Chapter 4

More Derivatives

Section 4.1 Chain Rule (pp. 155–163)

Quick Review 4.1

1. $f(g(x)) = f(x^2+1) = \sin(x^2+1)$

2. $f(g(h(x))) = f(g(7x))$
$= f((7x)^2+1)$
$= \sin[(7x)^2+1]$
$= \sin(49x^2+1)$

3. $(g \circ h)(x) = g(h(x))$
$= g(7x)$
$= (7x)^2+1$
$= 49x^2+1$

4. $(h \circ g)(x) = h(g(x))$
$= h(x^2+1)$
$= 7(x^2+1)$
$= 7x^2+7$

5. $f\left(\dfrac{g(x)}{h(x)}\right) = f\left(\dfrac{x^2+1}{7x}\right) = \sin\dfrac{x^2+1}{7x}$

6. $\sqrt{\cos x+2} = g(\cos x) = g(f(x))$

7. $\sqrt{3\cos^2 x+2} = g(3\cos^2 x)$
$= g(h(\cos x))$
$= g(h(f(x)))$

8. $3\cos x+6 = 3(\cos x+2)$
$= 3\left(\sqrt{\cos x+2}\right)^2$
$= h\left(\sqrt{\cos x+2}\right)$
$= h(g(\cos x))$
$= h(g(f(x)))$

9. $\cos 27x^4 = f(27x^4)$
$= f(3(3x^2)^2)$
$= f(h(3x^2))$
$= f(h(h(x)))$

10. $\cos\sqrt{2+3x^2} = \cos\sqrt{3x^2+2}$
$= f\left(\sqrt{3x^2+2}\right)$
$= f(g(3x^2))$
$= f(g(h(x)))$

Section 4.1 Exercises

1. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$y = \sin u \qquad u = 3x+1$
$\dfrac{dy}{du} = \cos u \qquad \dfrac{du}{dx} = 3$
$\dfrac{dy}{dx} = 3\cos(3x+1)$

2. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$y = \sin u \qquad u = 7-5x$
$\dfrac{dy}{du} = \cos u \qquad \dfrac{du}{dx} = -5$
$\dfrac{dy}{dx} = -5\cos(7-5x)$

3. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$y = \cos u \qquad u = \sqrt{3}x$
$\dfrac{dy}{du} = -\sin u \qquad \dfrac{du}{dx} = \sqrt{3}$
$\dfrac{dy}{dx} = -\sqrt{3}\sin(\sqrt{3}x)$

4. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$y = \tan u \qquad u = 2x-x^3$
$\dfrac{dy}{du} = \sec^2 u \qquad \dfrac{du}{dx} = 2-3x^2$
$\dfrac{dy}{dx} = (2-3x^2)\sec^2(2x-x^3)$

5. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$y = u^2 \qquad u = \dfrac{\sin x}{1+\cos x}$
$\dfrac{dy}{du} = 2u \qquad \dfrac{du}{dx} = \dfrac{1}{1+\cos x}$
$\dfrac{dy}{dx} = \dfrac{2\sin x}{(1+\cos x)^2}$

6. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$y = 5\cot u \qquad u = \frac{2}{x}$

$\frac{dy}{du} = -5\csc^2 u \qquad \frac{du}{dx} = -\frac{2}{x^2}$

$\frac{dy}{dx} = \frac{10}{x^2}\csc^2\left(\frac{2}{x}\right)$

7. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$y = \cos u \qquad u = \sin x$

$\frac{dy}{du} = -\sin u \qquad \frac{du}{dx} = \cos x$

$\frac{dy}{dx} = -\sin(\sin x)\cos x$

8. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$y = \sec u$

$\frac{dy}{du} = \sec u \tan u$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$\frac{dy}{dx} = \sec(\tan x)\tan(\tan x)\sec^2 x$

9. $\frac{ds}{dt} = \frac{d}{dt}\cos\left(\frac{\pi}{2} - 3t\right)$

$= \left[-\sin\left(\frac{\pi}{2} - 3t\right)\right]\frac{d}{dt}\left(\frac{\pi}{2} - 3t\right)$

$= \left[-\sin\left(\frac{\pi}{2} - 3t\right)\right](-3)$

$= 3\sin\left(\frac{\pi}{2} - 3t\right)$

10. $\frac{ds}{dt} = \frac{d}{dt}[t\cos(\pi - 4t)]$

$= (t)\frac{d}{dt}[\cos(\pi - 4t)] + \cos(\pi - 4t)\frac{d}{dt}(t)$

$= t[-\sin(\pi - 4t)]\frac{d}{dt}(\pi - 4t) + \cos(\pi - 4t)(1)$

$= t[-\sin(\pi - 4t)](-4) + \cos(\pi - 4t)$

$= 4t\sin(\pi - 4t) + \cos(\pi - 4t)$

11. $$\frac{ds}{dt} = \frac{d}{dt}\left(\frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t\right)$$
$$= \frac{4}{3\pi}(\cos 3t)\frac{d}{dt}(3t) + \frac{4}{5\pi}(-\sin 5t)\frac{d}{dt}(5t)$$
$$= \frac{4}{3\pi}(\cos 3t)(3) + \frac{4}{5\pi}(-\sin 5t)(5)$$
$$= \frac{4}{\pi}\cos 3t - \frac{4}{\pi}\sin 5t$$

12. $$\frac{ds}{dt} = \frac{d}{dt}\left[\sin\left(\frac{3\pi}{2}t\right) + \cos\left(\frac{7\pi}{4}t\right)\right]$$
$$= \cos\left(\frac{3\pi}{2}t\right)\frac{d}{dt}\left(\frac{3\pi}{2}t\right) - \sin\left(\frac{7\pi}{4}t\right)\frac{d}{dt}\left(\frac{7\pi}{4}t\right)$$
$$= \frac{3\pi}{2}\cos\left(\frac{3\pi}{2}t\right) - \frac{7\pi}{4}\sin\left(\frac{7\pi}{4}t\right)$$

13. $$\frac{dy}{dx} = \frac{d}{dx}(x+\sqrt{x})^{-2} = -2(x+\sqrt{x})^{-3}\frac{d}{dx}(x+\sqrt{x})$$
$$= -2(x+\sqrt{x})^{-3}\left(1+\frac{1}{2\sqrt{x}}\right)$$

14. $$\frac{dy}{dx} = \frac{d}{dx}(\csc x + \cot x)^{-1}$$
$$= -(\csc x + \cot x)^{-2}\frac{d}{dx}(\csc x + \cot x)$$
$$= -\frac{1}{(\csc x + \cot x)^2}(-\cot x \csc x - \csc^2 x)$$
$$= \frac{(\csc x)(\cot x + \csc x)}{(\csc x + \cot x)^2}$$
$$= \frac{\csc x}{\csc x + \cot x}$$

15. $$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-5} x - \cos^3 x)$$
$$= (-5\sin^{-6} x)\frac{d}{dx}(\sin x) - (3\cos^2 x)\frac{d}{dx}(\cos x)$$
$$= -5\sin^{-6} x \cos x + 3\cos^2 x \sin x$$

16. $\frac{dy}{dx} = \frac{d}{dx}[x^3(2x-5)^4]$

$= (x^3)\frac{d}{dx}(2x-5)^4 + (2x-5)^4\frac{d}{dx}(x^3)$

$= (x^3)(4)(2x-5)^3\frac{d}{dx}(2x-5) + (2x-5)^4(3x^2)$

$= (x^3)(4)(2x-5)^3(2) + 3x^2(2x-5)^4$

$= 8x^3(2x-5)^3 + 3x^2(2x-5)^4$

$= x^2(2x-5)^3[8x + 3(2x-5)]$

$= x^2(2x-5)^3(14x-15)$

17. $\frac{dy}{dx} = \frac{d}{dx}(\sin^3 x \tan 4x)$

$= (\sin^3 x)\frac{d}{dx}(\tan 4x) + (\tan 4x)\frac{d}{dx}(\sin^3 x)$

$= (\sin^3 x)(\sec^2 4x)\frac{d}{dx}(4x) + (\tan 4x)(3\sin^2 x)\frac{d}{dx}(\sin x)$

$= (\sin^3 x)(\sec^2 4x)(4) + (\tan 4x)(3\sin^2 x)(\cos x)$

$= 4\sin^3 x \sec^2 4x + 3\sin^2 x \cos x \tan 4x$

18. $\frac{dy}{dx} = \frac{d}{dx}(4\sqrt{\sec x + \tan x})$

$= 4 \cdot \frac{1}{2\sqrt{\sec x + \tan x}}\frac{d}{dx}(\sec x + \tan x)$

$= \frac{2}{\sqrt{\sec x + \tan x}}(\sec x \tan x + \sec^2 x)$

$= 2\sec x \frac{\sec x + \tan x}{\sqrt{\sec x + \tan x}}$

$= 2\sec x\sqrt{\sec x + \tan x}$

19. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3}{\sqrt{2x+1}}\right)$

$= \frac{\left(\sqrt{2x+1}\right)\frac{d}{dx}(3) - 3\frac{d}{dx}\left(\sqrt{2x+1}\right)}{(\sqrt{2x+1})^2}$

$= \frac{\left(\sqrt{2x+1}\right)(0) - 3\left(\frac{1}{2\sqrt{2x+1}}\right)\frac{d}{dx}(2x+1)}{2x+1}$

$= \frac{-3\left(\frac{1}{2\sqrt{2x+1}}\right)(2)}{2x+1}$

$= -\frac{3}{(2x+1)\sqrt{2x+1}}$

$= -3(2x+1)^{-3/2}$

20. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$$= \frac{\left(\sqrt{1+x^2}\right)\frac{d}{dx}(x) - x\frac{d}{dx}\left(\sqrt{1+x^2}\right)}{\left(\sqrt{1+x^2}\right)^2}$$

$$= \frac{\left(\sqrt{1+x^2}\right)(1) - x\left(\frac{1}{2\sqrt{1+x^2}}\right)\frac{d}{dx}(1+x^2)}{1+x^2}$$

$$= \frac{\sqrt{1+x^2} - x\left(\frac{1}{2\sqrt{1+x^2}}\right)(2x)}{1+x^2}$$

$$= \frac{(1+x^2) - x^2}{(1+x^2)(\sqrt{1+x^2})}$$

$$= (1+x^2)^{-3/2}$$

21. The last step here uses the identity $2 \sin a \cos a = \sin 2a$.

$$\frac{dy}{dx} = \frac{d}{dx}\sin^2(3x-2)$$

$$= 2\sin(3x-2)\frac{d}{dx}\sin(3x-2)$$

$$= 2\sin(3x-2)\cos(3x-2)\frac{d}{dx}(3x-2)$$

$$= 2\sin(3x-2)\cos(3x-2)(3)$$

$$= 6\sin(3x-2)\cos(3x-2)$$

$$= 3\sin(6x-4)$$

22. $\frac{dy}{dx} = \frac{d}{dx}(1+\cos 2x)^2$

$$= 2(1+\cos 2x)\frac{d}{dx}(1+\cos 2x)$$

$$= 2(1+\cos 2x)(-\sin 2x)\frac{d}{dx}(2x)$$

$$= 2(1+\cos 2x)(-\sin 2x)(2)$$

$$= -4(1+\cos 2x)(\sin 2x)$$

23. $\frac{dy}{dx} = \frac{d}{dx}(1+\cos^2 7x)^3$

$$= 3(1+\cos^2 7x)^2 \frac{d}{dx}(1+\cos^2 7x)$$

$$= 3(1+\cos^2 7x)^2(2\cos 7x)\frac{d}{dx}(\cos 7x)$$

$$= 3(1+\cos^2 7x)^2(2\cos 7x)(-\sin 7x)\frac{d}{dx}(7x)$$

$$= 3(1+\cos^2 7x)^2(2\cos 7x)(-\sin 7x)(7)$$

$$= -42(1+\cos^2 7x)^2 \cos 7x \sin 7x$$

24. $$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\tan 5x}\right)$$
$$= \frac{1}{2\sqrt{\tan 5x}} \frac{d}{dx} \tan 5x$$
$$= \frac{1}{2\sqrt{\tan 5x}} (\sec^2 5x) \frac{d}{dx}(5x)$$
$$= \frac{1}{2\sqrt{\tan 5x}} (\sec^2 5x)(5)$$
$$= \frac{5\sec^2 5x}{2\sqrt{\tan 5x}} \text{ or } \frac{5}{2}(\tan 5x)^{-1/2} \sec^2 5x$$

25. $$\frac{dr}{d\theta} = \frac{d}{d\theta} \tan(2-\theta) = \sec^2(2-\theta) \frac{d}{d\theta}(2-\theta)$$
$$= \sec^2(2-\theta)(-1)$$
$$= -\sec^2(2-\theta)$$

26. $$\frac{dr}{d\theta} = \frac{d}{d\theta}(\sec 2\theta \ \tan 2\theta)$$
$$= (\sec 2\theta) \frac{d}{d\theta}(\tan 2\theta) + (\tan 2\theta) \frac{d}{d\theta}(\sec 2\theta)$$
$$= (\sec 2\theta)(\sec^2 2\theta) \frac{d}{d\theta}(2\theta) + (\tan 2\theta)(\sec 2\theta \tan 2\theta) \frac{d}{d\theta}(2\theta)$$
$$= 2\sec^3 2\theta + 2\sec 2\theta \tan^2 2\theta$$

27. $$\frac{dr}{d\theta} = \frac{d}{d\theta}\sqrt{\theta \sin\theta}$$
$$= \frac{1}{2\sqrt{\theta\sin\theta}} \frac{d}{d\theta}(\theta \sin\theta)$$
$$= \frac{1}{2\sqrt{\theta\sin\theta}} \left[\theta \frac{d}{d\theta}(\sin\theta) + (\sin\theta) \frac{d}{d\theta}(\theta)\right]$$
$$= \frac{1}{2\sqrt{\theta\sin\theta}} (\theta\cos\theta + \sin\theta)$$
$$= \frac{\theta\cos\theta + \sin\theta}{2\sqrt{\theta\sin\theta}}$$

28. $$\frac{dr}{d\theta} = \frac{d}{d\theta}\left(2\theta\sqrt{\sec\theta}\right)$$
$$= (2\theta)\frac{d}{d\theta}\left(\sqrt{\sec\theta}\right) + \left(\sqrt{\sec\theta}\right)\frac{d}{d\theta}(2\theta)$$
$$= (2\theta)\left(\frac{1}{2\sqrt{\sec\theta}}\right)\frac{d}{d\theta}(\sec\theta) + 2\sqrt{\sec\theta}$$
$$= (2\theta)\left(\frac{1}{2\sqrt{\sec\theta}}\right)(\sec\theta\tan\theta) + 2\sqrt{\sec\theta}$$
$$= \theta\left(\sqrt{\sec\theta}\right)(\tan\theta) + 2\sqrt{\sec\theta}$$
$$= \sqrt{\sec\theta}(\theta\tan\theta + 2)$$

29. $y' = \frac{d}{dx}\tan x = \sec^2 x$

$$y'' = \frac{d}{dx}\sec^2 x = (2\sec x)\frac{d}{dx}(\sec x)$$
$$= (2\sec x)(\sec x \tan x)$$
$$= 2\sec^2 x \tan x$$

30. $y' = \frac{d}{dx}\cot x = -\csc^2 x$

$$y'' = \frac{d}{dx}(-\csc^2 x) = (-2\csc x)\frac{d}{dx}(\csc x)$$
$$= (-2\csc x)(-\csc x \cot x)$$
$$= 2\csc^2 x \cot x$$

31. $y' = \frac{d}{dx}\cot(3x-1)$

$$= -\csc^2(3x-1)\frac{d}{dx}(3x-1)$$
$$= -3\csc^2(3x-1)$$
$$y'' = \frac{d}{dx}[-3\csc^2(3x-1)]$$
$$= -3[2\csc(3x-1)]\frac{d}{dx}\csc(3x-1)$$
$$= -3[2\csc(3x-1)]\cdot[-\csc(3x-1)\cot(3x-1)]\frac{d}{dx}(3x-1)$$
$$= -3[2\csc(3x-1)][-\csc(3x-1)\cot(3x-1)](3)$$
$$= 18\csc^2(3x-1)\cot(3x-1)$$

32. $y' = \frac{d}{dx}\left[9\tan\left(\frac{x}{3}\right)\right]$

$$= 9\sec^2\left(\frac{x}{3}\right)\frac{d}{dx}\left(\frac{x}{3}\right)$$
$$= 3\sec^2\left(\frac{x}{3}\right)$$
$$y'' = \frac{d}{dx}\left[3\sec^2\left(\frac{x}{3}\right)\right]$$
$$= 3\left[2\sec\left(\frac{x}{3}\right)\right]\frac{d}{dx}\sec\left(\frac{x}{3}\right)$$
$$= 6\left[\sec\left(\frac{x}{3}\right)\right]\left[\sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)\right]\frac{d}{dx}\left(\frac{x}{3}\right)$$
$$= 2\sec^2\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)$$

33. $f'(u) = \frac{d}{du}(u^5 + 1) = 5u^4$

$g'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

$$\begin{aligned}(f \circ g)'(1) &= f'(g(1))g'(1) \\ &= f'(1)g'(1) \\ &= (5)\left(\frac{1}{2}\right) \\ &= \frac{5}{2}\end{aligned}$$

34. $f'(u) = \frac{d}{du}(1 - u^{-1}) = u^{-2} = \frac{1}{u^2}$

$$\begin{aligned}g'(x) &= \frac{d}{dx}(1-x)^{-1} \\ &= -(1-x)^{-2}\frac{d}{dx}(1-x) \\ &= -(1-x)^{-2}(-1) \\ &= \frac{1}{(1-x)^2}\end{aligned}$$

$$\begin{aligned}(f \circ g)'(-1) &= f'(g(-1))g'(-1) \\ &= f'\left(\frac{1}{2}\right)g'(-1) \\ &= (4)\left(\frac{1}{4}\right) \\ &= 1\end{aligned}$$

35. $$\begin{aligned}f'(u) &= \frac{d}{du}\left(\cot\frac{\pi u}{10}\right) \\ &= -\csc^2\left(\frac{\pi u}{10}\right)\frac{d}{du}\left(\frac{\pi u}{10}\right) \\ &= -\frac{\pi}{10}\csc^2\left(\frac{\pi u}{10}\right)\end{aligned}$$

$g'(x) = \frac{d}{dx}\left(5\sqrt{x}\right) = \frac{5}{2\sqrt{x}}$

$$\begin{aligned}(f \circ g)'(1) &= f'(g(1))g'(1) \\ &= f'(5)g'(1) \\ &= -\frac{\pi}{10}\left[\csc^2\left(\frac{\pi}{2}\right)\right]\left(\frac{5}{2}\right) \\ &= -\frac{\pi}{10}(1)\left(\frac{5}{2}\right) \\ &= -\frac{\pi}{4}\end{aligned}$$

36. $$\begin{aligned}f'(u) &= \frac{d}{du}\left[u + (\cos u)^{-2}\right] \\ &= 1 - 2(\cos u)^{-3}\frac{d}{du}\cos u \\ &= 1 + \frac{2\sin u}{\cos^3 u}\end{aligned}$$

$g'(x) = \frac{d}{dx}(\pi x) = \pi$

$$\begin{aligned}(f \circ g)'\left(\frac{1}{4}\right) &= f'\left(g\left(\frac{1}{4}\right)\right)g'\left(\frac{1}{4}\right) \\ &= f'\left(\frac{\pi}{4}\right)g'\left(\frac{1}{4}\right) \\ &= \left(1 + \frac{\frac{2}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^3}\right)(\pi) \\ &= 5\pi\end{aligned}$$

37. $$\begin{aligned}f'(u) &= \frac{d}{du}\frac{2u}{u^2+1} \\ &= \frac{(u^2+1)\frac{d}{du}(2u) - (2u)\frac{d}{du}(u^2+1)}{(u^2+1)^2} \\ &= \frac{(u^2+1)(2) - (2u)(2u)}{(u^2+1)^2} \\ &= \frac{-2u^2+2}{(u^2+1)^2}\end{aligned}$$

