} y = \infty$. There are no global extrema.

3.
$$y' = (x^2)(e^{1/x^2})(-2x^{-3}) + (e^{1/x^2})(2x)$$
$$= 2e^{1/x^2}\left(-\frac{1}{x} + x\right)$$
$$= \frac{2e^{1/x^2}(x-1)(x+1)}{x}$$

Intervals	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
Sign of y'	−	+	−	+
Behavior of y	Decreasing	Increasing	Decreasing	Increasing

$$y'' = \frac{d}{dx}[2e^{1/x^2}(-x^{-1} + x)]$$
$$= (2e^{1/x^2})(x^{-2} + 1) + (-x^{-1} + x)(2e^{1/x^2})(-2x^{-3})$$
$$= (2e^{1/x^2})(x^{-2} + 1 + 2x^{-4} - 2x^{-2})$$
$$= \frac{2e^{1/x^2}(x^4 - x^2 + 2)}{x^4}$$
$$= \frac{2e^{1/x^2}[(x^2 - 0.5)^2 + 1.75]}{x^4}$$

The second derivative is always positive (where defined), so the function is concave up for all $x \neq 0$.

(a) $[-1, 0)$ and $[1, \infty)$

(b) $(-\infty, -1]$ and $(0, 1]$

(c) $(-\infty, 0)$ and $(0, \infty)$

(d) None

(e) Local (and absolute) minima at $(1, e)$ and $(-1, e)$

(f) None

4. Note that the domain of the function is $[-2, 2]$.
$$y' = x\left(\frac{1}{2\sqrt{4-x^2}}\right)(-2x) + \left(\sqrt{4-x^2}\right)(1)$$
$$= \frac{-x^2 + (4 - x^2)}{\sqrt{4-x^2}}$$
$$= \frac{4 - 2x^2}{\sqrt{4-x^2}}$$

Intervals	$-2 < x < -\sqrt{2}$	$-\sqrt{2} < x < \sqrt{2}$	$\sqrt{2} < x < 2$
Sign of y'	−	+	−
Behavior of y	Decreasing	Increasing	Decreasing

$$y'' = \frac{(\sqrt{4-x^2})(-4x) - (4-2x^2)\left(\frac{1}{2\sqrt{4-x^2}}\right)(-2x)}{4-x^2}$$
$$= \frac{2x(x^2-6)}{(4-x^2)^{3/2}}$$

Note that the values $x = \pm\sqrt{6}$ are not zeros of y'' because they fall outside of the domain.

Intervals	$-2 < x < 0$	$0 < x < 2$
Sign of y''	+	−
Behavior of y	Concave up	Concave down

(a) $\left[-\sqrt{2}, \sqrt{2}\right]$

(b) $\left[-2, -\sqrt{2}\right]$ and $\left[\sqrt{2}, 2\right]$

(c) (–2, 0)

(d) (0, 2)

(e) Local maxima: (–2, 0), $(\sqrt{2}, 2)$
Local minima: (2, 0), $(-\sqrt{2}, -2)$
Note that the extrema at $x = \pm\sqrt{2}$ are also absolute extrema.

(f) (0, 0)

5. $y' = 1 - 2x - 4x^3$
Using grapher techniques, the zero of y' is $x \approx 0.385$.

Intervals	$x < 0.385$	$0.385 < x$
Sign of y'	+	−
Behavior of y	Increasing	Decreasing

$y'' = -2 - 12x^2 = -2(1 + 6x^2)$
The second derivative is always negative so the function is concave down for all x.

(a) Approximately $(-\infty, 0.385]$

(b) Approximately $[0.385, \infty)$

(c) None

(d) $(-\infty, \infty)$

(e) Local (and absolute) maximum at $\approx (0.385, 1.215)$

(f) None

6. $y' = e^{x-1} - 1$

Intervals	$x < 1$	$1 < x$
Sign of y'	$-$	$+$
Behavior of y	Decreasing	Increasing

$y'' = e^{x-1}$

The second derivative is always positive, so the function is concave up for all x.

(a) $[1, \infty)$

(b) $(-\infty, 1]$

(c) $(-\infty, \infty)$

(d) None

(e) Local (and absolute) minimum at (1, 0)

(f) None

7. Note that the domain is $(-1, 1)$.

$y = (1-x^2)^{-1/4}$

$$y' = -\frac{1}{4}(1-x^2)^{-5/4}(-2x) = \frac{x}{2(1-x^2)^{5/4}}$$

Intervals	$-1 < x < 0$	$0 < x < 1$
Sign of y'	$-$	$+$
Behavior of y	Decreasing	Increasing

$$y'' = \frac{2(1-x^2)^{5/4}(1) - (x)(2)\left(\frac{5}{4}\right)(1-x^2)^{1/4}(-2x)}{4(1-x^2)^{5/2}}$$

$$= \frac{(1-x^2)^{1/4}[2 - 2x^2 + 5x^2]}{4(1-x^2)^{5/2}}$$

$$= \frac{3x^2 + 2}{4(1-x^2)^{9/4}}$$

The second derivative is always positive, so the function is concave up on its domain $(-1, 1)$.

(a) $[0, 1)$

(b) $(-1, 0]$

(c) $(-1, 1)$

(d) None

(e) Local minimum at (0, 1)

(f) None

8. $y' = \frac{(x^3-1)(1)-(x)(3x^2)}{(x^3-1)^2} = \frac{-(2x^3+1)}{(x^3-1)^2}$

Intervals	$x < -2^{-1/3}$	$-2^{-1/3} < x < 1$	$1 < x$
Sign of y'	+	−	−
Behavior of y	Increasing	Decreasing	Decreasing

$$y'' = -\frac{(x^3-1)^2(6x^2)-(2x^3+1)(2)(x^3-1)(3x^2)}{(x^3-1)^4}$$
$$= -\frac{(x^3-1)(6x^2)-(2x^3+1)(6x^2)}{(x^3-1)^3}$$
$$= \frac{6x^2(x^3+2)}{(x^3-1)^3}$$

Intervals	$x < -2^{1/3}$	$-2^{1/3} < x < 0$	$0 < x < 1$	$1 < x$
Sign of y''	+	−	−	+
Behavior of y	Concave up	Concave down	Concave down	Concave up

(a) $(-\infty, -2^{-1/3}] \approx (-\infty, -0.794]$

(b) $[-2^{-1/3}, 1) \approx [-0.794, 1)$ and $(1, \infty)$

(c) $(-\infty, -2^{+1/3}) \approx (-\infty, -1.260)$ and $(1, \infty)$

(d) $(-2^{+1/3}, 1) \approx (-1.260, 1)$

(e) Local minimum at
$\left(-2^{-1/3}, \frac{2}{3}\cdot 2^{-1/3}\right) \approx (-0.794, 0.529)$

(f) $\left(-2^{1/3}, \frac{1}{3}\cdot 2^{1/3}\right) \approx (-1.260, 0.420)$

9. Note that the domain is $[-1, 1]$.

$y' = -\frac{1}{\sqrt{1-x^2}}$

Since y' is negative on $(-1, 1)$ and y is continuous, y is decreasing on its domain $[-1, 1]$.