$g'(x) = \frac{d}{dx}(10x^2 + x + 1) = 20x + 1$

$$\begin{aligned}(f \circ g)'(0) &= f'(g(0))g'(0) \\ &= f'(1)g'(0) \\ &= (0)(1) \\ &= 0\end{aligned}$$

38. $f'(u)$
$$= \frac{d}{du}\left(\frac{u-1}{u+1}\right)^2$$
$$= 2\left(\frac{u-1}{u+1}\right)\frac{d}{du}\left(\frac{u-1}{u+1}\right)$$
$$= 2\left(\frac{u-1}{u+1}\right)\frac{(u+1)\frac{d}{du}(u-1)-(u-1)\frac{d}{du}(u+1)}{(u+1)^2}$$
$$= 2\left(\frac{u-1}{u+1}\right)\frac{(u+1)-(u-1)}{(u+1)^2}$$
$$= \frac{4(u-1)}{(u+1)^3}$$
$$g'(x) = \frac{d}{dx}(x^{-2}-1) = -2x^{-3}$$
$$\begin{aligned}(f \circ g)'(-1) &= f'(g(-1))g'(-1)\\ &= f'(0)g'(-1)\\ &= (-4)(2)\\ &= -8\end{aligned}$$

39. **(a)** $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$$\begin{aligned}&= \frac{d}{du}(\cos u)\frac{d}{dx}(6x+2)\\ &= (-\sin u)(6)\\ &= -6\sin u\\ &= -6\sin(6x+2)\end{aligned}$$

(b) $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$$\begin{aligned}&= \frac{d}{du}(\cos 2u)\frac{d}{dx}(3x+1)\\ &= (-\sin 2u)(2)\bullet(3)\\ &= -6\sin 2u\\ &= -6\sin(6x+2)\end{aligned}$$

40. **(a)** $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$$\begin{aligned}&= \frac{d}{du}\sin(u+1)\frac{d}{dx}(x^2)\\ &= \cos(u+1)(1)\bullet 2x\\ &= 2x\cos(u+1)\\ &= 2x\cos(x^2+1)\end{aligned}$$

(b) $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$$\begin{aligned}&= \frac{d}{du}(\sin u)\frac{d}{dx}(x^2+1)\\ &= (\cos u)(2x)\\ &= 2x\cos u\\ &= 2x\cos(x^2+1)\end{aligned}$$

41. $\dfrac{dx}{dt} = \dfrac{d}{dt}(2\cos t) = -2\sin t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(2\sin t) = 2\cos t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-2\sin t} = -\cot t$$

This line passes through
$$\left(2\cos\frac{\pi}{4},\ 2\sin\frac{\pi}{4}\right) = \left(\sqrt{2},\ \sqrt{2}\right)$$
and has slope $-\cot\dfrac{\pi}{4} = -1.$ Its equation is
$y-\sqrt{2} = -1\left(x-\sqrt{2}\right).$

42. $\dfrac{dx}{dt} = \dfrac{d}{dt}(\sin 2\pi t)$
$$\begin{aligned}&= (\cos 2\pi t)\frac{d}{dt}(2\pi t)\\ &= 2\pi\cos 2\pi t\end{aligned}$$
$\dfrac{dy}{dt} = \dfrac{d}{dt}(\cos 2\pi t)$
$$\begin{aligned}&= (-\sin 2\pi t)\frac{d}{dt}(2\pi t)\\ &= -2\pi\sin 2\pi t\end{aligned}$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\pi\sin 2\pi t}{2\pi\cos 2\pi t} = -\tan 2\pi t$$

The line passes through
$\left(\sin\dfrac{2\pi}{-6},\ \cos\dfrac{2\pi}{-6}\right) = \left(-\dfrac{\sqrt{3}}{2},\ \dfrac{1}{2}\right)$ and has slope
$-\tan\dfrac{2\pi}{-6} = \sqrt{3}.$ Its equation is
$y-\dfrac{1}{2} = \sqrt{3}\left(x+\dfrac{\sqrt{3}}{2}\right).$

43. $\frac{dx}{dt}=\frac{d}{dt}(\sec^2 t-1)$

$=(2\sec t)\frac{d}{dt}(\sec t)$

$=(2\sec t)(\sec t\tan t)$

$=2\sec^2 t\tan t$

$\frac{dy}{dt}=\frac{d}{dt}\tan t=\sec^2 t$

$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sec^2 t}{2\sec^2 t\tan t}=\frac{1}{2}\cot t.$

The line passes through $\left(\sec^2\left(-\frac{\pi}{4}\right)-1,\ \tan\left(-\frac{\pi}{4}\right)\right)=(1,\ -1)$ and has slope $\frac{1}{2}\cot\left(-\frac{\pi}{4}\right)=-\frac{1}{2}$. Its equation is $y+1=-\frac{1}{2}(x-1).$

44. $\frac{dx}{dt}=\frac{d}{dt}\sec t=\sec t\tan t$

$\frac{dy}{dt}=\frac{d}{dt}\tan t=\sec^2 t$

$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sec^2 t}{\sec t\tan t}=\frac{\sec t}{\tan t}=\frac{1}{\sin t}=\csc t$

The line passes through $\left(\sec\frac{\pi}{6},\ \tan\frac{\pi}{6}\right)=\left(\frac{2}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$ and has slope $\csc\frac{\pi}{6}=2.$ Its equation is $y-\frac{1}{\sqrt{3}}=2\left(x-\frac{2}{\sqrt{3}}\right).$

45. $\frac{dx}{dt}=\frac{d}{dt}t=1$

$\frac{dy}{dt}=\frac{d}{dt}\sqrt{t}=\frac{1}{2\sqrt{t}}$

$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{1}{2\sqrt{t}}}{1}=\frac{1}{2\sqrt{t}}$

The line passes through $\left(\frac{1}{4},\ \sqrt{\frac{1}{4}}\right)=\left(\frac{1}{4},\ \frac{1}{2}\right)$ and has slope $\frac{1}{2\sqrt{\frac{1}{4}}}=1$. Its equation is $y-\frac{1}{2}=1\left(x-\frac{1}{4}\right).$

46. $\frac{dx}{dt}=\frac{d}{dt}(2t^2+3)=4t$

$\frac{dy}{dt}=\frac{d}{dt}(t^4)=4t^3$

$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{4t^3}{4t}=t^2$

The line passes through $(2(-1)^2+3, (-1)^4)=(5,1)$ and has slope $(-1)^2=1$. Its equation is $y-1=1(x-5)$.

47. $\frac{dx}{dt}=\frac{d}{dt}(t-\sin t)=1-\cos t$

$\frac{dy}{dt}=\frac{d}{dt}(1-\cos t)=\sin t$

$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sin t}{1-\cos t}$

The line passes through $\left(\frac{\pi}{3}-\sin\frac{\pi}{3},\ 1-\cos\frac{\pi}{3}\right)=\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2},\ \frac{1}{2}\right)$ and has slope $\frac{\sin\left(\frac{\pi}{3}\right)}{1-\cos\left(\frac{\pi}{3}\right)}=\sqrt{3}$. Its equation is $y-\frac{1}{2}=\sqrt{3}\left(x-\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right).$

48. $\frac{dx}{dt}=\frac{d}{dt}\cos t=-\sin t$

$\frac{dy}{dt}=\frac{d}{dt}(1+\sin t)=\cos t$

$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\cos t}{-\sin t}=-\cot t$

The line passes through $\left(\cos\frac{\pi}{2},\ 1+\sin\frac{\pi}{2}\right)=(0,\ 2)$ and has slope $-\cot\left(\frac{\pi}{2}\right)=0.$ Its equation is $y=2.$

49. **(a)** $\frac{dx}{dt}=\frac{d}{dt}(t^2+t)=2t+1$

$\frac{dy}{dt}=\frac{d}{dt}\sin t=\cos t$

$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\cos t}{2t+1}$

(b) $\frac{d}{dt}\left(\frac{dy}{dx}\right)$
$= \frac{d}{dt}\frac{\cos t}{2t+1}$
$= \frac{(2t+1)\frac{d}{dt}(\cos t) - (\cos t)\frac{d}{dt}(2t+1)}{(2t+1)^2}$
$= \frac{(2t+1)(-\sin t) - (\cos t)(2)}{(2t+1)^2}$
$= -\frac{(2t+1)(\sin t) + 2\cos t}{(2t+1)^2}$

(c) Let $u = \frac{dy}{dx}$.

Then $\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt}$, so $\frac{du}{dx} = \frac{du}{dt} \div \frac{dx}{dt}$.
Therefore,
$\frac{d}{dx}\left(\frac{dy}{dx}\right)$
$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt}$
$= -\frac{(2t+1)(\sin t) + 2\cos t}{(2t+1)^2} \div (2t+1)$
$= -\frac{(2t+1)(\sin t) + 2\cos t}{(2t+1)^3}$

(d) The expression in part (c).

50. Since the radius passes through (0, 0) and (2cos t, 2sin$t$), it has slope given by tan t. But the slope of the tangent line is

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-2\sin t} = -\cot t$, which is the negative reciprocal of tan t. This means that the radius and the tangent line are perpendicular. (The preceding argument breaks down when $t = \frac{k\pi}{2}$, where k is an integer. At these values, either the radius is horizontal and the tangent line is vertical or the radius is vertical and the tangent line is horizontal, so the result still holds.)

51. $\frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt}$
$= \frac{d}{d\theta}(\cos\theta)\frac{d\theta}{dt}$
$= (-\sin\theta)\left(\frac{d\theta}{dt}\right)$

When $\theta = \frac{3\pi}{2}$ and $\frac{d\theta}{dt} = 5$,

$\frac{ds}{dt} = \left(-\sin\frac{3\pi}{2}\right)(5) = 5.$

52. $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$
$= \frac{d}{dx}(x^2 + 7x - 5)\frac{dx}{dt}$
$= (2x+7)\left(\frac{dx}{dt}\right)$

When $x = 1$ and $\frac{dx}{dt} = \frac{1}{3}$,

$\frac{dy}{dt} = [2(1) + 7]\left(\frac{1}{3}\right) = 3.$

53. $\frac{dy}{dx} = \frac{d}{dx}\sin\frac{x}{2} = \left(\cos\frac{x}{2}\right)\frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}\cos\frac{x}{2}$
Since the range of the function
$f(x) = \frac{1}{2}\cos\frac{x}{2}$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$, the largest possible value of $\frac{dy}{dx}$ is $\frac{1}{2}$.

54. $\frac{dy}{dx} = \frac{d}{dx}(\sin mx)$
$= (\cos mx)\frac{d}{dx}(mx)$
$= m\cos mx$

The desired line has slope $y'(0) = m\cos 0 = m$ and passes through (0, 0), so its equation is $y = mx$.

55. $\frac{dy}{dx} = \frac{d}{dx}\left(2\tan\frac{\pi x}{4}\right)$
$= \left(2\sec^2\frac{\pi x}{4}\right)\frac{d}{dx}\left(\frac{\pi x}{4}\right)$
$= \frac{\pi}{2}\sec^2\left(\frac{\pi x}{4}\right)$

$y'(1) = \frac{\pi}{2}\sec^2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}(\sqrt{2})^2 = \pi.$

The tangent line has slope π and passes through $\left(1, 2\tan\frac{\pi}{4}\right) = (1, 2)$. Its equation is $y - 2 = \pi(x - 1)$.

The normal line has slope $-\frac{1}{\pi}$ and passes through (1, 2).

Its equation is $y-2=-\frac{1}{\pi}(x-1)$.

56. **(a)** $\frac{d}{dx}[2f(x)] = 2f'(x)$

At $x = 2$, the derivative is $2f'(2) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$.

(b) $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$

At $x = 3$, the derivative is $f'(3)+g'(3) = 2\pi+5$.

(c) $\frac{d}{dx}[f(x)\cdot g(x)] = f(x)g'(x)+g(x)f'(x)$

At $x = 3$, the derivative is

$$f(3)g'(3)+g(3)f'(3) = (3)(5)+(-4)(2\pi) = 15-8\pi.$$

(d) $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

At $x = 2$, the derivative is

$$\frac{g(2)f'(2)-f(2)g'(2)}{[g(2)]^2} = \frac{(2)\left(\frac{1}{3}\right)-(8)(-3)}{(2)^2} = \frac{\frac{74}{3}}{4} = \frac{37}{6}.$$

(e) $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

At $x = 2$, the derivative is

$$f'(g(2))g'(2) = f'(2)g'(2) = \left(\frac{1}{3}\right)(-3) = -1.$$

(f) $\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}}\frac{d}{dx}f(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

At $x = 2$, the derivative is $\frac{f'(2)}{2\sqrt{f(2)}} = \frac{\frac{1}{3}}{2\sqrt{8}} = \frac{1}{6(2\sqrt{2})} = \frac{1}{12\sqrt{2}}$.

(g)
$$\frac{d}{dx}\frac{1}{g^2(x)} = \frac{d}{dx}[g(x)]^{-2} = -2[g(x)]^{-3}\frac{d}{dx}g(x) = -\frac{2g'(x)}{[g(x)]^3}$$

At $x = 3$, the derivative is

$$-\frac{2g'(3)}{[g(3)]^3} = -\frac{2(5)}{(-4)^3} = -\frac{10}{-64} = \frac{5}{32}.$$

(h) $$\frac{d}{dx}\sqrt{f^2(x)+g^2(x)} = \frac{1}{2\sqrt{f^2(x)+g^2(x)}}\frac{d}{dx}[f^2(x)+g^2(x)]$$
$$= \frac{1}{2\sqrt{f^2(x)+g^2(x)}}[2f(x)\frac{d}{dx}f(x)+2g(x)\frac{d}{dx}g(x)]$$
$$= \frac{f(x)f'(x)+g(x)g'(x)}{\sqrt{f^2(x)+g^2(x)}}$$

At $x = 2$, the derivative is

$$\frac{f(2)f'(2)+g(2)g'(2)}{\sqrt{f^2(2)+g^2(2)}} = \frac{(8)\left(\frac{1}{3}\right)+(2)(-3)}{\sqrt{8^2+2^2}}$$
$$= \frac{-\frac{10}{3}}{\sqrt{68}}$$
$$= -\frac{\frac{10}{3}}{2\sqrt{17}}$$
$$= -\frac{5}{3\sqrt{17}}$$

57. $$\frac{d}{dx}\cos(x°) = \frac{d}{dx}\cos\left(\frac{\pi x}{180}\right)$$
$$= -\frac{\pi}{180}\sin\left(\frac{\pi x}{180}\right)$$
$$= -\frac{\pi}{180}\sin(x°)$$

58. **(a)** $$\frac{d}{dx}[5f(x)-g(x)] = 5f'(x)-g'(x)$$

At $x = 1$, the derivative is $5f'(1)-g'(1) = 5\left(-\frac{1}{3}\right)-\left(-\frac{8}{3}\right) = 1.$

(b) $$\frac{d}{dx}(f(x)g^3(x)) = f(x)\frac{d}{dx}g^3(x)+g^3(x)\frac{d}{dx}f(x)$$
$$= f(x)[3g^2(x)]\frac{d}{dx}g(x)+g^3(x)f'(x)$$
$$= 3f(x)g^2(x)g'(x)+g^3(x)f'(x)$$

At $x = 0$, the derivative is

$$3f(0)g^2(0)g'(0)+g^3(0)f'(0) = 3(1)(1)^2\left(\frac{1}{3}\right)+(1)^3(5)$$
$$= 6.$$

(c) $$\frac{d}{dx}\left(\frac{f(x)}{g(x)+1}\right) = \frac{[g(x)+1]\frac{d}{dx}f(x)-f(x)\frac{d}{dx}[g(x)+1]}{[g(x)+1]^2}$$
$$= \frac{[g(x)+1]f'(x)-f(x)g'(x)}{[g(x)+1]^2}$$

At $x = 1$, the derivative is

$$\frac{[g(1)+1]f'(1)-f(1)g'(1)}{[g(1)+1]^2}$$
$$=\frac{(-4+1)\left(-\frac{1}{3}\right)-(3)\left(-\frac{8}{3}\right)}{(-4+1)^2}$$
$$=\frac{9}{9}$$
$$=1.$$

(d) $\frac{d}{dx}f(g(x))=f'(g(x))g'(x)$

At $x=0$, the derivative is
$$f'(g(0))g'(0)=f'(1)g'(0)=\left(-\frac{1}{3}\right)\left(\frac{1}{3}\right)=-\frac{1}{9}.$$

(e) $\frac{d}{dx}g(f(x))=g'(f(x))f'(x)$

At $x=0$, the derivative is
$$g'(f(0))f'(0)=g'(1)f'(0)=\left(-\frac{8}{3}\right)(5)=-\frac{40}{3}$$

(f) $\frac{d}{dx}[g(x)+f(x)]^{-2}$
$$=-2[g(x)+f(x)]^{-3}\frac{d}{dx}[g(x)+f(x)]$$
$$=-\frac{2[g'(x)+f'(x)]}{[g(x)+f(x)]^3}$$

At $x=1$, the derivative is
$$-\frac{2[g'(1)+f'(1)]}{[g(1)+f(1)]^3}=-\frac{2\left(-\frac{8}{3}-\frac{1}{3}\right)}{(-4+3)^3}=-\frac{-6}{-1}=-6.$$

(g) $\frac{d}{dx}[f(x+g(x))]$
$$=f'(x+g(x))\frac{d}{dx}[x+g(x)]$$
$$=f'(x+g(x))(1+g'(x))$$

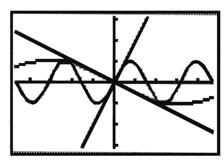
At $x=0$, the derivative is
$$f'(0+g(0))(1+g'(0))=f'(0+1)\left(1+\frac{1}{3}\right)=f'(1)\left(\frac{4}{3}\right)=\left(-\frac{1}{3}\right)\left(\frac{4}{3}\right)=-\frac{4}{9}.$$

59. For $y=\sin 2x$, $y'=(\cos 2x)\frac{d}{dx}(2x)=2\cos 2x$ and the slope at the origin is 2.

For $y=-\sin\frac{x}{2}$,

$y'=\left(-\cos\frac{x}{2}\right)\frac{d}{dx}\left(\frac{x}{2}\right)=-\frac{1}{2}\cos\frac{x}{2}$ and the slope at the origin is $-\frac{1}{2}$. Since the slopes of the two tangent lines are 2 and $-\frac{1}{2}$, the lines are perpendicular and the curves are orthogonal.

A graph of the two curves along with the tangents $y=2x$ and $y=-\frac{1}{2}x$ is shown.



[−4.7, 4.7] by [−3.1, 3.1]

60. Because the symbols $\frac{dy}{dx}$, $\frac{dy}{du}$, and $\frac{du}{dx}$ are not fractions. The individual symbols *dy*, *dx*, and *du* do not have numerical values.

61. Velocity: $s'(t)=-2\pi bA\sin(2\pi bt)$

acceleration: $s''(t)=-4\pi^2b^2A\cos(2\pi bt)$

jerk: $s'''(t)=8\pi^3b^3A\sin(2\pi bt)$

The velocity, amplitude, and jerk are proportional to b, b^2, and b^3 respectively. If the frequency b is doubled, then the amplitude of the velocity is doubled, the amplitude of the acceleration is quadrupled, and the amplitude of the jerk is multiplied by 8.

62. **(a)** $$y'(t) = \frac{d}{dt}19.3\sin\left[\frac{2\pi}{365}(x-101)\right] + \frac{d}{dt}(70)$$
$$= 19.3\cos\left[\frac{2\pi}{365}(x-101)\right] \cdot \frac{d}{dx}\left[\frac{2\pi}{365}(x-101)\right] + 0$$
$$= 19.3\cos\left[\frac{2\pi}{365}(x-101)\right] \cdot \frac{2\pi}{365}$$
$$= \frac{38.6\pi}{365}\cos\left[\frac{2\pi}{365}(x-101)\right]$$

Since cos u is greatest when $u = 0, \pm 2\pi$, and so on, $y'(t)$ is greatest when $\frac{2\pi}{365}(x-101) = 0$ or $x = 101$.

The temperature is increasing the fastest on day 101 (April 11).

(b) The rate of increase is $y'(101) = \frac{38.6\pi}{365} \approx 0.33$ degrees Fahrenheit per day.