$$y'' = \frac{d}{dx}[-(1-x^2)^{-1/2}]$$
$$= \frac{1}{2}(1-x^2)^{-3/2}(-2x)$$
$$= -\frac{x}{(1-x^2)^{3/2}}$$

Intervals	$-1 < x < 0$	$0 < x < 1$
Sign of y''	+	−
Behavior of y	Concave up	Concave down

(a) None

(b) $[-1, 1]$

(c) $(-1, 0)$

(d) $(0, 1)$

(e) Local (and absolute) maximum at $(-1, \pi)$;
local (and absolute) minimum at $(1, 0)$

(f) $\left(0, \dfrac{\pi}{2}\right)$

10. Note that the denominator of y is always positive because it is equivalent to $(x+1)^2 + 2$.

$$y' = \frac{(x^2+2x+3)(1)-(x)(2x+2)}{(x^2+2x+3)^2}$$
$$= \frac{-x^2+3}{(x^2+2x+3)^2}$$

Intervals	$x < -\sqrt{3}$	$-\sqrt{3} < x < \sqrt{3}$	$\sqrt{3} < x$
Sign of y'	−	+	−
Behavior of y	Decreasing	Increasing	Decreasing

$$y'' = \frac{(x^2+2x+3)^2(-2x)-(-x^2+3)(2)(x^2+2x+3)(2x+2)}{(x^2+2x+3)^4}$$
$$= \frac{(x^2+2x+3)(-2x)-2(2x+2)(-x^2+3)}{(x^2+2x+3)^3}$$
$$= \frac{2x^3-18x-12}{(x^2+2x+3)^3}$$

Using graphing techniques, the zeros of $2x^3-18x-12$ (and hence of y'') are at $x \approx -2.584$, $x \approx -0.706$, and $x \approx 3.290$.

Intervals	$(-\infty, -2.584)$	$(-2.584, -0.706)$	$(-0.706, 3.290)$	$(3.290, \infty)$
Sign of y''	−	+	−	+
Behavior of y	Concave down	Concave up	Concave down	Concave up

(a) $\left[-\sqrt{3}, \sqrt{3}\right]$

(b) $\left(-\infty, -\sqrt{3}\right]$ and $\left[\sqrt{3}, \infty\right)$

(c) Approximately (–2.584, –0.706) and (3.290, ∞)

(d) Approximately (–∞, –2.584) and (–0.706, 3.290)

(e) Local maximum at $\left(\sqrt{3}, \frac{\sqrt{3}-1}{4}\right) \approx (1.732, 0.183)$;

local minimum at $\left(-\sqrt{3}, \frac{-\sqrt{3}-1}{4}\right) \approx (-1.732, -0.683)$

(f) ≈(–2.584, –0.573), (–0.706, –0.338), and (3.290, 0.161)

11. For $x > 0$, $y' = \frac{d}{dx} \ln x = \frac{1}{x}$

For $x < 0$: $y' = \frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$

Thus $y' = \frac{1}{x}$ for all x in the domain.

Intervals	(–2, 0)	(0, 2)
Sign of y'	–	+
Behavior of y	Decreasing	Increasing

$y'' = -x^{-2}$.

The second derivative is always negative, so the function is concave down on each open interval of its domain.

(a) (0, 2]

(b) [–2, 0)

(c) None

(d) (–2, 0) and (0, 2)

(e) Local (and absolute) maxima at (–2, ln 2) and (2, ln 2)

(f) None

12. $y' = 3\cos 3x - 4\sin 4x$

Using graphing techniques, the zeros of y' in the domain $0 \le x \le 2\pi$ are $x \approx 0.176$, $x \approx 0.994$,

$x = \frac{\pi}{2} \approx 1.57$, $x \approx 2.148$, and $x \approx 2.965$, $x \approx 3.834$, $x = \frac{3\pi}{2}$, $x \approx 5.591$

Intervals	$0 < x < 0.176$	$0.176 < x < 0.994$	$0.994 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < 2.148$	$2.148 < x < 2.965$
Sign of y'	+	–	+	–	+
Behavior of y	Increasing	Decreasing	Increasing	Decreasing	Increasing

Intervals	$2.965 < x < 3.834$	$3.834 < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 5.591$	$5.591 < x < 2\pi$
Sign of y'	$-$	$+$	$-$	$+$
Behavior of y	Decreasing	Increasing	Decreasing	Increasing

$y'' = -9 \sin 3x - 16 \cos 4x$

Using graphing techniques, the zeros of y'' in the domain $0 \le x \le 2\pi$ are $x \approx 0.542$, $x \approx 1.266$, $x \approx 1.876$, $x \approx 2.600$, $x \approx 3.425$, $x \approx 4.281$, $x \approx 5.144$ and $x \approx 6.000$.

Intervals	$0 < x < 0.542$	$0.542 < x < 1.266$	$1.266 < x < 1.876$	$1.876 < x < 2.600$	$2.600 < x < 3.425$
Sign of y''	$-$	$+$	$-$	$+$	$-$
Behavior of y	Concave down	Concave up	Concave down	Concave up	Concave down

Intervals	$3.425 < x < 4.281$	$4.281 < x < 5.144$	$5.144 < x < 6.000$	$6.00 < x < 2\pi$
Sign of y''	$+$	$-$	$+$	$-$
Behavior of y	Concave up	Concave down	Concave up	Concave down

(a) Approximately $[0, 0.176]$, $\left[0.994, \frac{\pi}{2}\right]$, $[2.148, 2.965]$, $\left[3.834, \frac{3\pi}{2}\right]$, and $[5.591, 2\pi]$

(b) Approximately $[0.176, 0.994]$, $\left[\frac{\pi}{2}, 2.148\right]$, $[2.965, 3.834]$, and $\left[\frac{3\pi}{2}, 5.591\right]$

(c) Approximately $(0.542, 1.266)$, $(1.876, 2.600)$, $(3{,}425, 4.281)$, and $(5.144, 6.000)$

(d) Approximately $(0, 0.542)$, $(1.266, 1.876)$, $(2.600, 3.425)$, $(4.281, 5.144)$, and $(6.000, 2\pi)$

(e) Local maxima at $\approx (0.176, 1.266)$, $\left(\frac{\pi}{2}, 0\right)$ and $(2.965, 1.266)$, $\left(\frac{3\pi}{2}, 2\right)$, and $(2\pi, 1)$;

local minima at $\approx (0, 1)$, $(0.994, -0.513)$, $(2.148, -0.513)$, $(3.834, -1.806)$, and $(5.591, -1.806)$

Note that the local extrema at $x \approx 3.834$, $x = \frac{3\pi}{2}$, and $x \approx 5.591$ are also absolute extrema.

(f) $\approx (0.542, 0.437)$, $(1.266, -0.267)$, $(1.876, -0.267)$, $(2.600, 0.437)$, $(3.425, -0.329)$, $(4.281, 0.120)$, $(5.144, 0.120)$, and $(6.000, -0.329)$

13. $y' = \begin{cases} -e^{-x}, & x < 0 \\ 4 - 3x^2, & x > 0 \end{cases}$

Intervals	$x < 0$	$0 < x < \frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}} < x$
Sign of y' y'	–	+	–
Behavior of y	Decreasing	Increasing	Decreasing

$y'' = \begin{cases} e^{-x}, & x < 0 \\ -6x, & x > 0 \end{cases}$

Intervals	$x < 0$	$0 < x$
Sign of y''	+	–
Behavior of y	Concave up	Concave down

(a) $\left(0, \frac{2}{\sqrt{3}}\right]$

(b) $(-\infty, 0]$ and $\left[\frac{2}{\sqrt{3}}, \infty\right)$

(c) $(-\infty, 0)$

(d) $(0, \infty)$

(e) Local maximum at $\left(\frac{2}{\sqrt{3}}, \frac{16}{3\sqrt{3}}\right) \approx (1.155, 3.079)$

(f) None. Note that there is no point of inflection at $x = 0$ because the derivative is undefined and no tangent line exists at this point.

14. $y' = -5x^4 + 7x^2 + 10x + 4$

Using graphing techniques, the zeros of y' are $x \approx -0.578$ and $x \approx -1.692$.

Intervals	$x < -0.578$	$-0.578 < x < 1.692$	$1.692 < x$
Sign of y'	–	+	–
Behavior of y	Decreasing	Increasing	Decreasing

$y'' = -20x^3 + 14x + 10$

Using graphing techniques, the zero of y'' is $x \approx 1.079$.

Intervals	$x < 1.079$	$1.079 < x$
Sign of y''	+	−
Behavior of y	Concave up	Concave down

(a) Approximately $[-0.578, 1.692]$

(b) Approximately $(-\infty, -0.578]$ and $[1.692, \infty)$

(c) Approximately $(-\infty, 1.079)$

(d) Approximately $(1.079, \infty)$

(e) Local maximum at $\approx (1.692, 20.517)$; local minimum at $\approx (-0.578, 0.972)$

(f) $\approx (1.079, 13.601)$

15. $y = 2x^{4/5} - x^{9/5}$

$$y' = \frac{8}{5}x^{-1/5} - \frac{9}{5}x^{4/5} = \frac{8-9x}{5\sqrt[5]{x}}$$

Intervals	$x < 0$	$0 < x < \frac{8}{9}$	$\frac{8}{9} < x$
Sign of y'	−	+	−
Behavior of y	Decreasing	Increasing	Decreasing

$$y'' = -\frac{8}{25}x^{-6/5} - \frac{36}{25}x^{-1/5} = \frac{-4(2+9x)}{25x^{6/5}}$$

Intervals	$x < -\frac{2}{9}$	$-\frac{2}{9} < x < 0$	$0 < x$
Sign of y''	+	−	−
Behavior of y	Concave up	Concave down	Concave down

(a) $\left[0, \frac{8}{9}\right]$

(b) $(-\infty, 0]$ and $\left[\frac{8}{9}, \infty\right)$

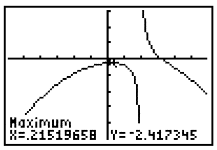
(c) $\left(-\infty, -\frac{2}{9}\right)$

(d) $\left(-\frac{2}{9}, 0\right)$ and $(0, \infty)$

(e) Local maximum at $\left(\frac{8}{9}, \frac{10}{9}\cdot\left(\frac{8}{9}\right)^{4/5}\right) \approx (0.889,\ 1.011)$; local minimum at (0, 0)

(f) $\left(-\frac{2}{9}, \frac{20}{9}\cdot\left(-\frac{2}{9}\right)^{4/5}\right) \approx \left(-\frac{2}{9},\ 0.667\right)$

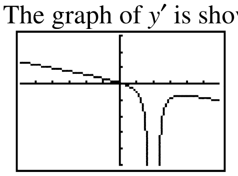
16. We use a combination of analytic and grapher techniques to solve this problem. Depending on the viewing windows chosen, graphs obtained using NDER may exhibit strange behavior near $x = 2$ because, for example, NDER(y, 2) ≈ 5,000,000 while y' is actually undefined at $x = 2$. The graph of $y = \frac{5-4x+4x^2-x^3}{x-2}$ is shown below.



[–5.875, 5.875] by [–50, 30]

$$y' = \frac{(x-2)(-4+8x-3x^2)-(5-4x+4x^2-x^3)(1)}{(x-2)^2}$$
$$= \frac{-2x^3+10x^2-16x+3}{(x-2)^2}$$

The graph of y' is shown below.



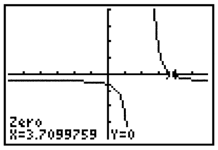
[–5.875, 5.875] by [–50, 30]

The zero of y' is $x \approx 0.215$.

Intervals	$x < 0.215$	$0.215 < x < 2$	$2 < x$
Sign of y'	+	–	–
Behavior of y	Increasing	Decreasing	Decreasing

$$y'' = \frac{(x-2)^2(-6x^2+20x-16)-(-2x^3+10x^2-16x+3)(2)(x-2)}{(x-2)^4}$$
$$= \frac{(x-2)(-6x^2+20x-16)-2(-2x^3+10x^2-16x+3)}{(x-2)^3}$$
$$= \frac{-2(x^3-6x^2+12x-13)}{(x-2)^3}$$

The graph of y'' is shown on the next page.



[–5.875, 5.875] by [–20, 20]

The zero of $x^3 - 6x^2 + 12x - 13$ (and hence of y'') is $x \approx 3.710$.

Intervals	$x < 2$	$2 < x < 3.710$	$3.710 < x$
Sign of y''	–	+	–
Behavior of y	Concave down	Concave up	Concave down

(a) Approximately $(-\infty, 0.215]$

(b) Approximately $[0.215, 2)$ and $(2, \infty)$

(c) Approximately $(2, 3.710)$

(d) $(-\infty, 2)$ and approximately $(3.710, \infty)$

(e) Local maximum at $\approx (0.215, -2.417)$

(f) $\approx (3.710, -3.420)$

17. $y' = 6(x+1)(x-2)^2$

Intervals	$x < -1$	$-1 < x < 2$	$2 < x$
Sign of y'	–	+	+
Behavior of y	Decreasing	Increasing	Increasing

$$\begin{aligned} y'' &= 6(x+1)(2)(x-2) + 6(x-2)^2(1) \\ &= 6(x-2)[(2x+2)+(x-2)] \\ &= 18x(x-2) \end{aligned}$$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y''	+	–	+
Behavior of y	Concave up	Concave down	Concave up

(a) There are no local maxima.

(b) There is a local (and absolute) minimum at $x = -1$.

(c) There are points of inflection at $x = 0$ and at $x = 2$.