63. Velocity: $$s'(t) = \frac{d}{dt}\sqrt{1+4t}$$
$$= \frac{1}{2\sqrt{1+4t}}\frac{d}{dt}(1+4t)$$
$$= \frac{4}{2\sqrt{1+4t}}$$
$$= \frac{2}{\sqrt{1+4t}}$$

At $t = 6$, the velocity is $\frac{2}{\sqrt{1+4(6)}} = \frac{2}{5}$ m/sec

Acceleration: $$s''(t) = \frac{d}{dt}\frac{2}{\sqrt{1+4t}}$$
$$= \frac{(\sqrt{1+4t})\frac{d}{dt}(2) - 2\frac{d}{dt}\sqrt{1+4t}}{(\sqrt{1+4t})^2}$$
$$= \frac{-2\left(\frac{1}{2\sqrt{1+4}}\right)\frac{d}{dt}(1+4t)}{1+4t}$$
$$= \frac{\frac{-4}{\sqrt{1+4t}}}{1+4t}$$
$$= -\frac{4}{(1+4t)^{3/2}}$$

At $t = 6$, the acceleration is $-\frac{4}{[1+4(6)]^{3/2}} = -\frac{4}{125}$ m/sec^2

64. Acceleration $$= \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \left(\frac{dv}{ds}\right)(v) = \left[\frac{d}{ds}(k\sqrt{s})\right](k\sqrt{s})$$
$$= \left(\frac{k}{2\sqrt{s}}\right)(k\sqrt{s}) = \frac{k^2}{2}, \text{ a constant.}$$

65. Note that this exercise concerns itself with the slowing down caused by the earth's atmosphere, *not* the acceleration caused by gravity.

Given: $v = \frac{k}{\sqrt{s}}$

$$\text{Acceleration} = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \left(\frac{dv}{ds}\right)(v) = (v)\left(\frac{dv}{ds}\right) = \left(\frac{k}{\sqrt{s}}\right)\frac{d}{ds}\frac{k}{\sqrt{s}} = \left(\frac{k}{\sqrt{s}}\right)\left(\frac{\sqrt{s}\,\frac{d}{ds}(k) - k\,\frac{d}{ds}\sqrt{s}}{(\sqrt{s})^2}\right) = \left(\frac{k}{\sqrt{s}}\right)\left(\frac{\frac{-k}{(2\sqrt{x})}}{s}\right) = -\frac{k^2}{2s^2},\ s \geq 0$$

Thus, the acceleration is inversely proportional to s^2.

66. Acceleration

$$= \frac{dv}{dt} = \frac{df(x)}{dt} = \frac{df(x)}{dx}\frac{dx}{dt} = f'(x)f(x)$$

67. $$\frac{dT}{du} = \frac{dT}{dL}\frac{dL}{du} = \left(\frac{d}{dL}2\pi\sqrt{\frac{L}{g}}\right)(kL) = \left(2\pi\frac{1}{2\sqrt{\frac{L}{g}}}\right)\left(\frac{d}{dL}\frac{L}{g}\right)(kL) = \left(\frac{\pi}{\sqrt{\frac{L}{g}}}\right)\left(\frac{1}{g}\right)(kL) = k\pi\sqrt{\frac{L}{g}} = \frac{kT}{2}$$

68. No, this does not contradict the Chain Rule. The Chain Rule states that if two functions are differentiable at the appropriate points, then their composite must also be differentiable. It does not say: If a composite is differentiable, then the functions which make up the composite must all be differentiable.

69. Yes; note that $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, then $f'(g(1))g'(1) = 0$, so either $g'(1) = 0$ or $f'(g(1)) = 0$. This means that either the graph of $y = g(x)$ has a horizontal tangent at $x = 1$, or the graph of $y = f(u)$ has a horizontal tangent at $u = g(1)$.

70. False; see example 8.

71. False. $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t}{-3\sin t} = -\cot(t)$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = -\cot\left(\frac{\pi}{4}\right) = -1$$

normal slope: $m_2 = -\frac{1}{m_1} = -\frac{1}{-1} = 1$

The slope of the normal is +1.

72. E; $\frac{dy}{dx} = \frac{d}{dx}\tan(4x)$

$y = \tan u \qquad u = 4x$

$\frac{dy}{du} = \sec^2 u \qquad \frac{du}{dx} = 4$

$\frac{dy}{dx} = 4\sec^2(4x)$

73. C; $\frac{dy}{dx} = \frac{d}{dx}\cos^2(x^3 + x^2)$

$y = \cos^2 u \qquad u = x^3 + x^2$

$\frac{dy}{du} = -2\sin u\cos u \qquad \frac{du}{dx} = 3x^2 + 2x$

$\frac{dy}{dx} = -2(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$

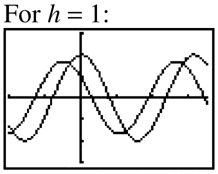
74. A; $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\frac{dy}{dt} = \frac{d}{dt}(-1+\sin t) = \cos t$$
$$\frac{dx}{dt} = \frac{d}{dt}(t-\cos t) = 1+\sin t$$
$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{1+\sin t}$$
$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{\cos 0}{1+\sin 0} = 1$$
$$x(0) = 0-\cos 0 = -1$$
$$y(0) = -1+\sin 0 = -1$$
$$y = 1(x-(-1))+(-1) = x$$

75. B; see problem 74.

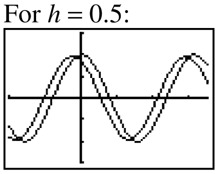
$$\frac{dy}{dx} = \frac{\cos t}{1+\sin t} = 0$$
$$\cos t = 0$$
$$t = \frac{\pi}{2}$$

76. For $h = 1$:



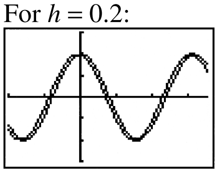
[–2, 3.5] by [–3, 3]

For $h = 0.5$:



[–2, 3.5] by [–3, 3]

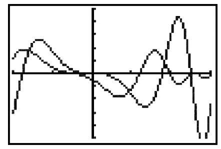
For $h = 0.2$:



[–2, 3.5] by [–3, 3]

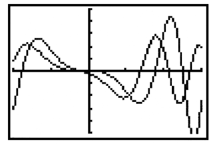
As $h \to 0$, the second curve (the difference quotient) approaches the first ($y = 2\cos 2x$). This is because $2\cos 2x$ is the derivative of $\sin 2x$, and the second curve is the difference quotient used to define the derivative of $\sin 2x$. As $h \to 0$, the difference quotient expression should be approaching the derivative.

77. For $h = 1$:



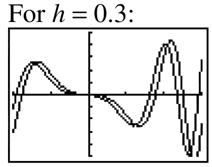
[–2, 3] by [–5, 5]

For $h = 0.3$:



[–2, 3] by [–5, 5]

For $h = 0.3$:



[–2, 3] by [–5, 5]

As $h \to 0$, the second curve (the difference quotient) approaches the first ($y = -2x\sin(x^2)$). This is because $-2x\sin(x^2)$ is the derivative of $\cos(x^2)$, and the second curve is the difference quotient used to define the derivative of $\cos(x^2)$. As $h \to 0$, the difference quotient expression should be approaching the derivative.

78. **(a)** Let $f(x) = |x| = \begin{cases} -x & x<0 \\ x & x \ge 0 \end{cases}$

Then $f'(x) = \frac{x}{|x|} = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases}$ and

$$\frac{d}{dx}|u| = \frac{d}{dx}f(u) = f'(u)\frac{du}{dx} = \frac{u}{|u|}\frac{du}{dx}$$

(b) $f'(x) = \left[\frac{d}{dx}(x^2-9)\right]\cdot\frac{x^2-9}{|x^2-9|}$

$$= \frac{(2x)(x^2-9)}{|x^2-9|}$$

$$g'(x) = \frac{d}{dx}(|x|\sin x)$$
$$= |x|\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}|x|$$
$$= |x|\cos x + \frac{x\sin x}{|x|}$$

Note: The expression for $g'(x)$ above is undefined at $x = 0$, but actually

$$g'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h}$$
$$= \lim_{h\to 0} \frac{|h|\sin h}{h}$$
$$= 0.$$

Therefore, we may express the derivative

as $g'(x) = \begin{cases} |x|\cos x + \dfrac{x\sin x}{|x|}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

79. $\dfrac{dG}{dx} = \dfrac{d}{dx}\sqrt{uv}$

$$= \frac{d}{dx}\sqrt{x(x+c)}$$
$$= \frac{d}{dx}\sqrt{x^2+cx}$$
$$= \frac{1}{2\sqrt{x^2+cx}}\frac{d}{dx}(x^2+cx)$$
$$= \frac{2x+c}{2\sqrt{x^2+cx}}$$
$$= \frac{x+(x+c)}{2\sqrt{x(x+c)}}$$
$$= \frac{u+v}{2\sqrt{uv}}$$
$$= \frac{A}{G}$$

Section 4.2 Implicit Differentiation (pp. 164–172)

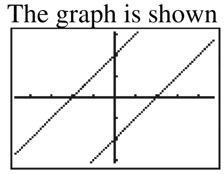
Exploration 1 An Unexpected Derivative

1. $2x - 2y - 2xy' + 2yy' = 0$. Solving for y', we find that $\dfrac{dy}{dx} = 1$ (provided $y \neq x$).

2. With a constant derivative of 1, the graph would seem to be a line with slope 1.

3. Letting $x = 0$ in the original equation, we find that $y = \pm 2$. This would seem to indicate that this equation defines two lines implicitly, both with slope 1. The two lines are $y = x + 2$ and $y = x - 2$.

4. Factoring the original equation, we have
$[(x-y)-2][(x-y)+2] = 0$
$\therefore x - y - 2 = 0$ or $x - y + 2 = 0$
$\therefore y = x - 2$ or $y = x + 2$.
The graph is shown below.



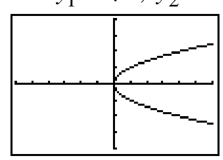
[−4.7, 4.7] by [−3.1, 3.1]

5. At each point (x, y) on either line, $\dfrac{dy}{dx} = 1$. The condition $y \neq x$ is true because both lines are parallel to the line $y = x$. The derivative is surprising because it does not depend on x or y, but there are no inconsistencies.

Quick Review 4.2

1. $x - y^2 = 0$

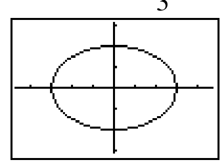
$$x = y^2$$
$$\pm\sqrt{x} = y$$
$$y_1 = \sqrt{x},\ y_2 = -\sqrt{x}$$



[-6, 6] by [-4, 4]

2. $4x^2 + 9y^2 = 36$

$$9y^2 = 36 - 4x^2$$
$$y^2 = \frac{36-4x^2}{9} = \frac{4}{9}(9-x^2)$$
$$y = \pm\frac{2}{3}\sqrt{9-x^2}$$
$$y_1 = \frac{2}{3}\sqrt{9-x^2},\ y_2 = -\frac{2}{3}\sqrt{9-x^2}$$



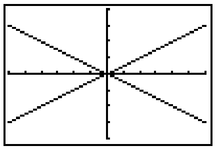
[-4.7, 4.7] by [-3.1, 3.1]

3. $x^2 - 4y^2 = 0$

$(x+2y)(x-2y) = 0$

$y = \pm\frac{x}{2}$

$y_1 = \frac{x}{2},\ y_2 = -\frac{x}{2}$



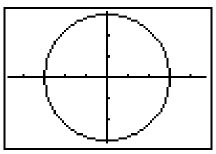
[-6, 6] by [-4, 4]

4. $x^2 + y^2 = 9$

$y^2 = 9 - x^2$

$y = \pm\sqrt{9-x^2}$

$y_1 = \sqrt{9-x^2},\ y_2 = -\sqrt{9-x^2}$



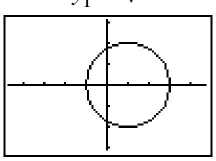
[-4.7, 4.7] by [-3.1, 3.1]

5. $x^2 + y^2 = 2x + 3$

$y^2 = 2x + 3 - x^2$

$y = \pm\sqrt{2x+3-x^2}$

$y_1 = \sqrt{2x+3-x^2},\ y_2 = -\sqrt{2x+3-x^2}$



[-4.7, 4.7] by [-3.1, 3.1]

6. $x^2y' - 2xy = 4x - y$

$x^2y' = 4x - y + 2xy$

$y' = \frac{4x - y + 2xy}{x^2}$

7. $y'\sin x - x\cos x = xy' + y$

$y'\sin x - xy' = y + x\cos x$

$(\sin x - x)y' = y + x\cos x$

$y' = \frac{y + x\cos x}{\sin x - x}$

8. $x(y^2 - y') = y'(x^2 - y)$

$xy^2 = y'(x^2 - y + x)$

$y' = \frac{xy^2}{x^2 - y + x}$

9. $\sqrt{x}(x - \sqrt[3]{x}) = x^{1/2}(x - x^{1/3})$

$= x^{1/2}x - x^{1/2}x^{1/3}$

$= x^{3/2} - x^{5/6}$

10. $\frac{x + \sqrt[3]{x^2}}{\sqrt{x^3}} = \frac{x + x^{2/3}}{x^{3/2}}$

$= \frac{x}{x^{3/2}} + \frac{x^{2/3}}{x^{3/2}}$

$= x^{-1/2} + x^{-5/6}$

Section 4.2 Exercises

1. $x^2y + xy^2 = 6$

$\frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) = \frac{d}{dx}(6)$

$x^2\frac{dy}{dx} + y(2x) + x(2y)\frac{dy}{dx} + y^2(1) = 0$

$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -(2xy + y^2)$

$(2xy + x^2)\frac{dy}{dx} = -(2xy + y^2)$

$\frac{dy}{dx} = -\frac{2xy + y^2}{2xy + x^2}$

2. $x^3 + y^3 = 18xy$

$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(18xy)$

$3x^2 + 3y^2\frac{dy}{dx} = 18x\frac{dy}{dx} + 18y(1)$

$3y^2\frac{dy}{dx} - 18x\frac{dy}{dx} = 18y - 3x^2$

$(3y^2 - 18x)\frac{dy}{dx} = 18y - 3x^2$

$\frac{dy}{dx} = \frac{18y - 3x^2}{3y^2 - 18x}$

$\frac{dy}{dx} = \frac{6y - x^2}{y^2 - 6x}$

3. $$y^2 = \frac{x-1}{x+1}$$
$$\frac{d}{dx} y^2 = \frac{d}{dx}\frac{x-1}{x+1}$$
$$2y\frac{dy}{dx} = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$$
$$2y\frac{dy}{dx} = \frac{2}{(x+1)^2}$$
$$\frac{dy}{dx} = \frac{1}{y(x+1)^2}$$

4. $$x^2 = \frac{x-y}{x+y}$$
$$\frac{d}{dx}(x^2) = \frac{d}{dx}\frac{x-y}{x+y}$$
$$2x = \frac{(x+y)\left(1-\frac{dy}{dx}\right)-(x-y)\left(1+\frac{dy}{dx}\right)}{(x+y)^2}$$
$$2x = \frac{\left[x-x\frac{dy}{dx}+y-y\frac{dy}{dx}\right]-\left[x+x\frac{dy}{dx}-y-y\frac{dy}{dx}\right]}{(x+y)^2}$$
$$2x = \frac{2y-2x\frac{dy}{dx}}{(x+y)^2}$$
$$x(x+y)^2 = y - x\frac{dy}{dx}$$
$$x\frac{dy}{dx} = y - x(x+y)^2$$
$$\frac{dy}{dx} = \frac{y-x(x+y)^2}{x} = \frac{y}{x}-(x+y)^2$$

Alternate solution:
$$x^2 = \frac{x-y}{x+y}$$
$$x^2(x+y) = x-y$$
$$x^3 + x^2y = x-y$$
$$\frac{d}{dx}(x^3)+\frac{d}{dx}(x^2y) = \frac{d}{dx}(x)-\frac{d}{dx}(y)$$
$$3x^2 + x^2\frac{dy}{dx} + y(2x) = 1-\frac{dy}{dx}$$
$$(x^2+1)\frac{dy}{dx} = 1-3x^2-2xy$$
$$\frac{dy}{dx} = \frac{1-3x^2-2xy}{x^2+1}$$

5. $$x = \tan y$$
$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$$
$$1 = \sec^2 y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$$

6. $$x = \sin y$$
$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$
$$1 = \cos y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\cos y} = \sec y$$

7. $$x + \tan xy = 0$$
$$\frac{d}{dx}(x)+\frac{d}{dx}(\tan xy) = \frac{d}{dx}(0)$$
$$1+\sec^2(xy)\frac{d}{dx}(xy) = 0$$
$$1+(\sec^2 xy)\left[x\frac{dy}{dx}+(y)(1)\right] = 0$$
$$(\sec^2 xy)(x)\frac{dy}{dx} = -1-(\sec^2 xy)(y)$$
$$\frac{dy}{dx} = \frac{-1-y\sec^2 xy}{x\sec^2 xy}$$
$$\frac{dy}{dx} = -\frac{1}{x}\cos^2 xy - \frac{y}{x}$$

8. $$x + \sin y = xy$$
$$\frac{d}{dx}(x)+\frac{d}{dx}(\sin y) = \frac{d}{dx}(xy)$$
$$1+(\cos y)\frac{dy}{dx} = x\frac{dy}{dx}+(y)(1)$$
$$(\cos y - x)\frac{dy}{dx} = -1+y$$
$$\frac{dy}{dx} = \frac{-1+y}{\cos y - x} = \frac{1-y}{x-\cos y}$$

9. $$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(13)$$
$$2x+2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y},\ -\frac{-2}{3} = \frac{2}{3}$$

10. $\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(9)$

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y},\ -\frac{0}{3}=0$$

11. $\frac{d}{dx}((x-1)^2+(y-1)^2)=\frac{d}{dx}(13)$

$$2(x-1)1+(2(y-1)1)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x-1}{y-1},-\frac{3-1}{4-1}=-\frac{2}{3}$$

12. $\frac{d}{dx}((x+2)^2+(y+3)^2)=\frac{d}{dx}(25)$

$$2(x+2)1+(2(y+3)1)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x+2}{y+3},-\frac{1+2}{-7+3}=\frac{3}{4}$$

13. $\frac{d}{dx}(x^2y-xy^2)=\frac{d}{dx}(4)$

$$x^2\frac{dy}{dx}+y\cdot 2x-\left(x\cdot 2y\frac{dy}{dx}+y^2\right)=0$$

$$(x^2-2xy)\frac{dy}{dx}+2xy-y^2=0$$

$$(x^2-2xy)\frac{dy}{dx}=y^2-2xy$$

$$\frac{dy}{dx}=\frac{y^2-2xy}{x^2-2xy}$$

defined at every point except where $x=0$ or $y=\frac{x}{2}$.

14. $\frac{d}{dx}(x)=\frac{d}{dx}(\cos y)$

$$1=-\sin y\frac{dy}{dx}$$

$$\frac{dy}{dx}=-\frac{1}{\sin y},$$

defined everywhere except where $\sin y=0$:
$y=\pm k\pi$
$x=\cos(k\pi)$
$x=1$ or -1

15. $\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(xy)$

$$3x^2+3y^2\frac{dy}{dx}=y+x\frac{dx}{dy}$$

$$3x^2-y=(x-3y^2)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{3x^2-y}{x-3y^2},$$

defined everywhere except where $y^2=\frac{x}{3}$

16. $\frac{d}{dx}(x^2+4xy+4y^2-3x)=\frac{d}{dx}(6)$

$$2x+4y-3+(4x+8y)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{3-2x-4y}{4x+8y},$$

defined everywhere except where $y=-\frac{1}{2}x$

17. $x^2+xy-y^2=1$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(xy)-\frac{d}{dx}(y^2)=\frac{d}{dx}(1)$$

$$2x+x\frac{dy}{dx}+(y)(1)-2y\frac{dy}{dx}=0$$

$$(x-2y)\frac{dy}{dx}=-2x-y$$

$$\frac{dy}{dx}=\frac{-2x-y}{x-2y}=\frac{2x+y}{2y-x}$$

Slope at (2, 3): $\frac{2(2)+3}{2(3)-2}=\frac{7}{4}$

(a) Tangent: $y-3=\frac{7}{4}(x-2)$

(b) Normal: $y-3=-\frac{4}{7}(x-2)$

18. $x^2+y^2=25$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(25)$$

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

Slope at $(3,-4):-\frac{3}{-4}=\frac{3}{4}$

(a) Tangent: $y+4=\frac{3}{4}(x-3)$

(b) Normal: $y+4=-\frac{4}{3}(x-3)$

19.
$$x^2y^2=9$$
$$\frac{d}{dx}(x^2y^2)=\frac{d}{dx}(9)$$
$$(x^2)(2y)\frac{dy}{dx}+(y^2)(2x)=0$$
$$2x^2y\frac{dy}{dx}=-2xy^2$$
$$\frac{dy}{dx}=-\frac{2xy^2}{2x^2y}=-\frac{y}{x}$$

Slope at $(-1, 3)$: $-\frac{3}{-1}=3$

(a) Tangent: $y-3=3(x+1)$

(b) Normal: $y-3=-\frac{1}{3}(x+1)$

20.
$$y^2-2x-4y-1=0$$
$$\frac{d}{dx}(y^2)-\frac{d}{dx}(2x)-\frac{d}{dx}(4y)-\frac{d}{dx}(1)=\frac{d}{dx}(0)$$
$$2y\frac{dy}{dx}-2-4\frac{dy}{dx}-0=0$$
$$(2y-4)\frac{dy}{dx}=2$$
$$\frac{dy}{dx}=\frac{1}{y-2}$$

Slope at $(-2, 1)$: $\frac{1}{1-2}=-1$

(a) Tangent: $y-1=-1(x+2)$

(b) Normal: $y-1=1(x+2)$

21.
$$6x^2+3xy+2y^2+17y-6=0$$
$$\frac{d}{dx}(6x^2)+\frac{d}{dx}(3xy)+\frac{d}{dx}(2y^2)+\frac{d}{dx}(17y)-\frac{d}{dx}(6)=\frac{d}{dx}(0)$$
$$12x+3x\frac{dy}{dx}+(3y)(1)+4y\frac{dy}{dx}+17\frac{dy}{dx}-0=0$$
$$3x\frac{dy}{dx}+4y\frac{dy}{dx}+17\frac{dy}{dx}=-12x-3y$$
$$(3x+4y+17)\frac{dy}{dx}=-12x-3y$$
$$\frac{dy}{dx}=\frac{-12x-3y}{3x+4y+17}$$

Slope at $(-1, 0)$: $\dfrac{-12(-1)-3(0)}{3(-1)+4(0)+17}=\dfrac{12}{14}=\dfrac{6}{7}$

(a) Tangent: $y=\dfrac{6}{7}(x+1)$

(b) Normal: $y=-\dfrac{7}{6}(x+1)+0$

22.
$$x^2-\sqrt{3}xy+2y^2=5$$
$$\frac{d}{dx}(x^2)-\sqrt{3}\frac{d}{dx}(xy)+2\frac{d}{dx}(y^2)=\frac{d}{dx}(5)$$
$$2x-\sqrt{3}(x)\frac{dy}{dx}-\sqrt{3}(y)(1)+4y\frac{dy}{dx}=0$$
$$\left(-x\sqrt{3}+4y\right)\frac{dy}{dx}=y\sqrt{3}-2x$$
$$\frac{dy}{dx}=\frac{y\sqrt{3}-2x}{-x\sqrt{3}+4y}$$

Slope at $(\sqrt{3}, 2)$: $\dfrac{2\sqrt{3}-2\sqrt{3}}{-\sqrt{3}\sqrt{3}+4(2)}=0$

(a) Tangent: $y=2$

(b) Normal: $x=\sqrt{3}$

23.
$$2xy+\pi\sin y=2\pi$$
$$2\frac{d}{dx}(xy)+\pi\frac{d}{dx}(\sin y)=\frac{d}{dx}(2\pi)$$
$$2x\frac{dy}{dx}+2y(1)+\pi\cos y\frac{dy}{dx}=0$$
$$(2x+\pi\cos y)\frac{dy}{dx}=-2y$$
$$\frac{dy}{dx}=-\frac{2y}{2x+\pi\cos y}$$

Slope at $\left(1, \dfrac{\pi}{2}\right)$: $-\dfrac{2\left(\frac{\pi}{2}\right)}{2(1)+\pi\cos\left(\frac{\pi}{2}\right)}=-\dfrac{\pi}{2}$

(a) Tangent: $y-\frac{\pi}{2}=-\frac{\pi}{2}(x-1)$

(b) Normal: $y-\frac{\pi}{2}=\frac{2}{\pi}(x-1)$

24.

$$x\sin 2y = y\cos 2x$$

$$\frac{d}{dx}(x\sin 2y)=\frac{d}{dx}(y\cos 2x)$$

$$(x)(\cos 2y)(2)\frac{dy}{dx}+(\sin 2y)(1)=(y)(-\sin 2x)\,(2)+(\cos 2x)\left(\frac{dy}{dx}\right)$$

$$(2x\cos 2y)\frac{dy}{dx}-(\cos 2x)\frac{dy}{dx}=-2y\,\sin 2x-\sin 2y$$

$$\frac{dy}{dx}=-\frac{2y\,\sin 2x+\sin 2y}{2x\,\cos 2y-\cos 2x}$$

$$(2x\cos 2y)\frac{dy}{dx}-(\cos 2x)\frac{dy}{dx}=-2y\,\sin 2x-\sin 2y$$

$$\frac{dy}{dx}=-\frac{2y\,\sin 2x+\sin 2y}{2x\,\cos 2y-\cos 2x}$$

Slope at $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$: $-\frac{2\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)+\sin(\pi)}{2\left(\frac{\pi}{4}\right)\cos(\pi)-\cos\left(\frac{\pi}{2}\right)}$

$$=-\frac{(\pi)(1)+0}{\left(\frac{\pi}{2}\right)(-1)-0}=2$$

(a) Tangent: $y-\frac{\pi}{2}=2\left(x-\frac{\pi}{4}\right)$

(b) Normal: $y-\frac{\pi}{2}=-\frac{1}{2}\left(x-\frac{\pi}{4}\right)$

25.

$$y=2\sin(\pi x-y)$$

$$\frac{dy}{dx}=\frac{d}{dx}2\sin(\pi x-y)$$

$$\frac{dy}{dx}=2\cos(\pi x-y)\left(\pi-\frac{dy}{dx}\right)$$

$$[1+2\cos(\pi x-y)]\frac{dy}{dx}=2\pi\cos(\pi x-y)$$

$$\frac{dy}{dx}=\frac{2\pi\cos(\pi x-y)}{1+2\cos(\pi x-y)}$$

Slope at $(1,\,0)$: $\frac{2\pi\cos\pi}{1+2\cos\pi}=\frac{2\pi(-1)}{1+2(-1)}=2\pi$

(a) Tangent: $y = 2\pi(x-1)$

(b) Normal: $y=-\frac{1}{2\pi}(x-1)+0$

26.

$$x^2\cos^2 y-\sin y=0$$

$$\frac{d}{dx}(x^2\cos^2 y)-\frac{d}{dx}(\sin y)=\frac{d}{dx}(0)$$

$$(x^2)(2\cos y)(-\sin y)\left(\frac{dy}{dx}\right)+(\cos^2 y)(2x)-(\cos y)\frac{dy}{dx}=0$$

$$-(2x^2\cos y\sin y+\cos y)\frac{dy}{dx}=-2x\cos^2 y$$

$$\frac{dy}{dx}=\frac{2x\cos^2 y}{\cos y+2x^2\cos y\sin y}=\frac{2x\cos y}{1+2x^2\sin y}$$

Slope at $(0,\pi)$: $\dfrac{2(0)\cos\pi}{1+2(0)^2\sin\pi}=0$

(a) Tangent: $y=\pi$

(b) Normal: $x=0$

27.

$$x^2+y^2=1$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(1)$$

$$2x+2yy'=0$$

$$2yy'=-2x$$

$$y'=-\frac{x}{y}$$

$$\begin{aligned}y''&=\frac{d}{dx}\left(-\frac{x}{y}\right)\\&=-\frac{(y)(1)-(x)(y')}{y^2}\\&=-\frac{y-x\left(-\frac{x}{y}\right)}{y^2}\\&=-\frac{x^2+y^2}{y^3}\end{aligned}$$

Since our original equation was $x^2+y^2=1$, we may substitute 1 for x^2+y^2, giving $y''=-\dfrac{1}{y^3}$.

28.

$$x^{2/3}+y^{2/3}=1$$

$$\frac{d}{dx}(x^{2/3})+\frac{d}{dx}(y^{2/3})=\frac{d}{dx}(1)$$

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0$$

$$y'=-\frac{x^{-1/3}}{y^{-1/3}}=-\left(\frac{y}{x}\right)^{1/3}$$

$$y'' = \frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{1/3}\right]$$
$$= -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\frac{d}{dx}\left(\frac{y}{x}\right)$$
$$= -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\frac{xy' - (y)(1)}{x^2}$$
$$= -\frac{1}{3}\frac{-(x)\left(\frac{y}{x}\right)^{1/3} - y}{x^{4/3}y^{2/3}}$$
$$= \frac{1}{3}\frac{x^{2/3}y^{1/3} + y}{x^{4/3}y^{2/3}}$$
$$= \frac{x^{2/3} + y^{2/3}}{3x^{4/3}y^{1/3}}$$

Since our original equation was $x^{2/3} + y^{2/3} = 1$, we may substitute 1 for $x^{2/3} + y^{2/3}$, giving $y'' = \frac{1}{3x^{4/3}y^{1/3}}$.

29. $$y^2 = x^2 + 2x$$
$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2x)$$
$$2yy' = 2x + 2$$
$$y' = \frac{2x+2}{2y} = \frac{x+1}{y}$$
$$y'' = \frac{d}{dx}\left(\frac{x+1}{y}\right)$$
$$= \frac{(y)(1) - (x+1)y'}{y^2}$$
$$= \frac{y - (x+1)\left(\frac{x+1}{y}\right)}{y^2}$$
$$= \frac{y^2 - (x+1)^2}{y^3}$$

Since our original equation was $y^2 = x^2 + 2x$, we may write $y^2 - (x+1)^2 = (x^2 + 2x) - (x^2 + 2x + 1) = -1$, which gives $y'' = -\frac{1}{y^3}$.

30. $$y^2 + 2y = 2x + 1$$
$$\frac{d}{dx}(y^2 + 2y) = \frac{d}{dx}(2x+1)$$
$$(2y+2)y' = 2$$
$$y' = \frac{1}{y+1}$$
$$y'' = \frac{d}{dx}\frac{1}{y+1}$$
$$= -(y+1)^{-2}y'$$
$$= -(y+1)^{-2}\left(\frac{1}{y+1}\right)$$
$$= -\frac{1}{(y+1)^3}$$

31. $\frac{dy}{dx} = \frac{d}{dx}x^{9/4} = \frac{9}{4}x^{(9/4)-1} = \frac{9}{4}x^{5/4}$

32. $\frac{dy}{dx} = \frac{d}{dx}x^{-3/5} = -\frac{3}{5}x^{(-3/5)-1} = -\frac{3}{5}x^{-8/5}$

33. $\frac{dy}{dx} = \frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx}x^{1/3} = \frac{1}{3}x^{(1/3)-1} = \frac{1}{3}x^{-2/3}$

34. $\frac{dy}{dx} = \frac{d}{dx}\sqrt[4]{x} = \frac{d}{dx}x^{1/4} = \frac{1}{4}x^{(1/4)-1} = \frac{1}{4}x^{-3/4}$

35. $$\frac{dy}{dx} = \frac{d}{dx}(2x+5)^{-1/2}$$
$$= -\frac{1}{2}(2x+5)^{(-1/2)-1}\frac{d}{dx}(2x+5)$$
$$= -\frac{1}{2}(2x+5)^{-3/2}(2)$$
$$= -(2x+5)^{-3/2}$$

36. $$\frac{dy}{dx} = \frac{d}{dx}(1-6x)^{2/3}$$
$$= \frac{2}{3}(1-6x)^{(2/3)-1}\frac{d}{dx}(1-6x)$$
$$= \frac{2}{3}(1-6x)^{-1/3}(-6)$$
$$= -4(1-6x)^{-1/3}$$

37. $\frac{dy}{dx} = \frac{d}{dx}\left(x\sqrt{x^2+1}\right)$

$= x\frac{d}{dx}\sqrt{x^2+1} + \sqrt{x^2+1}\frac{d}{dx}(x)$

$= x\frac{d}{dx}(x^2+1)^{1/2} + (x^2+1)^{1/2}$

$= x \bullet \frac{1}{2}(x^2+1)^{-1/2}(2x) + (x^2+1)^{1/2}$

$= x^2(x^2+1)^{-1/2} + (x^2+1)^{1/2}$

Note: This answer is equivalent to $\frac{2x^2+1}{\sqrt{x^2+1}}$.

38. $\frac{dy}{dx} = \frac{d}{dx}\frac{x}{\sqrt{x^2+1}}$

$= \frac{(x^2+1)^{1/2}\frac{d}{dx}x - x\frac{d}{dx}(x^2+1)^{1/2}}{x^2+1}$

$= \frac{(x^2+1)^{1/2} - x\cdot\frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1}$

$= \frac{x^2+1-x^2}{(x^2+1)(x^2+1)^{1/2}}$

$= \frac{1}{(x^2+1)^{3/2}}$

$= (x^2+1)^{-3/2}$

39. $\frac{dy}{dx} = \frac{d}{dx}(1-x^{1/2})^{1/2}$

$= \frac{1}{2}(1-x^{1/2})^{-1/2}\frac{d}{dx}(1-x^{1/2})$

$= \frac{1}{2}(1-x^{1/2})^{-1/2}\left(-\frac{1}{2}x^{-1/2}\right)$

$= -\frac{1}{4}(1-x^{1/2})^{-1/2}x^{-1/2}$

40. $\frac{dy}{dx} = \frac{d}{dx}3(2x^{-1/2}+1)^{-1/3}$

$= -(2x^{-1/2}+1)^{-4/3}\frac{d}{dx}(2x^{-1/2}+1)$

$= -(2x^{-1/2}+1)^{-4/3}(-x^{-3/2})$

$= x^{-3/2}(2x^{-1/2}+1)^{-4/3}$

41. $\frac{dy}{dx} = \frac{d}{dx}3(\csc x)^{3/2}$

$= \frac{9}{2}(\csc x)^{1/2}\frac{d}{dx}(\csc x)$

$= \frac{9}{2}(\csc x)^{1/2}(-\csc x\cot x)$

$= -\frac{9}{2}(\csc x)^{3/2}\cot x$

42. $\frac{dy}{dx} = \frac{d}{dx}[\sin(x+5)]^{5/4}$

$= \frac{5}{4}[\sin(x+5)]^{1/4}\frac{d}{dx}\sin(x+5)$

$= \frac{5}{4}[\sin(x+5)]^{1/4}\cos(x+5)$

43. **(a)** If $f(x) = \frac{3}{2}x^{2/3} - 3$, then $f'(x) = x^{-1/3}$ and $f''(x) = -\frac{1}{3}x^{-4/3}$ which contradicts the given equation $f''(x) = x^{-1/3}$.

(b) If $f(x) = \frac{9}{10}x^{5/3} - 7$, then $f'(x) = \frac{3}{2}x^{2/3}$ and $f''(x) = x^{-1/3}$, which matches the given equation.

(c) Differentiating both sides of the given equation $f''(x) = x^{-1/3}$ gives $f'''(x) = -\frac{1}{3}x^{-4/3}$, so it *must* be true that $f'''(x) = -\frac{1}{3}x^{-4/3}$.

(d) If $f'(x) = \frac{3}{2}x^{2/3} + 6$, then $f''(x) = x^{-1/3}$, which matches the given equation.

Conclusion: (b), (c), and (d) could be true.

44. **(a)** If $g'(t) = 4\sqrt[4]{t} - 4$, then

$g''(t) = \frac{d}{dx}(4t^{1/4} - 4) = t^{-3/4} = \frac{1}{t^{3/4}}$,

which matches the given equation.

(b) Differentiating both sides of the given equation $g''(t)=\frac{1}{t^{3/4}}=t^{-3/4}$ then $g'''(t)=-\frac{3}{4}t^{-7/4}$, which is not consistent with $g'''(t)=-\frac{4}{\sqrt[4]{t}}$.

(c) If $g(t)=t-7+\frac{16}{5}t^{5/4}$, then $g'(t)=1+4t^{1/4}$ and $g''(t)=t^{-3/4}=\frac{1}{t^{3/4}}$, which matches the given equation.

(d) If $g'(t)=\frac{1}{4}t^{1/4}$, then $g''(t)=\frac{1}{16}t^{-3/4}$, which contradicts the given equation.

Conclusion: (a) and (c) could be true.

45. (a)

$$y^4=y^2-x^2$$
$$\frac{d}{dx}(y^4)=\frac{d}{dx}(y^2)-\frac{d}{dx}x^2$$
$$4y^3\frac{dy}{dx}=2y\frac{dy}{dx}-2x$$
$$(4y^3-2y)\frac{dy}{dx}=-2x$$
$$\frac{dy}{dx}=\frac{-2x}{4y^3-2y}=\frac{x}{y-2y^3}$$

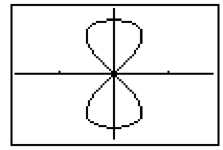
At $\left(\frac{\sqrt{3}}{4},\frac{\sqrt{3}}{2}\right)$:

$$\text{Slope}=\frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2}-2\left(\frac{\sqrt{3}}{2}\right)^3}$$
$$=\frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2}-\frac{3\sqrt{3}}{4}}\cdot\frac{\frac{4}{\sqrt{3}}}{\frac{4}{\sqrt{3}}}=\frac{1}{2-3}=-1$$

At $\left(\frac{\sqrt{3}}{4},\frac{1}{2}\right)$:

$$\text{Slope}=\frac{\frac{\sqrt{3}}{4}}{\frac{1}{2}-2\left(\frac{1}{2}\right)^3}=\frac{\frac{\sqrt{3}}{4}}{\frac{1}{2}-\frac{1}{4}}\cdot\frac{4}{4}=\frac{\sqrt{3}}{1}=\sqrt{3}$$

(b)



[–1.8, 1.8] by [–1.2, 1.2]

Parameter interval: $-1\le t\le 1$

46. (a)

$$y^2(2-x)=x^3$$
$$\frac{d}{dx}\left[y^2(2-x)\right]=\frac{d}{dx}(x^3)$$
$$(y^2)(-1)+(2-x)(2y)\frac{dy}{dx}=3x^2$$
$$2y(2-x)\frac{dy}{dx}=3x^2+y^2$$
$$\frac{dy}{dx}=\frac{3x^2+y^2}{2y(2-x)}$$

Slope at (1, 1): $\frac{3(1)^2+(1)^2}{2(1)(2-1)}=\frac{4}{2}=2$

Tangent: $y-1=2(x-1)$

Normal: $y-1=-\frac{1}{2}(x-1)$

(b) One way is to graph the equations $y=\pm\sqrt{\frac{x^3}{2-x}}$.

47. (a) $(-1)^3(1)^2=\cos(\pi)$ is true since both sides equal -1.

(b)

$$x^3y^2=\cos(\pi y)$$
$$\frac{d}{dx}(x^3y^2)=\frac{d}{dx}\cos(\pi y)$$
$$(x^3)(2y)\frac{dy}{dx}+(y^2)(3x^2)=(-\sin\pi y)(\pi)\frac{dy}{dx}$$
$$(2x^3y+\pi\sin\pi y)\frac{dy}{dx}=-3x^2y^2$$
$$\frac{dy}{dx}=-\frac{3x^2y^2}{2x^3y+\pi\sin\pi y}$$

Slope at (–1, 1):

$$-\frac{3(-1)^2(1)}{2(-1)^3(1)+\pi\sin\pi}=\frac{-3}{-2}=\frac{3}{2}$$

The slope of the tangent line is $\frac{3}{2}$.