18. $y' = 6(x+1)(x-2)$

Intervals	$x < -1$	$-1 < x < 2$	$2 < x$
Sign of y'	+	−	+
Behavior of y	Increasing	Decreasing	Increasing

$$y'' = \frac{d}{dx}6(x^2 - x - 2) = 6(2x-1)$$

Intervals	$x < \frac{1}{2}$	$\frac{1}{2} < x$
Sign of y''	−	+
Behavior of y	Concave down	Concave up

(a) There is a local maximum at $x = -1$.

(b) There is a local minimum at $x = 2$.

(c) There is a point of inflection at $x = \frac{1}{2}$.

19. Since $\frac{d}{dx}\left(-\frac{1}{4}x^{-4} - e^{-x}\right) = x^{-5} + e^{-x}$,

$f(x) = -\frac{1}{4}x^{-4} - e^{-x} + C.$

20. Since $\frac{d}{dx}\sec x = \sec x \tan x$, $f(x) = \sec x + C$.

21. Since $\frac{d}{dx}\left(2\ln x + \frac{1}{3}x^3 + x\right) = \frac{2}{x} + x^2 + 1$,

$f(x) = 2\ln x + \frac{1}{3}x^3 + x + C.$

22. Since $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + 2x^{1/2}\right) = \sqrt{x} + \frac{1}{\sqrt{x}}$,

$f(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} + C.$

23. $f(x) = -\cos x + \sin x + C$

$$\begin{aligned} f(\pi) &= 3 \\ 1 + 0 + C &= 3 \\ C &= 2 \\ f(x) &= -\cos x + \sin x + 2 \end{aligned}$$

24. $f(x)=\frac{3}{4}x^{4/3}+\frac{1}{3}x^3+\frac{1}{2}x^2+x+C$

$$f(1)=0$$
$$\frac{3}{4}+\frac{1}{3}+\frac{1}{2}+1+C=0$$
$$C=-\frac{31}{12}$$
$$f(x)=\frac{3}{4}x^{4/3}+\frac{1}{3}x^3+\frac{1}{2}x^2+x-\frac{31}{12}$$

25. $v(t)=s'(t)=9.8t+5$

$s(t)=4.9t^2+5t+C$
$s(0)=10$
$C=10$
$s(t)=4.9t^2+5t+10$

26. $a(t)=v'(t)=32$

$v(t)=32t+C_1$
$v(0)=20$
$C_1=20$
$v(t)=s'(t)=32t+20$
$s(t)=16t^2+20t+C_2$
$s(0)=5$
$C_2=5$
$s(t)=16t^2+20t+5$

27. $f(x)=\tan x$

$f'(x)=\sec^2 x$

$$L(x)=f\left(-\frac{\pi}{4}\right)+f'\left(-\frac{\pi}{4}\right)\left[x-\left(-\frac{\pi}{4}\right)\right]$$
$$=\tan\left(-\frac{\pi}{4}\right)+\sec^2\left(-\frac{\pi}{4}\right)\left(x+\frac{\pi}{4}\right)$$
$$=-1+2\left(x+\frac{\pi}{4}\right)$$
$$=2x+\frac{\pi}{2}-1$$

28. $f(x)=\sec x$

$f'(x)=\sec x\tan x$

$$L(x)=f\left(\frac{\pi}{4}\right)+f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)$$
$$=\sec\left(\frac{\pi}{4}\right)+\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)$$
$$=\sqrt{2}+\sqrt{2}(1)\left(x-\frac{\pi}{4}\right)$$
$$=\sqrt{2}x-\frac{\pi\sqrt{2}}{4}+\sqrt{2}$$

29. $f(x)=\frac{1}{1+\tan x}$

$$f'(x)=-(1+\tan x)^{-2}(\sec^2 x)$$
$$=-\frac{1}{\cos^2 x(1+\tan x)^2}$$
$$L(x)=f(0)+f'(0)(x-0)$$
$$=1-1(x-0)$$
$$=-x+1$$

30. $f(x)=e^x+\sin x$

$$f'(x)=e^x+\cos x$$
$$L(x)=f(0)+f'(0)(x-0)$$
$$=1+2(x-0)$$
$$=2x+1$$

31. The global minimum value of $\frac{1}{2}$ occurs at $x=2$.

32. **(a)** The values of y' and y'' are both negative where the graph is decreasing and concave down, at T.

(b) The value of y' is negative and the value of y'' is positive where the graph is decreasing and concave up, at P.

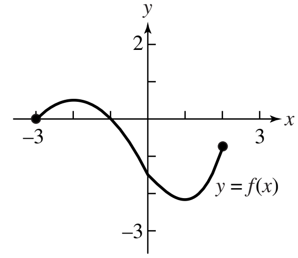
33. **(a)** The function is increasing on the interval (0, 2].

(b) The function is decreasing on the interval [−3, 0).

(c) The local extreme values occur only at the endpoints of the domain. A local maximum value of 1 occurs at $x=-3$, and a local maximum value of 3 occurs at $x=2$.

34. The 24th day

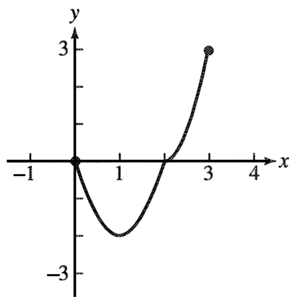
35.



36. **(a)** We know that f is decreasing on [0, 1] and increasing on [1, 3], the absolute minimum value occurs at $x = 1$ and the absolute maximum value occurs at an endpoint. Since $f(0) = 0$, $f(1) = -2$, and $f(3) = 3$, the absolute minimum value is -2 at $x = 1$ and the absolute maximum value is 3 at $x = 3$.

(b) The concavity of the graph does not change. There are no points of inflection.

(c)



37. **(a)** $f(x)$ is continuous on [0.5, 3] and differentiable on (0. 5, 3).

(b) $f'(x) = (x)\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$

Using $a = 0.5$ and $b = 3$, we solve as follows.

$$\begin{aligned} f'(c) &= \frac{f(3) - f(0.5)}{3 - 0.5} \\ 1 + \ln c &= \frac{3\ln 3 - 0.5\ln 0.5}{2.5} \\ \ln c &= \frac{\ln\left(\frac{3^3}{0.5^{0.5}}\right)}{2.5} - 1 \\ \ln c &= 0.4\ln(27\sqrt{2}) - 1 \\ c &= e^{-1}(27\sqrt{2})^{0.4} \\ c &= e^{-1}\sqrt[5]{1458} \approx 1.579 \end{aligned}$$

(c) The slope of the line is $m = \frac{f(b) - f(a)}{b - a}$

$$\begin{aligned} &= 0.4\ln\left(27\sqrt{2}\right) \\ &= 0.2 \ln 1458, \end{aligned}$$

and the line passes through (3, 3 ln 3). Its equation is $y = 0.2(\ln 1458)(x - 3) + 3\ln 3$, or approximately $y = 1.457x - 1.075$.