48. **(a)** When $x = 2$, we have $y^3 - 2y = -1$, or $y^3 - 2y + 1 = 0$. Clearly, $y = 1$ is one solution, and we may factor $y^3 - 2y + 1$ as $(y-1)(y^2 + y - 1)$. The solutions of $y^2 + y - 1 = 0$ are

$$y = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}.$$

Hence, there are three possible y-values: $1, \frac{-1-\sqrt{5}}{2}$, and $\frac{-1+\sqrt{5}}{2}$.

(b)

$$y^3 - xy = -1$$

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(xy) = \frac{d}{dx}(-1)$$

$$3y^2 y' - xy' - (y)(1) = 0$$

$$(3y^2 - x)y' = y$$

$$y' = \frac{y}{3y^2 - x}$$

$$y'' = \frac{d}{dx}\frac{y}{3y^2 - x}$$

$$= \frac{(3y^2 - x)(y') - (y)(6y\,y' - 1)}{(3y^2 - x)^2}$$

$$= \frac{y - xy' - 3y^2 y'}{(3y^2 - x)^2}$$

Since we are working with numerical information, there is no need to write a general expression for y'' in terms of x and y.

To evaluate $f'(2)$, evaluate the expression for y' using $x = 2$ and $y = 1$:

$$f'(2) = \frac{1}{3(1)^2 - 2} = \frac{1}{1} = 1$$

To evaluate $f''(2)$, evaluate the expression for y'' using $x = 2$, $y = 1$, and $y' = 1$:

$$f''(2) = \frac{(1) - 2(1) - 3(1)^2(1)}{[3(1)^2 - 2]^2} = \frac{-4}{1} = -4$$

49. Find the two points: The curve crosses the x-axis when $y = 0$, so the equation becomes $x^2 + 0x + 0 = 7$, or $x^2 = 7$. The solutions are $x = \pm\sqrt{7}$, so the points are $(\pm\sqrt{7}, 0)$. Show tangents are parallel:

$$x^2 + xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x + x\frac{dy}{dx} + (y)(1) + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Slope at $(\sqrt{7}, 0): -\frac{2(\sqrt{7}) + 0}{\sqrt{7} + 2(0)} = -2$

Slope at $(-\sqrt{7}, 0): -\frac{2(-\sqrt{7}) + 0}{-\sqrt{7} + 2(0)} = -2$

The tangents at these points are parallel because they have the same slope. The common slope is -2.

50.

$$x^2 + xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x + x\frac{dy}{dx} + (y)(1) + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

(a) The tangent is parallel to the x-axis when $\frac{dy}{dx} = -\frac{2x + y}{x + 2y} = 0$, or $y = -2x$.

Substituting $-2x$ for y in the original equation, we have

$$x^2 + xy + y^2 = 7$$

$$x^2 + (x)(-2x) + (-2x)^2 = 7$$

$$x^2 - 2x^2 + 4x^2 = 7$$

$$3x^2 = 7$$

$$x = \pm\sqrt{\frac{7}{3}}$$

The points are $\left(-\sqrt{\frac{7}{3}}, 2\sqrt{\frac{7}{3}}\right)$ and $\left(\sqrt{\frac{7}{3}}, -2\sqrt{\frac{7}{3}}\right)$.

(b) Since x and y are interchangeable in the original equation, $\frac{dx}{dy}$ can be obtained by interchanging x and y in the expression for $\frac{dy}{dx}$. That is, $\frac{dx}{dy} = -\frac{2y+x}{y+2x}$. The tangent is parallel to the y-axis when $\frac{dx}{dy} = 0$, or $x = -2y$. Substituting $-2y$ for x in the original equation, we have:

$$x^2 + xy + y^2 = 7$$
$$(-2y)^2 + (-2y)(y) + y^2 = 7$$
$$4y^2 - 2y^2 + y^2 = 7$$
$$3y^2 = 7$$
$$y = \pm\sqrt{\frac{7}{3}}$$

The points are $\left(-2\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right)$ and $\left(2\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right)$.

Note that these are the same points that would be obtained by interchanging x and y in the solution to part (a).

51. First curve:

$$2x^2 + 3y^2 = 5$$
$$\frac{d}{dx}(2x^2) + \frac{d}{dx}(3y^2) = \frac{d}{dx}(5)$$
$$4x + 6y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{4x}{6y} = -\frac{2x}{3y}$$

Second curve:

$$y^2 = x^3$$
$$\frac{d}{dx}y^2 = \frac{d}{dx}x^3$$
$$2y\frac{dy}{dx} = 3x^2$$
$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

At (1, 1), the slopes are $-\frac{2}{3}$ and $\frac{3}{2}$ respectively. At $(1, -1)$, the slopes are $\frac{2}{3}$ and $-\frac{3}{2}$ respectively. In both cases, the tangents are perpendicular. To graph the curves and normal lines, we may use the following parametric equations for $-\pi \le t \le \pi$:

First curve: $x = \sqrt{\frac{5}{2}}\cos t,\ y = \sqrt{\frac{5}{3}}\sin t$

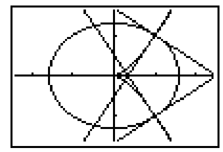
Second curve: $x = \sqrt[3]{t^2},\ y = t$

Tangents at (1, 1): $x = 1+3t,\ y = 1-2t$

$x = 1+2t,\ y = 1+3t$

Tangents at (1, −1): $x = 1+3t,\ y = -1+2t$

$x = 1+2t,\ y = -1-3t$



[−2.4, 2.4] by [−1.6, 1.6]

52.
$$v(t) = s'(t)$$
$$= \frac{d}{dt}(4+6t)^{3/2}$$
$$= \frac{3}{2}(4+6t)^{1/2}(6)$$
$$= 9(4+6t)^{1/2}$$
$$a(t) = v'(t)$$
$$= \frac{d}{dt}[9(4+6t)^{1/2}]$$
$$= \frac{9}{2}(4+6t)^{-1/2}(6)$$
$$= 27(4+6t)^{-1/2}$$

At $t = 2$, the velocity is $v(2) = 36$ m/sec and the acceleration is $a(2) = \frac{27}{4}\text{ m/sec}^2$.

53.
$$\text{Acceleration} = \frac{dv}{dt} = \frac{d}{dt}[8(s-t)^{1/2} + 1]$$
$$= 4(s-t)^{-1/2}\left(\frac{ds}{dt} - 1\right)$$
$$= 4(s-t)^{-1/2}\ (v-1)$$
$$= 4(s-t)^{-1/2}\ [(8(s-t)^{1/2} + 1) - 1]$$
$$= 32(s-t)^{-1/2}\ (s-t)^{1/2}$$
$$= 32\text{ ft/sec}^2$$

54.
$$y^4 - 4y^2 = x^4 - 9x^2$$
$$\frac{d}{dx}(y^4) - \frac{d}{dx}(4y^2) = \frac{d}{dx}(x^4) - \frac{d}{dx}(9x^2)$$
$$4y^3\frac{dy}{dx} - 8y\frac{dy}{dx} = 4x^3 - 18x$$
$$\frac{dy}{dx} = \frac{4x^3 - 18x}{4y^3 - 8y} = \frac{2x^3 - 9x}{2y^3 - 4y}$$

Slope at $(3, 2)$: $\dfrac{2(3)^3 - 9(3)}{2(2)^3 - 4(2)} = \dfrac{27}{8}$

Slope at $(-3, 2)$: $\dfrac{2(-3)^3 - 9(-3)}{2(2)^3 - 4(2)} = -\dfrac{27}{8}$

Slope at $(-3, -2)$: $\dfrac{2(-3)^3 - 9(-3)}{2(-2)^3 - 4(-2)} = \dfrac{27}{8}$

Slope at $(3, -2)$: $\dfrac{2(3)^3 - 9(3)}{2(-2)^3 - 4(-2)} = -\dfrac{27}{8}$

55. **(a)**
$$x^3 + y^3 - 9xy = 0$$
$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - 9\frac{d}{dx}(xy) = \frac{d}{dx}(0)$$
$$3x^2 + 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9(y)(1) = 0$$
$$(3y^2 - 9x)\frac{dy}{dx} = 9y - 3x^2$$
$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$$

Slope at $(4, 2)$: $\dfrac{3(2) - (4)^2}{(2)^2 - 3(4)} = \dfrac{-10}{-8} = \dfrac{5}{4}$

Slope at $(2, 4)$: $\dfrac{3(4) - (2)^2}{(4)^2 - 3(2)} = \dfrac{8}{10} = \dfrac{4}{5}$

(b) The tangent is horizontal when $\dfrac{dy}{dx} = \dfrac{3y - x^2}{y^2 - 3x} = 0$, or $y = \dfrac{x^2}{3}$.

Substituting $\dfrac{x^2}{3}$ for y in the original equation, we have:
$$x^3 + y^3 - 9xy = 0$$
$$x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0$$
$$x^3 + \frac{x^6}{27} - 3x^3 = 0$$
$$\frac{x^3}{27}(x^3 - 54) = 0$$
$x = 0$ or $x = \sqrt[3]{54} = 3\sqrt[3]{2}$

At $x = 0$, we have $y = \dfrac{0^2}{3} = 0$, which gives the point $(0, 0)$, which is the origin.

At $x = 3\sqrt[3]{2}$, we have
$y = \dfrac{1}{3}\left(3\sqrt[3]{2}\right)^2 = \dfrac{1}{3}\left(9\sqrt[3]{4}\right) = 3\sqrt[3]{4}$, so the point other than the origin is $\left(3\sqrt[3]{2}, 3\sqrt[3]{4}\right)$ or approximately $(3.780, 4.762)$.

(c) The equation $x^3 + y^3 - 9xy = 0$ is not affected by interchanging x and y, so its graph is symmetric about the line $y = x$ and we may find the desired point by interchanging the x-value and the y-value in the answer to part (b). The desired point is $\left(3\sqrt[3]{4}, 3\sqrt[3]{2}\right)$ or approximately $(4.762, 3.780)$.

56.
$$x^2 + 2xy - 3y^2 = 0$$
$$\frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(0)$$
$$2x + 2x\frac{dy}{dx} + 2(y)(1) - 6y\frac{dy}{dx} = 0$$
$$(2x - 6y)\frac{dy}{dx} = -2x - 2y$$
$$\frac{dy}{dx} = \frac{-2x - 2y}{2x - 6y} = \frac{x + y}{3y - x}$$

At $(1, 1)$ the curve has slope $\dfrac{1+1}{3(1) - 1} = \dfrac{2}{2} = 1$, so the normal line is $y = -1(x - 1) + 1$ or $y = -x + 2$.

Substituting $-x + 2$ for y in the original equation, we have:
$$x^2 + 2xy - 3y^2 = 0$$
$$x^2 + 2x(-x + 2) - 3(-x + 2)^2 = 0$$
$$x^2 - 2x^2 + 4x - 3(x^2 - 4x + 4) = 0$$
$$-4x^2 + 16x - 12 = 0$$
$$-4(x - 1)(x - 3) = 0$$
$x = 1$ or $x = 3$

Since the given point $(1, 1)$ had $x = 1$, we choose $x = 3$ and so $y = -(3) + 2 = -1$. The desired point is $(3, -1)$.

57. $$xy + 2x - y = 0$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(2x) - \frac{d}{dx}(y) = \frac{d}{dx}(0)$$

$$x\frac{dy}{dx} + (y)(1) + 2 - \frac{dy}{dx} = 0$$

$$(x-1)\frac{dy}{dx} = -2 - y$$

$$\frac{dy}{dx} = \frac{-2-y}{x-1} = \frac{2+y}{1-x}$$

Since the slope of the line $2x + y = 0$ is –2, we wish to find points where the normal has slope –2, that is, where the tangent has slope $\frac{1}{2}$.

Thus, we have

$$\frac{2+y}{1-x} = \frac{1}{2}$$
$$2(2+y) = 1-x$$
$$4 + 2y = 1 - x$$
$$x = -2y - 3$$

Substituting $-2y - 3$ in the original equation, we have:

$$xy + 2x - y = 0$$
$$(-2y-3)y + 2(-2y-3) - y = 0$$
$$-2y^2 - 8y - 6 = 0$$
$$-2(y+1)(y+3) = 0$$

$y = -1$ or $y = -3$

At $y = -1$, $x = -2y - 3 = 2 - 3 = -1$.
At $y = -3$: $x = -2y - 3 = 6 - 3 = 3$.
The desired points are (–1, –1) and (3, –3).
Finally, we find the desired normals to the curve, which are the lines of slope –2 passing through each of these points.
At (–1, –1), the normal line is $y = -2(x + 1) - 1$ or $y = -2x - 3$. At (3, –3), the normal line is $y = -2(x - 3) - 3$ or $y = -2x + 3$.

58. $$x = y^2$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(y^2)$$

$$1 = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

The normal line at (x, y) has slope $-2y$. Thus, the normal line at (b^2, b) is

$y = -2b(x - b^2) + b$, or $y = -2bx + 2b^3 + b$.

This line intersects the x-axis at

$x = \frac{2b^3 + b}{2b} = b^2 + \frac{1}{2}$, which is the value of a and must be greater than $\frac{1}{2}$ if $b \neq 0$.

The two normals at $(b^2, \pm b)$ will be perpendicular when they have slopes ± 1, which gives $-2y = \pm 1$ or $y = \pm\frac{1}{2}\left(\text{or } b = \pm\frac{1}{2}\right)$.

The corresponding value of a is $b^2 + \frac{1}{2} = \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{3}{4}$. Thus, the two nonhorizontal normals are perpendicular when $a = \frac{3}{4}$.

59. False.

$$\frac{d}{dx}(xy^2 + x) = \frac{d}{dx}(1)$$

$$y^2 + 1 + 2xy\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-1-y^2}{2xy},$$

$$\left.\frac{dy}{dx}\right|_{(1/2,\,1)} = \frac{-1-1^2}{2\left(\frac{1}{2}\right)1} = -2$$

60. True. By the power rule.

$$y = (x)^{1/3}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x)^{1/3} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

61. A; $\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx}(1)$

$$2x - y + (-x + 2y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

62. A;

$$\frac{d^2y}{dx^2}$$
$$= \frac{d}{dx}\left[\frac{y-2x}{2y-x}\right]$$
$$= \frac{(2y-x)(y'-2)-(y-2x)(2y'-1)}{(2y-x)^2}$$
$$= \frac{(2yy'-4y-xy'+2x)-(2yy'-y-4xy'+2x)}{(2y-x)^2}$$
$$= \frac{-3y+3xy'}{(2y-x)^2}$$
$$= \frac{-3y+3x\left(\frac{y-2x}{2y-x}\right)}{(2y-x)^2}$$
$$= \frac{-6y^2+3xy+3xy-6x^2}{(2y-x)^3}$$
$$= \frac{-6(x^2-xy+y^2)}{(2y-x)^3}$$
$$= -\frac{6}{(2y-x)^3}$$

63. E; $\frac{d}{dx}(y) = \frac{d}{dx}x^{3/4}$

$$\frac{dy}{dx} = \frac{3}{4}x^{-1/4} = \frac{3}{4x^{1/4}}$$

64. C; $\frac{d}{dx}(y^2 - x^2) = \frac{d}{dx}(1)$

$$-2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{x}{y} = \frac{1}{\sqrt{2}}$$

65. **(a)**
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$b^2x^2 + a^2y^2 = a^2b^2$$
$$\frac{d}{dx}(b^2x^2) + \frac{d}{dx}(a^2y^2) = \frac{d}{dx}(a^2b^2)$$
$$2b^2x + 2a^2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{2b^2x}{2a^2y} = -\frac{b^2x}{a^2y}$$

The slope at (x_1, y_1) is $-\frac{b^2x_1}{a^2y_1}$.

The tangent line is

$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$. This gives:

$$a^2y_1y - a^2y_1^2 = -b^2x_1x + b^2x_1^2$$
$$a^2y_1y + b^2x_1x = a^2y_1^2 + b^2x_1^2.$$

But $a^2y_1^2 + b^2x_1^2 = a^2b^2$ since (x_1, y_1) is on the ellipse. Therefore, $a^2y_1y + b^2x_1x = a^2b^2$, and dividing by a^2b^2 gives $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$.

(b)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$b^2x^2 - a^2y^2 = a^2b^2$$
$$\frac{d}{dx}(b^2x^2) - \frac{d}{dx}(a^2y^2) = \frac{d}{dx}(a^2b^2)$$
$$2b^2x - 2a^2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-2b^2x}{-2a^2y} = \frac{b^2x}{a^2y}$$

The slope at (x_1, y_1) is $\frac{b^2x_1}{a^2y_1}$.

The tangent line is $y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$.

This gives:

$$a^2y_1y - a^2y_1^2 = b^2x_1x - b^2x_1^2$$
$$b^2x_1^2 - a^2y_1^2 = b^2x_1x - a^2y_1y$$

But $b^2x_1^2 - a^2y_1^2 = a^2b^2$ since (x_1, y_1) is on the hyperbola. Therefore, $b^2x_1x - a^2y_1y = a^2b^2$, and dividing by a^2b gives $\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$.

66. **(a)** Solve for y:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$-\frac{y^2}{b^2} = -\frac{x^2}{a^2} + 1$$
$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$
$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

(b) $\lim_{|x|\to\infty} \frac{f(x)}{g(x)} = \lim_{|x|\to\infty} \frac{\frac{b}{a}\sqrt{x^2-a^2}}{\frac{b}{a}|x|}$

$$= \lim_{|x|\to\infty} \frac{\sqrt{x^2-a^2}}{\sqrt{x^2}}$$
$$= \lim_{|x|\to\infty} \sqrt{1-\frac{a^2}{x^2}}$$
$$= 1$$

(c) $\lim_{|x|\to\infty} \frac{f(x)}{g(x)} = \lim_{|x|\to\infty} \frac{-\frac{b}{a}\sqrt{x^2-a^2}}{-\frac{b}{a}|x|}$

$$= \lim_{|x|\to\infty} \frac{\sqrt{x^2-a^2}}{\sqrt{x^2}}$$
$$= \lim_{|x|\to\infty} \sqrt{1-\frac{a^2}{x^2}}$$
$$= 1$$

Quick Quiz Sections 4.1–4.2

1. B; $y = \sin^4 u$ $\quad u = 3x$

$$\frac{dy}{du} = 4\sin^3 u \cos u \quad \frac{du}{dx} = 3$$
$$\frac{dy}{dx} = 12\sin^3(3x)\cos(3x)$$

2. B; $2x^2 - 3y^2 = 2xy - 6$

$$4x - 6y\frac{dy}{dx} = 2x\frac{dy}{dx} + 2y$$
$$4x - 2y = (2x+6y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{4x-2y}{2x+6y}$$

Slope at (3, 2): $\frac{4(3)-2(2)}{2(3)+6(2)} = \frac{4}{9}$

3. C; $x = 3\sin t$ $\quad y = 2\cos t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$\frac{dy}{dt} = \frac{d}{dt}(2\cos t) = -2\sin t$$
$$\frac{dx}{dt} = \frac{d}{dt}(3\sin t) = 3\cos t$$
$$\frac{dy}{dx} = \frac{-2\sin t}{3\cos t} = -\frac{2}{3}\tan t.$$

4. **(a)** Differentiate implicitly:

$$\frac{d}{dx}(xy^2 - x^3y) = \frac{d}{dx}(6)$$
$$1\cdot y^2 + x\cdot 2y\frac{dy}{dx} - \left(3x^2y + x^3\frac{dy}{dx}\right) = 0$$
$$2xy\frac{dy}{dx} - x^3\frac{dy}{dx} = 3x^2y - y^2$$
$$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$$

(b) If $x = 1$, then $y^2 - y = 6$, so $y = -2$ or $y = 3$.

At $(1, -2)$, $\frac{dy}{dx} = \frac{3(1)^2(-2)-(-2)^2}{2(1)(-2)-(1)^3} = 2.$

The tangent line is $y + 2 = 2(x - 1)$.