(d) The slope of the line is $m = 0.2\ln 1458$, and the line passes through

$$\begin{aligned} (c, f(c)) &= (e^{-1}\sqrt[5]{1458},\ e^{-1}\sqrt[5]{1458}(-1 + 0.2\ \ln\ 1458)) \\ &\approx (1.579,\ 0.722). \end{aligned}$$

Its equation is

$y = 0.2(\ln 1458)(x - c) + f(c)$, $y = 0.2\ln 1458(x - e^{-1}\sqrt[5]{1458}) + e^{-1}\sqrt[5]{1458}(-1 + 0.2 \ln 1458)$,

$y = 0.2(\ln 1458)x - e^{-1}\sqrt[5]{1458}$, or approximately $y = 1.457x - 1.579$.

38. **(a)** $v(t) = s'(t) = 4 - 6t - 3t^2$

(b) $a(t) = v'(t) = -6 - 6t$

(c) The particle starts at position 3 moving in the positive direction, but decelerating. At approximately t = 0.528, it reaches position 4.128 and changes direction, beginning to move in the negative direction. After that, it continues to accelerate while moving in the negative direction.

39. **(a)** $L(x) = f(0) + f'(0)(x-0)$
$= -1 + 0(x-0)$
$= -1$

(b) $f(0.1) \approx L(0.1) = -1$

(c) Greater than the approximation in (b), since $f'(x)$ is actually positive over the interval (0, 0.1) and the estimate is based on the derivative being 0.

40. **(a)** Since $\frac{dy}{dx} = (x^2)(-e^{-x}) + (e^{-x})(2x)$
$= (2x - x^2)e^{-x}$,
$dy = (2x - x^2)e^{-x}dx.$

(b) $dy = [2(1) - (1)^2](e^{-1})(0.01)$
$= 0.01e^{-1}$
≈ 0.00368

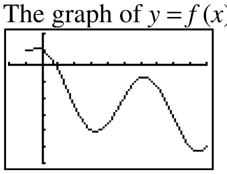
41. $f(x) = 2\cos x - \sqrt{1+x}$

$f'(x) = -2\sin x - \frac{1}{2\sqrt{1+x}}$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \frac{2\cos x_n - \sqrt{1+x_n}}{-2\sin x_n - \frac{1}{2\sqrt{1+x_n}}}$

The graph of $y = f(x)$ shows that $f(x) = 0$ has one solution, near $x = 1$.



[–2, 10] by [–6, 2]

$x_1 = 1$
$x_2 \approx 0.8361848$
$x_3 \approx 0.8283814$
$x_4 \approx 0.8283608$
$x_5 \approx 0.8283608$
Solution: $x \approx 0.828361$

42. Let t represent time in seconds, where the rocket lifts off at $t = 0$. Since $a(t) = v'(t) = 20 \text{ m/sec}^2$ and $v(0) = 0$ m / sec, we have $v(t) = 20t$, and so $v(60) = 1200$ m/sec. The speed after 1 minute (60 seconds) will be 1200 m/sec.

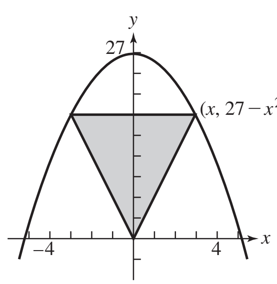
43. Let t represent time in seconds, where the rock is blasted upward at $t = 0$. Since $a(t) = v'(t) = -3.72$ m/sec^2 and $v(0) = 93$ m/sec, we have $v(t) = -3.72t + 93$. Since $s'(t) = -3.72t + 93$ and $s(0) = 0$, we have $s(t) = -1.86t^2 + 93t$. Solving $v(t) = 0$, we find that the rock attains its maximum height at $t = 25$ sec and its height at that time is $s(25) = 1162.5$ m.

44. Note that $s = 100 - 2r$ and the sector area is given by

$$A = \pi r^2\left(\frac{s}{2\pi r}\right) = \frac{1}{2}rs = \frac{1}{2}r(100-2r) = 50r - r^2.$$

To find the domain of $A(r) = 50r - r^2$, note that $r > 0$ and $0 < s < 2\pi r$, which implies $0 < 100 - 2r < 2\pi r$, which gives $12.1 \approx \frac{50}{\pi+1} < r < 50$. Since $A'(r) = 50 - 2r$, the critical point occurs at $r = 25$. This value is in the domain and corresponds to the maximum area because $A''(r) = -2$, which is negative for all r. The greatest area is attained when $r = 25$ ft and $s = 50$ ft.

45.



For $0 < x < \sqrt{27}$, the triangle with vertices at $(0, 0)$ and $(\pm x,\ 27 - x^2)$ has an area given by $A(x) = \frac{1}{2}(2x)(27 - x^2) = 27x - x^3$. Since $A' = 27 - 3x^2 = 3(3 - x)(3 + x)$ and $A'' = -6x$, the critical point in the interval $(0, \sqrt{27})$ occurs at $x = 3$ and corresponds to the maximum area because $A''(x)$ is negative in this interval. The largest possible area is $A(3) = 54$ square units.

46. If the dimensions are x ft by x ft by h ft, then the total amount of steel used is $x^2 + 4xh$ ft^2. Therefore, $x^2 + 4xh = 108$ and so $h = \frac{108 - x^2}{4x}$. The volume is given by $V(x) = x^2h = \frac{108x - x^3}{4} = 27x - 0.25x^3$. Then $V'(x) = 27 - 0.75x^2 = 0.75(6 + x)(6 - x)$ and $V''(x) = -1.5x$. The critical point occurs at $x = 6$, and it corresponds to the maximum volume because $V''(x) < 0$ for $x > 0$. The corresponding height is $\frac{108 - 6^2}{4(6)} = 3$ ft. The base measures 6 ft by 6 ft, and the height is 3 ft.

47. If the dimensions are x ft by x ft by h ft, then we have $x^2h = 32$ and so $h = \frac{32}{x^2}$. Neglecting the quarter-inch thickness of the steel, the area of the steel used is $A(x) = x^2 + 4xh = x^2 + \frac{128}{x}$. We can minimize the weight of the vat by minimizing this quantity. Now $A'(x) = 2x - 128x^{-2} = \frac{2}{x^2}(x^3 - 4^3)$ and $A''(x) = 2 + 256x^{-3}$. The critical point occurs at $x = 4$ and corresponds to the minimum possible area because $A''(x) > 0$ for $x > 0$. The corresponding height is $\frac{32}{4^2} = 2$ ft. The base should measure 4 ft by 4 ft, and the height should be 2 ft.

48. We have $r^2 + \left(\frac{h}{2}\right)^2 = 3$, so $r^2 = 3 - \frac{h^2}{4}$. We wish to minimize the cylinder's volume $V = \pi r^2 h = \pi\left(3 - \frac{h^2}{4}\right)h = 3\pi h - \frac{\pi h^3}{4}$ for $0 < h < 2\sqrt{3}$. Since $\frac{dV}{dh} = 3\pi - \frac{3\pi h^2}{4} = \frac{3\pi}{4}(2 + h)(2 - h)$ and $\frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$, the critical point occurs at $h = 2$ and it corresponds to the maximum value because $\frac{d^2V}{dh^2} < 0$ for $h > 0$. The corresponding value of r is $\sqrt{3 - \frac{2^2}{4}} = \sqrt{2}$. The largest possible cylinder has height 2 and radius $\sqrt{2}$.

49. Note that, from similar cones, $\frac{r}{6} = \frac{12-h}{12}$, so $h = 12–2r$. The volume of the smaller cone is given by

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(12-2r) = 4\pi r^2 - \frac{2\pi}{3}r^3$$

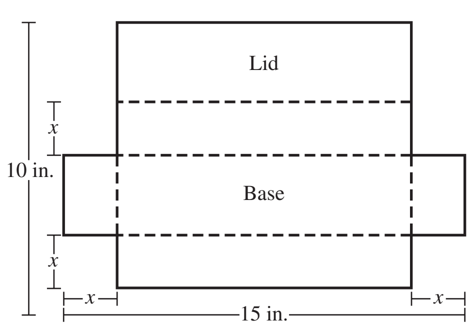
for $0 < r < 6$. Then

$\frac{dV}{dr} = 8\pi r - 2\pi r^2 = 2\pi r(4-r)$, so the critical point occurs at $r = 4$. This critical point corresponds to the maximum volume because $\frac{dV}{dr} > 0$ for

$$\begin{aligned} a - mx &= f(x) - f'(x)\cdot x \\ &= B + \frac{B}{C}\sqrt{C^2 - x^2} + \frac{Bx^2}{C\sqrt{C^2 - x^2}} \\ &= B + \frac{B}{C}\sqrt{C^2 - \frac{3C^2}{4}} + \frac{B\left(\frac{3C^2}{4}\right)}{C\sqrt{C^2 - \frac{3C^2}{4}}} \\ &= B + \frac{B}{C}\left(\frac{C}{2}\right) + \frac{\frac{3BC^2}{4}}{\frac{C^2}{2}} \\ &= B + \frac{B}{2} + \frac{3B}{2} \\ &= 3B \end{aligned}$$

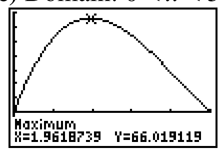
and $\frac{dV}{dr} < 0$ for $4 < r < 6$. The smaller cone has the largest possible value when $r = 4$ ft and $h = 4$ ft.

50.



(a) $V(x) = x(15-2x)(5-x)$

(b, c) Domain: $0 < x < 5$



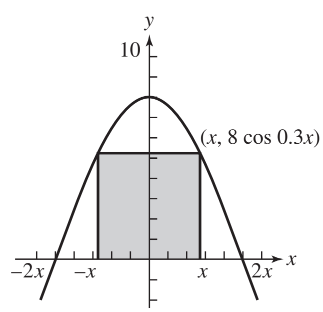
The maximum volume is approximately $66.019\ \text{in}^3$ and it occurs when $x \approx 1.962$ in.

(d) Note that $V(x) = 2x^3 - 25x^2 + 75x$, so $V'(x) = 6x^2 - 50x + 75$. Solving $V'(x) = 0$, we have

$$\begin{aligned} x &= \frac{50 \pm \sqrt{(-50)^2 - 4(6)(75)}}{2(6)} \\ &= \frac{50 \pm \sqrt{700}}{12} \\ &= \frac{50 \pm 10\sqrt{7}}{12} \\ &= \frac{25 \pm 5\sqrt{7}}{6}. \end{aligned}$$

These solutions are approximately $x \approx 1.962$ and $x = 6.371$, so the critical point in the appropriate domain occurs at $x = \frac{25 - 5\sqrt{7}}{6}$.

51.



For $0 < x < \frac{5\pi}{3}$, the area of the rectangle is given by

$A(x) = (2x)(8\cos 0.3x) = 16x\cos 0.3x$.

Then

$$\begin{aligned} A'(x) &= 16x(-0.3\sin 0.3x) + 16(\cos 0.3x)(1) \\ &= 16(\cos 0.3x - 0.3x\sin 0.3x) \end{aligned}$$

Solving $A'(x) = 0$ graphically, we find that the critical point occurs at $x \approx 2.868$ and the corresponding area is approximately 29.925 square units.

52. The cost (in thousands of dollars) is given by

$$C(x) = 40x + 30(20 - y)$$
$$= 40x + 600 - 30\sqrt{x^2 - 144}.$$

Then $C'(x) = 40 - \frac{30}{2\sqrt{x^2 - 144}}(2x)$

$$= 40 - \frac{30x}{\sqrt{x^2 - 144}}.$$

Solving $C'(x) = 0$, we have:

$$\frac{30x}{\sqrt{x^2 - 144}} = 40$$
$$3x = 4\sqrt{x^2 - 144}$$
$$9x^2 = 16x^2 - 2304$$
$$2304 = 7x^2$$

Choose the positive solution:

$$x = +\frac{48}{\sqrt{7}} \approx 18.142 \text{ mi}$$

$$y = \sqrt{x^2 - 12^2} = \frac{36}{\sqrt{7}} \approx 13.607 \text{ mi}$$

53. The length of the track is given by $2x + 2\pi r$, so we have $2x + 2\pi r = 400$ and therefore $x = 200 - \pi r$. Then the area of the rectangle is

$$A(r) = 2rx$$
$$= 2r(200 - \pi r)$$
$$= 400r - 2\pi r^2, \text{ for } 0 < r < \frac{200}{\pi}.$$

Therefore, $A'(r) = 400 - 4\pi r$ and $A''(r) = -4\pi$, so the critical point occurs at $r = \frac{100}{\pi}$ m and this point corresponds to the maximum rectangle area because $A''(r) < 0$ for all r.

The corresponding value of x is

$$x = 200 - \pi\left(\frac{100}{\pi}\right) = 100 \text{ m}.$$

The rectangle will have the largest possible area when $x = 100$ m and $r = \frac{100}{\pi}$ m.

54. Assume the profit is k dollars per hundred grade B tires and $2k$ dollars per hundred grade A tires. Then the profit is given by

$$P(x) = 2kx + k \cdot \frac{40 - 10x}{5 - x}$$
$$= 2k \cdot \frac{(20 - 5x) + x(5 - x)}{5 - x}$$
$$= 2k \cdot \frac{20 - x^2}{5 - x}$$

$$P'(x) = 2k \cdot \frac{(5 - x)(-2x) - (20 - x^2)(-1)}{(5 - x)^2}$$
$$= 2k \cdot \frac{x^2 - 10x + 20}{(5 - x)^2}$$

The solutions of $P'(x) = 0$ are

$$x = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(20)}}{2(1)} = 5 \pm \sqrt{5},$$ so the solution in the appropriate domain is $x = 5 - \sqrt{5} \approx 2.76$.

Check the profit for the critical point and endpoints:

Critical point: $x \approx 2.76$ $P(x) \approx 11.06k$

End points: $x = 0$ $P(x) = 8k$

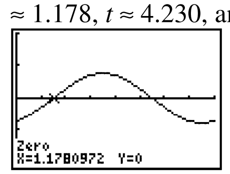
$x = 4$ $P(x) = 8k$

The highest profit is obtained when $x \approx 2.76$ and $y \approx 5.53$, which corresponds to 276 grade A tires and
553 grade B tires.