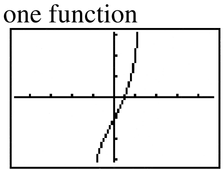
At $(1, 3)$, $\frac{dy}{dx} = \frac{3(1)^2(3)-3^2}{2(1)(3)-1^3} = 0.$ The tangent line is $y = 3$.

(c) The tangent line is vertical where $2xy - x^3 = 0$, which implies $x = 0$ or $y = \frac{x^2}{2}$. There is no point on the curve where $x = 0$. If $y = \frac{x^2}{2}$, then $x\left(\frac{x^2}{2}\right)^2 - x^3\left(\frac{x^2}{2}\right) = 6$. The only solution to this equation is $x = \sqrt[5]{-24}$.

Section 4.3 Derivatives of Inverse Trigonometric Functions (pp. 173–179)

Exploration 1 Finding a Derivative on an Inverse Graph Geometrically

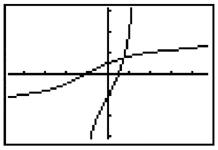
1. The graph is shown; it appears to be a one-to-one function



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

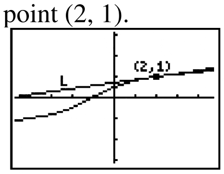
2. $f'(x) = 5x^4 + 2$. The fact that this function is always positive enables us to conclude that f is everywhere increasing, and hence one-to-one.

3. The graph of f^{-1} is shown, along with the graph of f. The graph of f^{-1} is obtained from the graph of f by reflecting it in the line $y = x$.



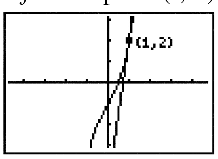
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

4. The line L is tangent to the graph of f^{-1} at the point (2, 1).



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

5. The reflection of line L is tangent to the graph of f at the point $(1, 2)$



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

6. The reflection of the line L is the tangent line to the graph of $y = x^5 + 2x - 1$ at the point $(1, 2)$. The slope is $\frac{dy}{dx}$ at $x = 1$, which is 7.

7. The slope of L is the reciprocal of the slope of its reflection $\left(\text{since } \frac{\Delta y}{\Delta x} \text{ gets reflected to become } \frac{\Delta x}{\Delta y}.\right)$ It is $\frac{1}{7}$.

8. $\frac{1}{7}$

Quick Review 4.3

1. Domain: $[-1, 1]$
Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
At 1: $\frac{\pi}{2}$

2. Domain: $[-1, 1]$
Range: $[0, \pi]$
At $1: 0$

3. Domain: all reals
Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
At $1: \frac{\pi}{4}$

4. Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
At $1: 0$

5. Domain: all reals
Range: all reals
At 1:1

6. $f(x) = y = 3x - 8$
$y + 8 = 3x$
$x = \frac{y+8}{3}$
Interchange x and y:
$y = \frac{x+8}{3}$
$f^{-1}(x) = \frac{x+8}{3}$

7. $f(x) = y = \sqrt[3]{x+5}$
$y^3 = x + 5$
$x = y^3 - 5$
Interchange x and y:
$y = x^3 - 5$
$f^{-1}(x) = x^3 - 5$

8. $f(x) = y = \frac{8}{x}$
$x = \frac{8}{y}$
Interchange x and y:
$y = \frac{8}{x}$
$f^{-1}(x) = \frac{8}{x}$

9. $f(x) = y = \frac{3x-2}{x}$

$$xy = 3x - 2$$
$$(y-3)x = -2$$
$$x = \frac{-2}{y-3} = \frac{2}{3-y}$$

Interchange x and y:

$$y = \frac{2}{3-x}$$
$$f^{-1}(x) = \frac{2}{3-x}$$

10. $f(x) = y = \arctan\frac{x}{3}$

$$\tan y = \frac{x}{3}, -\frac{\pi}{2} < y < \frac{\pi}{2}$$
$$x = 3\tan y, -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Interchange x and y:

$$y = 3\tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$f^{-1}(x) = 3\tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Section 4.3 Exercises

1. $\frac{dy}{dx} = \frac{d}{dx}\cos^{-1}(x^2)$

$$= -\frac{1}{\sqrt{1-(x^2)^2}}\frac{d}{dx}(x^2)$$
$$= -\frac{1}{\sqrt{1-x^4}}(2x)$$
$$= -\frac{2x}{\sqrt{1-x^4}}$$

2. $\frac{dy}{dx} = \frac{d}{dx}\cos^{-1}\left(\frac{1}{x}\right)$

$$= -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\left(-\frac{1}{x^2}\right)$$
$$= \frac{1}{|x|\sqrt{x^2-1}}$$

3. $\frac{dy}{dt} = \frac{d}{dt}\sin^{-1}\sqrt{2}t$

$$= \frac{1}{\sqrt{1-\left(\sqrt{2}t\right)^2}}\frac{d}{dt}\left(\sqrt{2}t\right)$$
$$= \frac{\sqrt{2}}{\sqrt{1-2t^2}}$$

4. $\frac{dy}{dt} = \frac{d}{dt}\sin^{-1}(1-t)$

$$= \frac{1}{\sqrt{1-(1-t)^2}}\frac{d}{dt}(1-t)$$
$$= -\frac{1}{\sqrt{2t-t^2}}$$

5. $\frac{dy}{dt} = \frac{d}{dt}\sin^{-1}\left(\frac{3}{t^2}\right)$

$$= \frac{1}{\sqrt{1-\left(\frac{3}{t^2}\right)^2}}\frac{d}{dt}\left(\frac{3}{t^2}\right)$$
$$= \frac{1}{\sqrt{1-\frac{9}{t^4}}}\left(-\frac{6}{t^3}\right)$$
$$= -\frac{6}{t\sqrt{t^4-9}}$$

6. $\frac{dy}{ds}$

$$= \frac{d}{ds}\left(s\sqrt{1-s^2}\right) + \frac{d}{ds}(\cos^{-1}s)$$
$$= (s)\left(\frac{1}{2\sqrt{1-s^2}}\right)(-2s) + \left(\sqrt{1-s^2}\right)(1) - \frac{1}{\sqrt{1-s^2}}$$
$$= -\frac{s^2}{\sqrt{1-s^2}} + \sqrt{1-s^2} - \frac{1}{\sqrt{1-s^2}}$$
$$= -\frac{-s^2+(1-s^2)-1}{\sqrt{1-s^2}}$$
$$= -\frac{2s^2}{\sqrt{1-s^2}}$$

7. $\frac{dy}{dx}$

$= \frac{d}{dx}(x \sin^{-1} x) + \frac{d}{dx}\left(\sqrt{1-x^2}\right)$

$= (x)\left(\frac{1}{\sqrt{1-x^2}}\right) + (\sin^{-1} x)(1) + \frac{1}{2\sqrt{1-x^2}}(-2x)$

$= \sin^{-1} x$

8. $\frac{dy}{dx} = \frac{d}{dx}[\sin^{-1}(2x)]^{-1}$

$= -[\sin^{-1}(2x)]^{-2} \frac{d}{dx} \sin^{-1}(2x)$

$= -[\sin^{-1}(2x)]^{-2} \frac{1}{\sqrt{1-4x^2}}(2)$

$= -\frac{2}{[\sin^{-1}(2x)]^2 \sqrt{1-4x^2}}$

9. $x(t) = \sin^{-1}\left(\frac{t}{4}\right)$

$\frac{dx}{dt} = \frac{d}{dt}\left[\sin^{-1}\left(\frac{t}{4}\right)\right]$

$= \frac{1}{\sqrt{1-\left(\frac{t}{4}\right)^2}} \frac{d}{dt}\left(\frac{t}{4}\right)$

$= \frac{1}{\sqrt{1-\frac{t^2}{16}}} \cdot \frac{1}{4}$

$= \frac{1}{\sqrt{16-t^2}}$

$v(3) = \left.\frac{dx}{dt}\right|_{t=3} = \frac{1}{\sqrt{16-3^2}} = \frac{\sqrt{7}}{7}$

10. $\frac{dx}{dt} = \frac{d}{dt}\left[\sin^{-1}\left(\frac{\sqrt{t}}{4}\right)\right]$

$= \frac{1}{\sqrt{1-\left(\frac{\sqrt{t}}{4}\right)^2}} \frac{d}{dt}\left(\frac{\sqrt{t}}{4}\right)$

$= \frac{1}{\sqrt{1-\frac{t}{16}}} \cdot \frac{1}{8\sqrt{t}}$

$v(4) = \left.\frac{dx}{dt}\right|_{t=4} = \frac{1}{\sqrt{1-\frac{4}{16}}} \cdot \frac{1}{8\sqrt{4}}$

$= \frac{1}{\sqrt{1-\frac{1}{4}}} \cdot \frac{1}{16}$

$= \frac{2}{\sqrt{3}} \cdot \frac{1}{16}$

$= \frac{\sqrt{3}}{24}$

11. $\frac{dx}{dt} = \frac{d}{dt}[\tan^{-1} t] = \frac{1}{1+t^2}$

$v(2) = \left.\frac{dx}{dt}\right|_{t=2} = \frac{1}{1+2^2} = \frac{1}{5}$

12. $\frac{dx}{dt} = \frac{d}{dt}[\tan^{-1}(t^2)]$

$= \frac{1}{1+(t^2)^2} \cdot \frac{d}{dt}(t^2)$

$= \frac{2t}{1+t^4}$

$v(1) = \left.\frac{dx}{dt}\right|_{t=1} = \frac{2(1)}{1+1^4} = 1$

13. $\frac{dy}{ds} = \frac{d}{ds} \sec^{-1}(2s+1)$

$= \frac{1}{|2s+1|\sqrt{(2s+1)^2-1}} \frac{d}{ds}(2s+1)$

$= \frac{1}{|2s+1|\sqrt{4s^2+4s}}(2)$

$= \frac{1}{|2s+1|\sqrt{s^2+s}}$

14. $$\frac{dy}{ds} = \frac{d}{ds}\sec^{-1} 5s = \frac{1}{|5s|\sqrt{(5s)^2-1}}\frac{d}{ds}(5s) = \frac{1}{|s|\sqrt{25s^2-1}}$$

15. $$\frac{dy}{dx} = \frac{d}{dx}\csc^{-1}(x^2+1) = -\frac{1}{|x^2+1|\sqrt{(x^2+1)^2-1}}\frac{d}{dx}(x^2+1) = -\frac{2x}{(x^2+1)\sqrt{x^4+2x^2}} = -\frac{2}{(x^2+1)\sqrt{x^2+2}}$$

Note that the condition $x > 0$ is required in the last step.

16. $$\frac{dy}{dx} = \frac{d}{dx}\csc^{-1}\left(\frac{x}{2}\right) = -\frac{1}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}}\frac{d}{dx}\left(\frac{x}{2}\right) = -\frac{2}{|x|\sqrt{x^2-4}}$$

17. $$\frac{dy}{dt} = \frac{d}{dt}\sec^{-1}\left(\frac{1}{t}\right) = \frac{1}{\left|\frac{1}{t}\right|\sqrt{\left(\frac{1}{t}\right)^2-1}}\frac{d}{dt}\left(\frac{1}{t}\right) = \frac{1}{\left|\frac{1}{t}\right|\sqrt{\left(\frac{1}{t}\right)^2-1}}\left(-\frac{1}{t^2}\right) = -\frac{1}{\sqrt{1-t^2}}$$

Note that the condition $t > 0$ is required in the last step.

18. $$\frac{dy}{dt} = \frac{d}{dt}\cot^{-1}\sqrt{t} = -\frac{1}{1+\left(\sqrt{t}\right)^2}\frac{d}{dt}\sqrt{t} = -\frac{1}{2\sqrt{t}(t+1)}$$

19. $$\frac{dy}{dt} = \frac{d}{dt}\cot^{-1}\sqrt{t-1} = -\frac{1}{1+\left(\sqrt{t-1}\right)^2}\frac{d}{dt}\sqrt{t-1} = -\left(\frac{1}{1+t-1}\right)\left(\frac{1}{2\sqrt{t-1}}\right) = -\frac{1}{2t\sqrt{t-1}}$$

20. $$\frac{dy}{ds} = \frac{d}{ds}\sqrt{s^2-1} - \frac{d}{ds}\sec^{-1} s = \frac{1}{2\sqrt{s^2-1}}(2s) - \frac{1}{|s|\sqrt{s^2-1}} = \frac{s|s|-1}{|s|\sqrt{s^2-1}}$$

21. $$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\sqrt{x^2-1}\right) + \frac{d}{dx}(\csc^{-1} x) = \frac{1}{1+\left(\sqrt{x^2-1}\right)^2}\frac{d}{dx}\left(\sqrt{x^2-1}\right) - \frac{1}{|x|\sqrt{x^2-1}} = \frac{1}{x^2}\frac{1}{2\sqrt{x^2-1}}(2x) - \frac{1}{|x|\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}} - \frac{1}{|x|\sqrt{x^2-1}} = 0$$

Note that the condition $x > 1$ is required in the last step.

22. $$\frac{dy}{dx} = \frac{d}{dx}\left(\cot^{-1}\frac{1}{x}\right) - \frac{d}{dx}(\tan^{-1} x) = -\frac{1}{1+\left(\frac{1}{x^2}\right)}\frac{d}{dx}\left(\frac{1}{x}\right) - \frac{1}{1+x^2} = \left(-\frac{1}{1+\frac{1}{x^2}}\right)\left(-\frac{1}{x^2}\right) - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0,\ x \neq 0$$

The condition $x \neq 0$ is required because the original function was undefined when $x = 0$.

23. $y = \sec^{-1} x$

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$y'(2) = \frac{1}{|2|\sqrt{2^2 - 1}} = \frac{1}{2\sqrt{3}}$$

$$y(2) = \sec^{-1}(2) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$y = \frac{1}{2\sqrt{3}}(x - 2) + \frac{\pi}{3}$$

or $y = 0.289(x - 2) + 1.047$

$y = 0.289x + 0.469$

24. $y = \tan^{-1} x$

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

$$y'(2) = \frac{1}{1 + 2^2} = \frac{1}{5}$$

$$y(2) = \tan^{-1}(2)$$

$$y = \frac{1}{5}(x - 2) + \tan^{-1}(2)$$

or $y = 0.2(x - 2) + 1.107$

$y = 0.2x + 0.707$

25. $y = \sin^{-1}\left(\frac{x}{4}\right)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(\frac{x}{4}\right)^2}} \frac{d}{dx}\left(\frac{x}{4}\right)$$

$$= \frac{1}{\sqrt{1 - \frac{x^2}{16}}} \cdot \frac{1}{4}$$

$$= \frac{1}{\sqrt{16 - x^2}}$$

$$y'(3) = \frac{1}{\sqrt{16 - 3^2}} = \frac{1}{\sqrt{7}}$$

$$y(3) = \sin^{-1}\left(\frac{3}{4}\right)$$

$$y = \frac{1}{\sqrt{7}}(x - 3) + \sin^{-1}\left(\frac{3}{4}\right)$$

or $y = 0.378(x - 3) + 0.848$

$y = 0.378x - 0.286$

26. $y = \tan^{-1}(x^2)$

$$\frac{dy}{dx} = \frac{1}{1 + (x^2)^2} \cdot \frac{d}{dx}(x^2)$$

$$= \frac{1}{1 + x^4} \cdot 2x$$

$$= \frac{2x}{1 + x^4}$$

$$y'(1) = \frac{2(1)}{1 + 1^4} = \frac{2}{2} = 1$$

$$y(1) = \tan^{-1}(1^2)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$y = 1(x - 1) + \frac{\pi}{4}$$

$$y = x - 1 + \frac{\pi}{4}$$

or $y = x - 0.215$

27. (a) Since $\frac{dy}{dx} = \sec^2 x$, the slope at $\left(\frac{\pi}{4}, 1\right)$ is $\sec^2\left(\frac{\pi}{4}\right) = 2.$

The tangent line is given by $y - 1 = 2\left(x - \frac{\pi}{4}\right).$

(b) Since $\frac{dy}{dx} = \frac{1}{1 + x^2}$, the slope at $\left(1, \frac{\pi}{4}\right)$ is $\frac{1}{1 + 1^2} = \frac{1}{2}.$

The tangent line is given by $y - \frac{\pi}{4} = \frac{1}{2}(x - 1).$

28. (a) Note that $f'(x) = 5x^4 + 6x^2 + 1$. Thus $f(1) = 3$ and $f'(1) = 12$.

(b) Since the graph of $y = f(x)$ includes the point (1, 3) and the slope of the graph is 12 at this point, the graph of $y = f^{-1}(x)$ will include (3, 1) and the slope will be $\frac{1}{12}$. Thus, $f^{-1}(3) = 1$ and $(f^{-1})'(3) = \frac{1}{12}.$

(We have assumed that $f^{-1}(x)$ is defined and differentiable at $x = 3$. This is true by Theorem 3, because $f'(x) = 5x^4 + 6x^2 + 1$, which is never zero.)

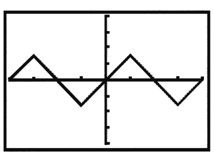
29. **(a)** Note that $f'(x) = -\sin x + 3$, which is always between 2 and 4. Thus f is differentiable at every point on the interval $(-\infty, \infty)$ and $f'(x)$ is never zero on this interval, so f has a differentiable inverse by Theorem 3.

(b) $f(0) = \cos 0 + 3(0) = 1$;
$f'(0) = -\sin 0 + 3 = 3$

(c) Since the graph of $y = f(x)$ includes the point (0, 1) and the slope of the graph is 3 at this point, the graph of $y = f^{-1}(x)$ will include (1, 0) and the slope will be $\frac{1}{3}$,

Thus, $f^{-1}(1) = 0$ and $(f^{-1})'(1) = \frac{1}{3}$.

30.



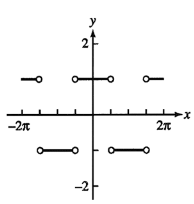
$[-2\pi, 2\pi]$ by $[-4, 4]$

(a) All reals

(b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) At the points $x = k\frac{\pi}{2}$, where k is an odd integer.

(d)



(e) $f'(x) = \frac{d}{dx}\sin^{-1}(\sin x)$

$$= \frac{1}{\sqrt{1-\sin^2 x}}\frac{d}{dx}\sin x$$

$$= \frac{\cos x}{\sqrt{1-\sin^2 x}}$$

which is ± 1 depending on whether $\cos x$ is positive or negative.

31. **(a)** $v(t) = \frac{dx}{dt} = \frac{1}{1+t^2}$ which is always positive.

(b) $a(t) = \frac{dv}{dt} = -\frac{2t}{(1+t^2)^2}$ which is always negative.

(c) $\frac{\pi}{2}$

32. $\frac{d}{dx}\cos^{-1}(x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right)$

$$= 0 - \frac{d}{dx}\sin^{-1}(x)$$

$$= -\frac{1}{\sqrt{1-x^2}}$$

33. $\frac{d}{dx}\cot^{-1} x = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1}(x)\right)$

$$= 0 - \frac{d}{dx}\tan^{-1}(x)$$

$$= -\frac{1}{1+x^2}$$

34. $\frac{d}{dx}\csc^{-1}(x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sec^{-1}(x)\right)$

$$= 0 - \frac{d}{dx}\sec^{-1}(x)$$

$$= -\frac{1}{|x|\sqrt{x^2-1}}$$

35. True. By definition of the function.

36. False. The domain is all real numbers.

37. E; $\frac{d}{dx}\sin^{-1}\left(\frac{x}{2}\right)=\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{x}{2}\right)$

$$=\frac{1}{\sqrt{1-\frac{x^2}{4}}}\cdot\frac{1}{2}$$

$$=\frac{1}{\sqrt{4-x^2}}$$

38. D; $\frac{d}{dx}\tan^{-1}(3x)=\frac{1}{1+(3x)^2}\frac{d}{dx}(3x)$

$$=\frac{1}{1+9x^2}\cdot 3$$

$$=\frac{3}{1+9x^2}$$

39. A; $\frac{d}{dx}\sec^{-1}(x^2)=\frac{1}{\left|x^2\right|\sqrt{(x^2)^2-1}}\frac{d}{dx}(x^2)$

$$=\frac{1}{x^2\sqrt{x^4-1}}\cdot 2x$$

$$=\frac{2}{x\sqrt{x^4-1}}$$

40. C; $\frac{dy}{dx}=\frac{d}{dx}(\tan^{-1}(2x))$

$$=\frac{1}{1+(2x)^2}\cdot\frac{d}{dx}(2x)$$

$$=\frac{1}{1+4x^2}\cdot 2$$

$$=\frac{2}{1+4x^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1}=\frac{2}{1+4(1)^2}=\frac{2}{5}$$

41. **(a)** $y=\frac{\pi}{2}$

(b) $y=-\frac{\pi}{2}$

(c) None, since $\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}\neq 0.$

42. **(a)** $y=0$

(b) $y=\pi$

(c) None, since $\frac{d}{dx}\cot^{-1}x=-\frac{1}{1+x^2}\neq 0.$

43. **(a)** $y=\frac{\pi}{2}$

(b) $y=\frac{\pi}{2}$

(c) None, since $\frac{d}{dx}\sec^{-1}x=\frac{1}{|x|\sqrt{x^2-1}}\neq 0.$

44. **(a)** $y=0$

(b) $y=0$

(c) None, since $\frac{d}{dx}\csc^{-1}x=-\frac{1}{|x|\sqrt{x^2-1}}\neq 0.$

45. **(a)** None, since $\sin^{-1}x$ is undefined for $x>1$.

(b) None, since $\sin^{-1}x$ is undefined for $x<-1$.

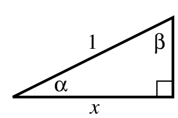
(c) None, since $\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}}\neq 0.$

46. **(a)** None, since $\cos^{-1}x$ is undefined for $x>1$.

(b) None, since $\cos^{-1}x$ is undefined for $x<-1$.

(c) None, since $\frac{d}{dx}\cos^{-1}x=-\frac{1}{\sqrt{1-x^2}}\neq 0.$

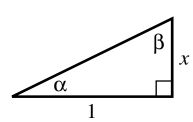
47. **(a)**



$\alpha=\cos^{-1}x,\ \beta=\sin^{-1}x$

Therefore, $\cos^{-1}x+\sin^{-1}x=\alpha+\beta=\frac{\pi}{2}.$

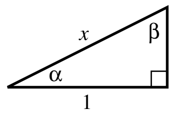
(b)



$\alpha=\tan^{-1}x,\ \beta=\cot^{-1}x$

Therefore, $\tan^{-1}x+\cot^{-1}x=\alpha+\beta=\frac{\pi}{2}.$

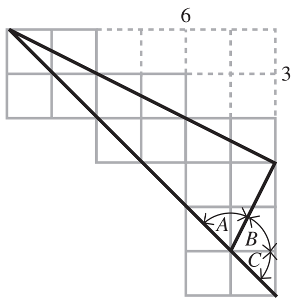
(c)



$\alpha = \sec^{-1} x,\ \beta = \csc^{-1} x$

Therefore, $\sec^{-1} x + \csc^{-1} x = \alpha + \beta = \dfrac{\pi}{2}.$

48.



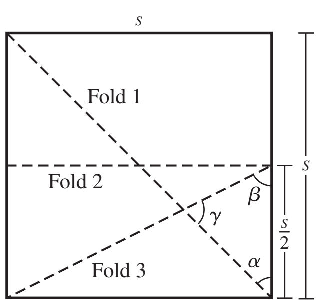
The "straight angle" with the arrows in it is the sum of the three angles *A*, *B*, and *C*.

A is equal to $\tan^{-1} 3$ since the opposite side is 3 times as long as the adjacent side.

B is equal to $\tan^{-1} 2$ since the side opposite it is 2 units and the adjacent side is one unit.

C is equal to $\tan^{-1} 1$ since both the opposite and adjacent sides are one unit long.
But the sum of these three angles is the "straight angle," which has measure π radians.

49.



If *s* is the length of a side of the square, then

$\tan \alpha = \dfrac{s}{s} = 1,$ so $\alpha = \tan^{-1} 1$ and

$\tan \beta = \dfrac{s}{\frac{s}{2}} = 2,$ so $\beta = \tan^{-1} 2.$

$$\begin{aligned}\text{Then } \gamma &= \pi - \alpha - \beta \\ &= \pi - \tan^{-1} 1 - \tan^{-1} 2 \\ &= \tan^{-1} 3.\end{aligned}$$

(In the last step, we used Exercise 48.)

Section 4.4 Derivatives of Exponential and Logarithmic Functions (pp. 180–188)

Exploration 1 Leaving Milk on the Counter

1. The temperature of the refrigerator is 42°F, the temperature of the milk at time $t = 0$.

2. The temperature of the room is 72°F, the limit to which *y* tends as *t* increases.

3. The milk is warming up the fastest at $t = 0$. The second derivative $y'' = -30(\ln(0.98))^2(0.98)^t$ is negative, so y' (the rate at which the milk is warming) is maximized at the lowest value of *t*.

4. We set $y = 55$ and solve;
$$\begin{aligned}72 - 30(0.98)^t &= 55 \\ (0.98)^t &= \frac{17}{30} \\ t\ln(0.98) &= \ln\left(\frac{17}{30}\right) \\ t &= \frac{\ln\left(\frac{17}{30}\right)}{\ln(0.98)} = 28.114\end{aligned}$$
The milk reaches a temperature of 55° F after about 28 minutes.

5. $\dfrac{dy}{dt} = -30\ln(0.98) \bullet (0.98)^t$. At $t = 28.114$, $\dfrac{dy}{dt} \approx 0.343$ degrees/minute.

Quick Review 4.4

1. $\log_5 8 = \dfrac{\ln 8}{\ln 5}$

2. $7^x = e^{\ln 7^x} = e^{x \ln 7}$

3. $\ln(e^{\tan x}) = \tan x$

4. $$\begin{aligned}\ln(x^2 - 4) - \ln(x + 2) &= \ln\frac{x^2 - 4}{x + 2} \\ &= \ln\frac{(x+2)(x-2)}{x+2} \\ &= \ln(x - 2)\end{aligned}$$

5. $\log_2(8^{x-5}) = \log_2(2^3)^{x-5}$
$= \log_2 2^{3x-15}$
$= 3x - 15$

6. $\dfrac{\log_4 x^{15}}{\log_4 x^{12}} = \dfrac{15\log_4 x}{12\log_4 x} = \dfrac{15}{12} = \dfrac{5}{4},\ x > 0$

7. $3\ln x - \ln 3x + \ln(12x^2)$
$= \ln x^3 - \ln 3x + \ln(12x^2)$
$= \ln \dfrac{(x^3)(12x^2)}{3x}$
$= \ln(4x^4)$

8. $3^x = 19$
$\ln 3^x = \ln 19$
$x \ln 3 = \ln 19$
$x = \dfrac{\ln 19}{\ln 3} \approx 2.68$

9. $5^t \ln 5 = 18$
$5^t = \dfrac{18}{\ln 5}$
$\ln 5^t = \ln \dfrac{18}{\ln 5}$
$t \ln 5 = \ln 18 - \ln(\ln 5)$
$t = \dfrac{\ln 18 - \ln(\ln 5)}{\ln 5} \approx 1.50$

10. $3^{x+1} = 2x$
$\ln 3^{x+1} = \ln 2^x$
$(x+1)\ln 3 = x \ln 2$
$x(\ln 3 - \ln 2) = -\ln 3$
$x = \dfrac{\ln 3}{\ln 2 - \ln 3} \approx -2.71$

Section 4.4 Exercises

1. $\dfrac{dy}{dx} = \dfrac{d}{dx}(2e^x) = 2e^x$

2. $\dfrac{dy}{dx} = \dfrac{d}{dx}(e^{2x}) = e^{2x}\dfrac{d}{dx}(2x) = 2e^{2x}$

3. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{-x} = e^{-x}\dfrac{d}{dx}(-x) = -e^{-x}$

4. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{-5x} = e^{-5x}\dfrac{d}{dx}(-5x) = -5e^{-5x}$

5. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{2x/3} = e^{2x/3}\dfrac{d}{dx}\left(\dfrac{2x}{3}\right) = \dfrac{2}{3}e^{2x/3}$

6. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{-x/4} = e^{-x/4}\dfrac{d}{dx}\left(-\dfrac{x}{4}\right) = -\dfrac{1}{4}e^{-x/4}$

7. $\dfrac{dy}{dx} = \dfrac{d}{dx}(xe^2) - \dfrac{d}{dx}(e^x) = e^2 - e^x$

8. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2e^x) - \dfrac{d}{dx}(xe^x)$
$= (x^2)(e^x) + (e^x)(2x) - [(x)(e^x) + (e^x)(1)]$
$= x^2e^x + xe^x - e^x$

9. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}}\dfrac{d}{dx}(\sqrt{x}) = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

10. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{(x^2)} = e^{(x^2)}\dfrac{d}{dx}(x^2) = 2xe^{(x^2)}$

11. $\dfrac{dy}{dx} = \dfrac{d}{dx}8^x = 8^x \ln 8$

12. $\dfrac{dy}{dx} = \dfrac{d}{dx}9^{-x} = 9^{-x}(\ln 9)\dfrac{d}{dx}(-x) = -9^{-x}\ln 9$

13. $\dfrac{dy}{dx} = \dfrac{d}{dx}3^{\csc x} = 3^{\csc x}(\ln 3)\dfrac{d}{dx}(\csc x)$
$= 3^{\csc x}(\ln 3)(-\csc x \cot x)$
$= -3^{\csc x}(\ln 3)(\csc x \cot x)$

14. $\dfrac{dy}{dx} = \dfrac{d}{dx}3^{\cot x} = 3^{\cot x}(\ln 3)\dfrac{d}{dx}(\cot x)$
$= 3^{\cot x}(\ln 3)(-\csc^2 x)$
$= -3^{\cot x}(\ln 3)(\csc^2 x)$

15. $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(x^2) = \dfrac{1}{x^2}\dfrac{d}{dx}(x^2) = \dfrac{1}{x^2}(2x) = \dfrac{2}{x}$

16. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\ln x)^2 = 2\ln x \dfrac{d}{dx}(\ln x) = \dfrac{2\ln x}{x}$

17. $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(x^{-1}) = \dfrac{d}{dx}(-\ln x) = -\dfrac{1}{x},\ x > 0$

18. $\frac{dy}{dx} = \frac{d}{dx} \ln \frac{10}{x}$
$= \frac{d}{dx}(\ln 10 - \ln x)$
$= 0 - \frac{1}{x}$
$= -\frac{1}{x}, x > 0$

19. $\frac{d}{dx} \ln (\ln x) = \frac{1}{\ln x} \frac{d}{dx} \ln x = \frac{1}{\ln x} \bullet \frac{1}{x} = \frac{1}{x \ln x}$

20. $\frac{dy}{dx} = \frac{d}{dx}(x \ln x - x)$
$= (x)\left(\frac{1}{x}\right) + (\ln x)(1) - 1$
$= 1 + \ln x - 1$
$= \ln x$

21. $\frac{dy}{dx} = \frac{d}{dx}(\log_4 x^2)$
$= \frac{d}{dx} \frac{\ln x^2}{\ln 4}$
$= \frac{d}{dx}\left[\left(\frac{2}{\ln 4}\right)(\ln x)\right]$
$= \frac{2}{\ln 4} \cdot \frac{1}{x}$
$= \frac{2}{x \ln 4}$
$= \frac{1}{x \ln 2}$

22. $\frac{dy}{dx} = \frac{d}{dx} \left(\log_5 \sqrt{x}\right)$
$= \frac{d}{dx} \frac{\ln x^{1/2}}{\ln 5}$
$= \frac{d}{dx} \frac{\frac{1}{2} \ln x}{\ln 5}$
$= \frac{1}{2 \ln 5} \frac{d}{dx}(\ln x)$
$= \frac{1}{2 \ln 5} \cdot \frac{1}{x}$
$= \frac{1}{2x \ln 5}, \ x > 0$

23. $\frac{dy}{dx} = \frac{d}{dx} \log_2 \left(\frac{1}{x}\right)$
$= \frac{d}{dx}(-\log_2 x)$
$= -\frac{1}{x \ln 2}, x > 0$

24. $\frac{dy}{dx} = \frac{d}{dx} \frac{1}{\log_2 x}$
$= -\frac{1}{(\log_2 x)^2} \frac{d}{dx}(\log_2 x)$
$= -\frac{1}{(\log_2 x)^2} \ \frac{1}{x \ln 2}$
$= -\frac{1}{x(\ln 2)(\log_2 x)^2}$ or $-\frac{\ln 2}{x(\ln x)^2}$

25. $\frac{dy}{dx} = \frac{d}{dx}(\ln 2 \cdot \log_2 x)$
$= (\ln 2) \frac{d}{dx}(\log_2 x)$
$= (\ln 2)\left(\frac{1}{x \ln 2}\right)$
$= \frac{1}{x}, x > 0$

26. $\frac{dy}{dx} = \frac{d}{dx} \log_3 (1 + x \ln 3)$
$= \frac{1}{(1 + x \ln 3) \ln 3} \frac{d}{dx}(1 + x \ln 3)$
$= \frac{\ln 3}{(1 + x \ln 3) \ln 3}$
$= \frac{1}{1 + x \ln 3}, \ x > -\frac{1}{\ln 3}$

27. $\frac{dy}{dx} = \frac{d}{dx}(\log_{10} e^x)$
$= \frac{d}{dx}(x \log_{10} e)$
$= \log_{10} e$
$= \frac{\ln e}{\ln 10}$
$= \frac{1}{\ln 10}$

28. $\frac{dy}{dx} = \frac{d}{dx} \ln 10^x = \frac{d}{dx} (x \ln 10) = \ln 10$

29. $m = 5$

$y = 3^x + 1$

$y' = 3^x \ln 3 = 5$

$x = 1.379$

$y = 3^{1.379} + 1 = 5.551$

(1.379, 5.551)

30. $m_2 = -\frac{1}{m_1} = \frac{1}{3}$

$\frac{d}{dx}(2e^x - 1) = 2e^x$

$\frac{1}{3} = 2e^x$

$\frac{1}{6} = e^x$

$x = -\ln 6$

$y = 2e^x - 1$

$y = \frac{2}{6} - 1 = -\frac{2}{3}$

$\left(-\ln 6,\ -\frac{2}{3}\right)$ or $(-1.792,\ -0.667)$

31. Equation of line: $y = mx$

Slope: $m = \frac{d}{dx}\ln(2x) = \frac{1}{2x} \cdot 2 = \frac{1}{x}$

At the point where the tangent line touches the graph, $y = mx$ and $y = \ln(2x)$

$mx = \ln(2x)$

$\frac{1}{x} \cdot x = \ln(2x)$

$1 = \ln(2x)$

$e^1 = 2x$

$x = \frac{e}{2}$

Therefore, $m = \frac{1}{x} = \frac{2}{e}$.

32. Equation of line: $y = mx$

Slope: $m = \frac{d}{dx}\left(\ln \frac{x}{3}\right)$

$= \frac{1}{\frac{x}{3}} \frac{d}{dx}\left(\frac{x}{3}\right)$

$= \frac{3}{x} \cdot \frac{1}{3}$

$= \frac{1}{x}$

At the point where the tangent line touches the graph, $y = mx$ and $y = \ln\left(\frac{x}{3}\right)$

$mx = \ln\left(\frac{x}{3}\right)$

$\frac{1}{x} \cdot x = \ln\left(\frac{x}{3}\right)$

$1 = \ln\left(\frac{x}{3}\right)$

$e^1 = \frac{x}{3}$

$x = 3e$

Therefore, $m = \frac{1}{x} = \frac{1}{3e}$.

33. $\frac{dy}{dx} = \frac{d}{dx}(x^{\pi}) = \pi x^{\pi - 1}$

34. $\frac{dy}{dx} = \frac{d}{dx}(x^{1+\sqrt{2}})$

$= \left(1+\sqrt{2}\right) x^{1+\sqrt{2}-1}$

$= \left(1+\sqrt{2}\right) x^{\sqrt{2}}$

35. $\frac{dy}{dx} = \frac{d}{dx} x^{-\sqrt{2}} = -\sqrt{2} x^{-\sqrt{2}-1}$

36. $\frac{dy}{dx} = \frac{d}{dx} x^{1-e} = (1-e)x^{1-e-1} = (1-e)x^{-e}$

37. $\frac{d}{dx}\ln(x+2) = \frac{1}{x+2}\frac{d}{dx}(x+2) = \frac{1}{x+2}$

Domain of f: $x + 2 > 0$

$x > -2$

Domain of f': $x \neq -2$ and $x > -2$, so $x > -2$.

38. $\frac{d}{dx}\ln(2x+2) = \frac{1}{2x+2} \cdot \frac{d}{dx}(2x+2)$

$= \frac{1}{2(x+1)} \cdot 2$

$= \frac{1}{x+1}$

Domain of f: $2x + 2 > 0$

$2x > -2$

$x > -1$

Domain of f': $x \neq -1$ and $x > -1$, so $x > -1$

39. $$\frac{d}{dx}\ln(2-\cos x)=\frac{1}{2-\cos x}\cdot\frac{d}{dx}(2-\cos x)$$
$$=\frac{\sin x}{2-\cos x}$$

Domain of f: $2-\cos x>0$
$$-\cos x>-2$$
$$\cos x<2$$
which is true for all x.

Domain of f': $\cos x\neq 2$ which is true for all x.

All real numbers.

40. $$\frac{d}{dx}\ln(x^2+1)=\frac{1}{x^2+1}\frac{d}{dx}(x^2+1)=\frac{2x}{x^2+1}$$

Since $x^2+1>0$ for all x,
Domain of f = Domain of f' = all real numbers.

41. $$\frac{d}{dx}\log_2(3x+1)=\frac{1}{(3x+1)\ln 2}\cdot\frac{d}{dx}(3x+1)$$
$$=\frac{3}{(3x+1)\ln 2}$$

Domain of f: $3x+1>0$
$$x>-\frac{1}{3}$$

Domain of f': $3x+1\neq 0$ and $x>-\frac{1}{3}$, so

$x>-\frac{1}{3}$

42. First, note that
$$\log_{10}\sqrt{x+1}=\log_{10}(x+1)^{1/2}$$
$$=\frac{1}{2}\log_{10}(x+1)$$
$$\frac{d}{dx}\log_{10}\sqrt{x+1}=\frac{d}{dx}\left[\frac{1}{2}\log_{10}(x+1)\right]$$
$$=\frac{1}{2}\cdot\frac{1}{(x+1)\ln 10}\cdot\frac{d}{dx}(x+1)$$
$$=\frac{1}{2(x+1)\ln 10}$$

Domain of f: $x+1>0$
$$x>-1$$

Domain of f': $x\neq -1$ and $x>-1$, so $x>-1$.