55. **(a)** The distance between the particles is $|f(t)|$ where $f(t) = -\cos t + \cos\left(t + \frac{\pi}{4}\right)$. Then

$$f'(t) = \sin t - \sin\left(t + \frac{\pi}{4}\right)$$

Solving $f'(t) = 0$ graphically, we obtain $t \approx 1.178$, $t \approx 4.230$, and so on.



$[0, 2\pi]$ by $[-2, 2]$

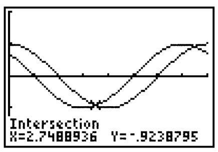
Alternatively, $f'(t) = 0$ may be solved analytically as follows.

$$f'(t) = \sin\left[\left(t+\frac{\pi}{8}\right)-\frac{\pi}{8}\right] - \sin\left[\left(t+\frac{\pi}{8}\right)+\frac{\pi}{8}\right]$$
$$= \left[\sin\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8} - \cos\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8}\right] - \left[\sin\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8} + \cos\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8}\right]$$
$$= -2\sin\frac{\pi}{8}\cos\left(t+\frac{\pi}{8}\right),$$

so the critical points occur when $\cos\left(t+\frac{\pi}{8}\right)=0$, or $t=\frac{3\pi}{8}+k\pi$. At each of these values,

$f(t)=\pm 2\cos\frac{3\pi}{8}\approx \pm 0.765$ units, so the maximum distance between the particles is 0.765 unit.

(b) Solving $\cos t = \cos\left(t+\frac{\pi}{4}\right)$ graphically, we obtain $t\approx 2.749$, $t\approx 5.890$, and so on.



$[0, 2\pi]$ by $[-2, 2]$

Alternatively, this problem may be solved analytically as follows.

$$\cos t = \cos\left(t+\frac{\pi}{4}\right)$$
$$\cos\left[\left(t+\frac{\pi}{8}\right)-\frac{\pi}{8}\right] = \cos\left[\left(t+\frac{\pi}{8}\right)+\frac{\pi}{8}\right]$$
$$\cos\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8}+\sin\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8} = \cos\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8}-\sin\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8}$$
$$2\sin\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8}=0$$
$$\sin\left(t+\frac{\pi}{8}\right)=0$$
$$t=\frac{7\pi}{8}+k\pi$$

The particles collide when $t=\frac{7\pi}{8}\approx 2.749$ (plus multiples of π if they keep going.)

56. The dimensions will be x in. by $10-2x$ in. by $16-2x$ in., so

$$V(x) = x(10-2x)(16-2x)$$
$$= 4x^3 - 52x^2 + 160x$$

for $0 < x < 5$.

Then $V'(x) = 12x^2 - 104x + 160 = 4(x-2)(3x-20)$, so the critical point in the correct domain is $x = 2$. This critical point corresponds to the maximum possible volume because $V'(x) > 0$ for $0 < x < 2$ and $V'(x) < 0$ for $2 < x < 5$. The box of largest volume has a height of 2 in. and a base measuring 6 in. by 12 in., and its volume is 144 in^3.

57. Step 1:
r = radius of circle
A = area of circle
Step 2:
At the instant in question, $\frac{dr}{dt} = -\frac{2}{\pi}$ m/sec and $r = 10$ m.
Step 3:
We want to find $\frac{dA}{dt}$.
Step 4:
$A = \pi r^2$
Step 5:
$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$
Step 6:
$\frac{dA}{dt} = 2\pi(10)\left(-\frac{2}{\pi}\right) = -40$
The area is changing at the rate of $-40\ \text{m}^2/\text{sec}$.

58. Step 1:
x = x-coordinate of particle
y = y-coordinate of particle
D = distance from origin to particle
Step 2:
At the instant in question, $x = 5$ m, $y = 12$ m, $\frac{dx}{dt} = -1$ m/sec, and $\frac{dy}{dt} = -5$ m/sec.
Step 3:
We want to find $\frac{dD}{dt}$.
Step 4:
$D = \sqrt{x^2 + y^2}$
Step 5:
$$\frac{dD}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}}$$
Step 6:
$$\frac{dD}{dt} = \frac{(5)(-1)+(12)(-5)}{\sqrt{5^2+12^2}} = -5 \text{ m/sec}$$
Since $\frac{dD}{dt}$ is negative, the particle is *approaching* the origin at the *positive* rate of 5 m/sec.

59. Step 1:
x = edge of length of cube
V = volume of cube
Step 2:
At the instant in question,
$\frac{dV}{dt} = 1200\ \text{cm}^3/\text{min}$ and $x = 20$ cm.
Step 3:
We want to find $\frac{dx}{dt}$.
Step 4:
$V = x^3$
Step 5:
$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$
Step 6:
$1200 = 3(20)^2\frac{dx}{dt}$
$\frac{dx}{dt} = 1$ cm/min
The edge length is increasing at the rate of 1 cm/min.

60. Step 1:
x = x-coordinate of point
y = y-coordinate of point
D = distance from origin to point
Step 2:
At the instant in question, $x = 3$ and $\frac{dD}{dt} = 11$ units per sec.
Step 3:
We want to find $\frac{dx}{dt}$.
Step 4:
Since $D^2 = x^2 + y^2$ and $y = x^{3/2}$, we have $D = \sqrt{x^2 + x^3}$ for $x \ge 0$.
Step 5:
$$\frac{dD}{dt} = \frac{1}{2\sqrt{x^2+x^3}}(2x+3x^2)\frac{dx}{dt} = \frac{2x+3x^2}{2x\sqrt{1+x}}\frac{dx}{dt} = \frac{3x+2}{2\sqrt{1+x}}\frac{dx}{dt}$$
Step 6:
$$11 = \frac{3(3)+2}{2\sqrt{4}}\frac{dx}{dt}$$
$\frac{dx}{dt} = 4$ units per sec

61. **(a)** Since $\frac{h}{r} = \frac{10}{4}$, we may write

$h = \frac{5r}{2}$ or $r = \frac{2h}{5}$.

(b) Step 1:
h = depth of water in tank
r = radius of surface of water
V = volume of water in tank
Step 2:
At the instant in question,

$\frac{dV}{dt} = -5 \text{ ft}^3/\text{min}$ and $h = 6$ ft.

Step 3:

We want to find $-\frac{dh}{dt}$.

Step 4:

$V = \frac{1}{3}\pi r^2 h = \frac{4}{75}\pi h^3$

Step 5:

$\frac{dV}{dt} = \frac{4}{25}\pi h^2 \frac{dh}{dt}$

Step 6:

$-5 = \frac{4}{25}\pi(6)^2 \frac{dh}{dt}$

$\frac{dh}{dt} = -\frac{125}{144\pi} \approx -0.276$ ft/min

Since $\frac{dh}{dt}$ is negative, the water level is *dropping* at the positive rate of ≈ 0.276 ft/min.

62. Step 1:
r= radius of outer layer of cable on the spool
θ= clockwise angle turned by spool
s= length of cable that has been unwound
Step 2:

At the instant in question, $\frac{ds}{dt} = 6$ ft/sec and

$r = 1.2$ ft
Step 3:

We want to find $\frac{d\theta}{dt}$.

Step 4:
$s = r\theta$
Step 5:

Since r is essentially constant, $\frac{ds}{dt} = r\frac{d\theta}{dt}$

Step 6:

$6 = 1.2\frac{d\theta}{dt}$

$\frac{d\theta}{dt} = 5$ radians/sec

The spool is turning at the rate of 5 radians per second.

63. $a(t) = v'(t) = -g = -32 \text{ ft/sec}^2$

Since $v(0) = 32$ ft/sec, $v(t) = s'(t) = -32t + 32$.

Since $s(0) = -17$ ft, $s(t) = -16t^2 + 32t - 17$.

The shovelful of dirt reaches its maximum height when $v(t) = 0$, at $t = 1$ sec. Since $s(1) = -1$, the shovelful of dirt is still below ground level at this time. There was not enough speed to get the dirt out of the hole. Duck!

64. We have $V = \frac{1}{3}\pi r^2 h$, so $\Delta V = \frac{2}{3}\pi rh\, \Delta r$.

When the radius changes from a to $a + \Delta r$, the volume change is approximately

$\Delta V \approx \frac{2}{3}\pi ah\, \Delta r$.

65. **(a)** Let x = edge of length of cube and S = surface area of cube. Then $S = 6x^2$, which means $\Delta S = 12x\, \Delta x$. We want $|\Delta S| \le 0.02S$, which gives

$|12x\, \Delta x| \le 0.02(6x^2)$ or $|\Delta x| \le 0.01x$. The edge should be measured with an error of no more than 1%.

(b) Let V = volume of cube. Then $V = x^3$, which means $\Delta V = 3x^2 \Delta x$. We have $|\Delta x| \le 0.01x$, which means

$\left|3x^2 \Delta x\right| \le 3x^2(0.01x) = 0.03V$,

so $|\Delta V| \le 0.03V$. The volume calculation will be accurate to within approximately 3% of the correct volume.

66. Let C = circumference, r = radius, S = surface area, and V = volume.

(a) Since $C = 2\pi r$, we have $\Delta C = 2\pi\, \Delta r$. Therefore,

$\left|\frac{\Delta C}{C}\right| = \left|\frac{2\pi\, \Delta r}{2\pi r}\right| = \left|\frac{\Delta r}{r}\right| < \frac{0.4\text{cm}}{10\text{cm}} = 0.04$

The calculated radius will be within approximately 4% of the correct radius.

(b) Since $S = 4\pi r^2$, we have so $\Delta S = 8\pi r\ \Delta r$. Therefore,

$$\left|\frac{\Delta S}{S}\right| = \left|\frac{8\pi r\ \Delta r}{4\pi r^2}\right| = \left|\frac{2\ \Delta r}{r}\right| \le 2(0.04) = 0.08.$$

The calculated surface area will be within approximately 8% of the correct surface area.

(c) Since $V = \frac{4}{3}\pi r^3$, we have $\Delta V = 4\pi r^2\ \Delta r$. Therefore

$$\left|\frac{\Delta V}{V}\right| = \left|\frac{4\pi r^2\ \Delta r}{\frac{4}{3}\pi r^3}\right| = \left|\frac{3\ \Delta r}{r}\right| \le 3(0.04) = 0.12.$$

The calculated volume will be within approximately 12% of the correct volume.

67. By similar triangles, we have $\frac{a}{6} = \frac{a+20}{h}$, which gives $ah = 6a + 120$ or $h = 6 + 120a^{-1}$ The height of the lamp post is approximately $6 + 120(15)^{-1} = 14$ ft. The estimated error in measuring a was $|\Delta a| \le 1$ in. $= \frac{1}{12}$ ft. Since $\frac{dh}{da} = -120a^{-2}$, we have

$$|\Delta h| = \left|-120a^{-2}\ \Delta a\right| \le 120(15)^{-2}\left(\frac{1}{12}\right) = \frac{2}{45} \text{ ft},$$

so the estimated possible error is $\pm\frac{2}{45}$ ft or $\pm\frac{8}{15}$ in.

68. $\frac{dy}{dx} = 2 \sin x \cos x - 3.$ Since $\sin x$ and $\cos x$ are both between 1and –1, the value of $2 \sin x \cos x$ is never greater than 2. Therefore, $\frac{dy}{dx} \le 2 - 3 = -1$ for all values of x.

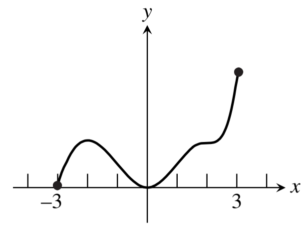
Since $\frac{dy}{dx}$ is always negative, the function decreases on every interval.

69. (a) f has a relative maximum at $x = -2$. This is where $f'(x) = 0$, causing f' to go from positive to negative.

(b) f has a relative minimum at $x = 0$. This is where $f'(x) = 0$, causing f' to go from negative to positive.

(c) The graph of f is concave up on $(-1, 1)$ and on $(2, 3)$. These are the intervals on which the derivative of f is increasing.

(d)



70. (a)

$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r dr$$
$$\frac{dA}{dt} = 2\pi(2)\left(\frac{1}{3}\right) = \frac{4}{3}\pi \frac{\text{in.}^2}{\text{sec}}$$

(b) $V = \frac{1}{3}\pi(2^2)h = 8\pi \Rightarrow h = 6$

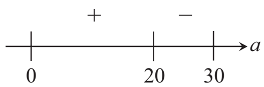
$$4\pi = \frac{dV}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$
$$4 = \frac{1}{3}\left(2(2)6\left(\frac{1}{3}\right) + 2^2\frac{dh}{dt}\right)$$
$$12 = \left(8 + 4\frac{dh}{dt}\right)$$
$$1 = \frac{dh}{dt}$$

(c) $\frac{dA}{dh} = \frac{\frac{4}{3}\pi}{1} = \frac{4}{3}\pi\frac{\text{in.}^2}{\text{in.}}$

71. (a) $V = \pi\left(\frac{a}{2\pi}\right)^2 b$, and $b = \frac{60-2a}{4} = 15 - \frac{a}{2}$, so $V = \frac{30a^2 - a^3}{8\pi}$.

Thus $\frac{dV}{da} = \frac{1}{8\pi}\left(60a - 3a^2\right) = \frac{3}{8\pi}a(20-a)$. The relevant domain for a in this problem is $(0, 30)$, so $a = 20$ is the only critical number. The cylinder of maximum volume is formed when $a = 20$ and $b = 5$.

(b) The sign graph for the derivative
$\frac{dV}{da} = \frac{3}{8\pi} a(20 - a)$ on the interval (0, 30)
is as follows:



By the First Derivative Test, there is a maximum at $a = 20$.