43.
$$y=(\sin x)^x$$
$$\ln y=\ln(\sin x)^x$$
$$\ln y=x\ln(\sin x)$$
$$\frac{d}{dx}\ln y=\frac{d}{dx}\left[x\ln(\sin x)\right]$$
$$\frac{1}{y}\frac{dy}{dx}=(x)\left(\frac{1}{\sin x}\right)(\cos x)+\ln(\sin x)(1)$$
$$\frac{dy}{dx}=y\left[x\cot x+\ln(\sin x)\right]$$
$$\frac{dy}{dx}=(\sin x)^x[x\cot x+\ln(\sin x)]$$

44.
$$y=x^{\tan x}$$
$$\ln y=\ln(x^{\tan x})$$
$$\ln y=(\tan x)(\ln x)$$
$$\frac{d}{dx}\ln y=\frac{d}{dx}\left[(\tan x)(\ln x)\right]$$
$$\frac{1}{y}\frac{dy}{dx}=(\tan x)\left(\frac{1}{x}\right)+(\ln x)(\sec^2 x)$$
$$\frac{dy}{dx}=y\left[\frac{\tan x}{x}+(\ln x)(\sec^2 x)\right]$$
$$\frac{dy}{dx}=x^{\tan x}\left[\frac{\tan x}{x}+(\ln x)(\sec^2 x)\right]$$

45. $y = \sqrt[5]{\frac{(x-3)^4(x^2+1)}{(2x+5)^3}} = \left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5}$

$$\ln y = \ln\left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5}$$

$$\ln y = \frac{1}{5}\ln\frac{(x-3)^4(x^2+1)}{(2x+5)^3}$$

$$\ln y = \frac{1}{5}[4\ln(x-3)+\ln(x^2+1)-3\ln(2x+5)]$$

$$\frac{d}{dx}(\ln y) = \frac{4}{5}\frac{d}{dx}\ln(x-3)+\frac{1}{5}\frac{d}{dx}\ln(x^2+1)-\frac{3}{5}\frac{d}{dx}\ln(2x+5)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{5}\frac{1}{x-3}+\frac{1}{5}\frac{1}{x^2+1}(2x)-\frac{3}{5}\frac{1}{2x+5}(2)$$

$$\frac{dy}{dx} = y\left(\frac{4}{5(x-3)}+\frac{2x}{5(x^2+1)}-\frac{6}{5(2x+5)}\right)$$

$$\frac{dy}{dx} = \left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5}\cdot\left(\frac{4}{5(x-3)}+\frac{2x}{5(x^2+1)}-\frac{6}{5(2x+5)}\right)$$

46. $y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} = \frac{x(x^2+1)^{1/2}}{(x+1)^{2/3}}$

$$\ln y = \ln\frac{x(x^2+1)^{1/2}}{(x+1)^{2/3}}$$

$$\ln y = \ln x+\frac{1}{2}\ln(x^2+1)-\frac{2}{3}\ln(x+1)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\ln x+\frac{1}{2}\frac{d}{dx}\ln(x^2+1)-\frac{2}{3}\frac{d}{dx}\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}+\frac{1}{2}\frac{1}{x^2+1}(2x)-\frac{2}{3}\frac{1}{x+1}(1)$$

$$\frac{dy}{dx} = y\left(\frac{1}{x}+\frac{x}{x^2+1}-\frac{2}{3(x+1)}\right)$$

$$\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left(\frac{1}{x}+\frac{x}{x^2+1}-\frac{2}{3(x+1)}\right)$$

47. $y = x^{\ln x}$

$$\ln y = \ln(x^{\ln x}) = \ln x \bullet \ln x = (\ln x)^2$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\ln x)^2$$

$$= 2\ln x\cdot\frac{d}{dx}(\ln x)$$

$$= 2\ln x\cdot\frac{1}{x}$$

$$= \frac{2\ln x}{x}$$

$$\frac{dy}{dx} = y\cdot\frac{2\ln x}{x} = \frac{2x^{\ln x}\ln x}{x}$$

48. $y = x^{(1/\ln x)}$

$\ln y = \ln x^{(1/\ln x)} = \frac{1}{\ln x} \cdot \ln x = 1$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(1)$

$\frac{1}{y}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 0,\ x > 0$

49. $\frac{dy}{dx} = e^x$

The tangent line passes through (0, 0) and (a, e^a) for some value of a, and has slope e^a.

Thus $\frac{e^a - 0}{a - 0} = e^a$

$\frac{e^a}{a} = e^a,$

so $a = 1$. Therefore the line has slope e^1 and passes through (1, e). It has equation $y - e = e(x-1)$, or $y = ex$.

Therefore, $y = e^1(x-1) + e^1$

$y = ex$

50. For $y = xe^x$, we have

$y' = (x)(e^x) + (e^x)(1) = (x+1)e^x$, so the normal line through the point (a, ae^a) has slope $m = -\frac{1}{(a+1)e^a}$ and its equation is

$y = -\frac{1}{(a+1)e^a}(x-a) + ae^a$. The desired normal line includes the point (0, 0), so we have:

$0 = -\frac{1}{(a+1)e^a}(0-a) + ae^a$

$0 = \frac{a}{(a+1)e^a} + ae^a$

$0 = a\left(\frac{1}{(a+1)e^a} + e^a\right)$

$a = 0$ or $\frac{1}{(a+1)e^a} + e^a = 0$

The equation $\frac{1}{(a+1)e^a} + e^a = 0$ has no solution (as can be seen by graphing $y = \frac{1}{(x+1)e^x} + e^x$ on a calculator), so we need to use $a = 0$. The equation of the normal line is $y = \frac{-1}{(0+1)e^0}(x-0) + 0e^0$, or $y = -x$.

51. **(a)** $P(0) = \frac{300}{1 + 2^{4-0}} \approx 18$

(b) $P'(t) = 300\frac{d}{dt}(1 + 2^{4-t})^{-1}$

$= -300(1 + 2^{4-t})^{-2} \cdot (\ln 2)2^{4-t}(-1)$

$= \frac{300\,(\ln 2)\,2^{4-t}}{(1 + 2^{4-t})^2}$

$P'(4) = \frac{300\,(\ln 2)}{4} = 52$

(c) Graph $P'(t)$ on a graphing calculator. Use TRACE or CALC→ MAXIMUM to find that the maximum of $P'(t)$ is at $t = 4$. The rumor spreads at its maximum rate after 4 days; at that time the rumor is spreading at a rate of 52 students per day.

52. **(a)** $P(0) = \frac{200}{1 + e^{5-0}} = 1$

(b) $\frac{d}{dt}200((1 + e^{5-t})^{-1})$

$= 200(-1)(1 + e^{5-t})^{-2}\frac{d}{dt}(1 + e^{5-t})$

$= 200(-1)(1 + e^{5-t})^{-2}(e^{5-t})(-1)$

$= \frac{200e^{5-t}}{(1 + e^{5-t})^2}$

$P'(4) = \frac{200e^{5-4}}{(1 + e^{5-4})^2} = 39$

(c) Graph $P'(t)$ on a graphing calculator. Use TRACE or CALC→ MAXIMUM to find that $P'(t)$ has a maximum at $t = 5$.

$P'(5) = \frac{200e^{5-5}}{(1 + e^{5-5})^2} = 50$

The flu spreads at its maximum rate after 5 days. At that time, the flu is spreading to 50 students per day.

53. $\frac{dA}{dt} = 20\frac{d}{dt}\left(\frac{1}{2}\right)^{t/140}$

$= 20\frac{d}{dt}2^{-t/140}$

$= 20(2^{-t/140})(\ln 2)\frac{d}{dt}\left(-\frac{t}{140}\right)$

$= 20(2^{-t/140})(\ln 2)\left(-\frac{1}{140}\right)$

$= -\frac{(2^{-t/140})(\ln 2)}{7}$

At $t = 2$ days, we have

$\frac{dA}{dt} = -\frac{(2^{-1/70})(\ln 2)}{7} \approx -0.098$ grams/day.

This means that the rate of *decay* is the positive rate of approximately 0.098 gram/day.

54. **(a)** $\frac{d}{dx}\ln(kx) = \frac{1}{kx}\frac{d}{dx}kx = \frac{k}{kx} = \frac{1}{x}$

(b) $\frac{d}{dx}\ln(kx) = \frac{d}{dx}(\ln k + \ln x)$

$= 0 + \frac{d}{dx}\ln x$

$= \frac{1}{x}$

55. **(a)** Since $f'(x) = 2^x \ln 2$,

$f'(0) = 2^0 \ln 2 = \ln 2$.

(b) $f'(0) = \lim_{h\to 0}\frac{f(h) - f(0)}{h}$

$= \lim_{h\to 0}\frac{2^h - 2^0}{h}$

$= \lim_{h\to 0}\frac{2^h - 1}{h}$

(c) Since quantities in parts (a) and (b) are equal, $\lim_{h\to 0}\frac{2^h - 1}{h} = \ln 2$.

(d) By following the same procedure as above using $g(x) = 7^x$, we may see that

$\lim_{h\to 0}\frac{7^h - 1}{h} = \ln 7$.

56. Recall that a point (a, b) is on the graph of $y = e^x$ if and only if the point (b, a) is on the graph of $y = \ln x$. Since there are points (x, e^x) on the graph of $y = e^x$ with arbitrarily large x-coordinates, there will be points $(x, \ln x)$ on the graph of $y = \ln x$ with arbitrarily large y-coordinates.

57. False; it is $(\ln 2)2^x$.

58. False; it is $2e^{2x}$.

59. B; $P(0) = \frac{150}{1 + e^{4-0}} = 3$

60. D; $x + 3 > 0$

$x > -3$

61. A; $y = \log_{10}(2x - 3)$

$\frac{dy}{dx} = \frac{1}{(\ln 10)(2x-3)}\frac{d}{dx}(2x - 3)$

$= \frac{2}{(\ln 10)(2x-3)}$

62. E; $y = 2^{1-x}$

$y' = 2^{1-x}(\ln 2)(-1)$

$y'(2) = -2^{1-2}(\ln 2)$

$y'(2) = -\frac{(\ln 2)}{2}$

63. **(a)** The graph y_4 is a horizontal line at $y = a$.

(b) The graph of y_3 is always a horizontal line.

a	2	3	4	5
y_3	0.693147	1.098613	1.386295	1.609439
$\ln a$	0.693147	1.098612	1.386294	1.609438

We conclude that the graph of y_3 is a horizontal line at $y = \ln a$.

(c) $\frac{d}{dx}a^x = a^x$ if and only if $y_3 = \frac{y_2}{y_1} = 1$.

So if $y_3 = \ln a$, then $\frac{d}{dx}a^x$ will equal a^x if and only if $\ln a = 1$, or $a = e$.

(d) $y_2 = \frac{d}{dx}a^x = a^x \ln a$. This will equal $y_1 = a^x$ if and only if ln a = 1, or $a = e$.

64. $\frac{d}{dx}\left(-\frac{1}{2}x^2 + k\right) = -x$ and $\frac{d}{dx}(\ln x + c) = \frac{1}{x}$.
Therefore, at any value of x, where the two curves intersect, the two tangent lines will be perpendicular.

65. **(a)** Since the line passes through the origin and has slope $\frac{1}{e}$, its equation is $y = \frac{x}{e}$.

(b) The graph of $y = \ln x$ lies below the graph of the line $y = \frac{x}{e}$ for all positive $x \neq e$.

Therefore, $\ln x < \frac{x}{e}$ for all positive $x \neq e$.

(c) Multiplying by e, $e \ln x < x$ or $\ln x^e < x$.

(d) Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

(e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

Quick Quiz Sections 4.3–4.4

1. A; $f'(x) = \frac{1}{x + 4 + e^{-3x}}(1 - 3e^{-3x})$

$$f'(0) = \frac{1}{5}(-2) = -\frac{2}{5}$$

2. B; the slope of g at (2, 1) is the reciprocal of the slope of f at (1, 2).

$$g'(2) = \frac{1}{f'(1)} = \left.\frac{1}{3x^2 + 1}\right|_{x=1} = \frac{1}{4}$$

3. C; $\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}(2x))$

$$= \frac{1}{\sqrt{1 - (2x)^2}} \frac{d}{dx}(2x)$$
$$= \frac{2}{\sqrt{1 - 4x^2}}$$

4. **(a)** $v(t) = x'(t)$

$$= e^t \cdot \sin t + e^t \cdot \cos t$$
$$= e^t(\sin t + \cos t)$$

The particle is at rest when $v(t) = 0$, which requires that $\sin t + \cos t = 0$, or equivalently that $\tan t = -1$. The only values of t in the interval $[0, 2\pi]$ satisfying this condition are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

(b) From part (a) we have

$x'(t) = e^t(\sin t + \cos t)$, so

$$x''(t) = e^t(\sin t + \cos t) + e^t(\cos t - \sin t)$$
$$= 2e^t \cos t.$$

Substitution gives:

$$Ax''(t) + x'(t) = x(t)$$
$$A(2e^t \cos t) + e^t(\sin t + \cos t) = e^t \sin t$$
$$2Ae^t \cos t + e^t \cos t = 0$$
$$2A + 1 = 0$$
$$A = -\frac{1}{2}$$

Chapter 4 Review Exercises (pp. 189–191)

1. $\frac{dy}{dx} = \frac{d}{dx}(e^{3x-7}) = e^{3x-7}\frac{d}{dx}(3x - 7) = 3e^{3x-7}$

2. $\frac{dy}{dx} = \frac{d}{dx}(\tan(e^x))$

$$= \sec^2(e^x)\frac{d}{dx}(e^x)$$
$$= e^x \sec^2(e^x)$$

3. $\frac{dy}{dx} = \frac{d}{dx}(\sin^3 x)$

$$= \frac{d}{dx}((\sin x)^3)$$
$$= 3(\sin x)^2 \frac{d}{dx}(\sin x)$$
$$= 3\sin^2 x \cos x$$

4. $\frac{dy}{dx} = \frac{d}{dx}(\ln(\csc x))$

$$= \frac{1}{\csc x}\frac{d}{dx}(\csc x)$$
$$= \frac{1}{\csc x}(-\csc x \cot x)$$
$$= -\cot x$$

5. $\dfrac{ds}{dt}=\dfrac{d}{dt}\cos(1-2t)$
$=-\sin(1-2t)(-2)$
$=2\sin(1-2t)$

6. $\dfrac{ds}{dt}=\dfrac{d}{dt}\cot\left(\dfrac{2}{t}\right)$
$=-\csc^2\left(\dfrac{2}{t}\right)\dfrac{d}{dt}\left(\dfrac{2}{t}\right)$
$=-\csc^2\left(\dfrac{2}{t}\right)\left(-\dfrac{2}{t^2}\right)$
$=\dfrac{2}{t^2}\csc^2\left(\dfrac{2}{t}\right)$

7. $\dfrac{dy}{dx}=\dfrac{d}{dx}\left(\sqrt{1+\cos x}\right)$
$=\dfrac{d}{dx}((1+\cos x)^{1/2})$
$=\dfrac{1}{2}(1+\cos x)^{-1/2}\dfrac{d}{dx}(1+\cos x)$
$=-\dfrac{\sin x}{2\sqrt{1+\cos x}}$

8. $\dfrac{dy}{dx}=\dfrac{d}{dx}\left(x\sqrt{2x+1}\right)$
$=(x)\left(\dfrac{1}{2\sqrt{2x+1}}\right)(2)+\left(\sqrt{2x+1}\right)(1)$
$=\dfrac{x+(2x+1)}{\sqrt{2x+1}}$
$=\dfrac{3x+1}{\sqrt{2x+1}}$

9. $\dfrac{dr}{d\theta}=\dfrac{d}{d\theta}\sec(1+3\theta)$
$=\sec(1+3\theta)\tan(1+3\theta)(3)$
$=3\sec(1+3\theta)\tan(1+3\theta)$

10. $\dfrac{dr}{d\theta}=\dfrac{d}{d\theta}\tan^2(3-\theta^2)$
$=2\tan(3-\theta^2)\dfrac{d}{d\theta}\tan(3-\theta^2)$
$=2\tan(3-\theta^2)\sec^2(3-\theta^2)(-2\theta)$
$=-4\theta\tan(3-\theta^2)\sec^2(3-\theta^2)$

11. $\dfrac{dy}{dx}=\dfrac{d}{dx}(x^2\csc 5x)$
$=(x^2)(-\csc 5x\cot 5x)(5)+(\csc 5x)(2x)$
$=-5x^2\csc 5x\cot 5x+2x\csc 5x$

12. $\dfrac{dy}{dx}=\dfrac{d}{dx}\ln\sqrt{x}$
$=\dfrac{1}{\sqrt{x}}\dfrac{d}{dx}\sqrt{x}$
$=\dfrac{1}{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}}$
$=\dfrac{1}{2x},\ x>0$

13. $\dfrac{dy}{dx}=\dfrac{d}{dx}\ln(1+e^x)=\dfrac{1}{1+e^x}\dfrac{d}{dx}(1+e^x)=\dfrac{e^x}{1+e^x}$

14. $\dfrac{dy}{dx}=\dfrac{d}{dx}(xe^{-x})$
$=(x)(e^{-x})(-1)+(e^{-x})(1)$
$=-xe^{-x}+e^{-x}$

15. $\dfrac{dy}{dx}=\dfrac{d}{dx}(e^{1+\ln x})=\dfrac{d}{dx}(e^1e^{\ln x})=\dfrac{d}{dx}(ex)=e$

16. $\dfrac{dy}{dx}=\dfrac{d}{dx}\ln(\sin x)$
$=\dfrac{1}{\sin x}\dfrac{d}{dx}(\sin x)$
$=\dfrac{\cos x}{\sin x}$
$=\cot x,$
for values of x in the intervals $(k\pi,\ (k+1)\pi)$, where k is even.

17. $\dfrac{dr}{dx}=\dfrac{d}{dx}\ln(\cos^{-1}x)$
$=\dfrac{1}{\cos^{-1}x}\dfrac{d}{dx}\cos^{-1}x$
$=\dfrac{1}{\cos^{-1}x}\left(-\dfrac{1}{\sqrt{1-x^2}}\right)$
$=-\dfrac{1}{\cos^{-1}x\sqrt{1-x^2}}$ for $-1<x<1$

18. $\dfrac{dr}{d\theta}=\dfrac{d}{d\theta}\log_2(\theta^2)$
$=\dfrac{1}{\theta^2\ln 2}\dfrac{d}{d\theta}(\theta^2)$
$=\dfrac{2\theta}{\theta^2\ln 2}$
$=\dfrac{2}{\theta\ln 2}$

19. $$\frac{ds}{dt} = \frac{d}{dt}\log_5(t-7) = \frac{1}{(t-7)\ln 5}\frac{d}{dt}(t-7) = \frac{1}{(t-7)\ln 5},\ t > 7$$

20. $$\frac{ds}{dt} = \frac{d}{dt}(8^{-t}) = 8^{-t}(\ln 8)\frac{d}{dt}(-t) = -8^{-t}\ln 8$$

21. Use logarithmic differentiation.

$$\begin{aligned} y &= x^{\ln x} \\ \ln y &= \ln(x^{\ln x}) \\ \ln y &= (\ln x)(\ln x) \\ \frac{d}{dx}\ln y &= \frac{d}{dx}(\ln x)^2 \\ \frac{1}{y}\frac{dy}{dx} &= 2\ln x\frac{d}{dx}\ln x \\ \frac{dy}{dx} &= \frac{2y\ln x}{x} \\ \frac{dy}{dx} &= \frac{2x^{\ln x}\ln x}{x} \end{aligned}$$

22. $$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\frac{(2x)2^x}{\sqrt{x^2+1}} \\ &= \frac{\sqrt{x^2+1}\frac{d}{dx}[(2x)2^x] - (2x)(2^x)\frac{d}{dx}\sqrt{x^2+1}}{x^2+1} \\ &= \frac{\sqrt{x^2+1}\left[(2x)(2^x)(\ln 2) + (2^x)(2)\right] - (2x)(2^x)\frac{1}{2\sqrt{x^2+1}}(2x)}{x^2+1} \\ &= \frac{(x^2+1)(2^x)(2x\ln 2 + 2) - 2x^2(2^x)}{(x^2+1)^{3/2}} \\ &= \frac{(2\cdot 2^x)[(x^2+1)(x\ln 2+1) - x^2]}{(x^2+1)^{3/2}} \\ &= \frac{(2\cdot 2^x)(x^3\ln 2 + x^2 + x\ln 2 + 1 - x^2)}{(x^2+1)^{3/2}} \\ &= \frac{(2\cdot 2^x)(x^3\ln 2 + x\ln 2 + 1)}{(x^2+1)^{3/2}} \end{aligned}$$

Alternate solution, using logarithmic differentiation:

$$y = \frac{(2x)2^x}{\sqrt{x^2+1}}$$

$$\ln y = \ln(2x) + \ln(2^x) - \ln\sqrt{x^2+1}$$

$$\ln y = \ln 2 + \ln x + x \ln 2 - \frac{1}{2}\ln(x^2+1)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}[\ln 2 + \ln x + x\ln 2 - \frac{1}{2}\ln(x^2+1)]$$

$$\frac{1}{y}\frac{dy}{dx} = 0 + \frac{1}{x} + \ln 2 - \frac{1}{2}\frac{1}{x^2+1}(2x)$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \ln 2 - \frac{x}{x^2+1}\right)$$

$$\frac{dy}{dx} = \frac{(2x)2^x}{\sqrt{x^2+1}}\left(\frac{1}{x} + \ln 2 - \frac{x}{x^2+1}\right)$$

23. $\frac{dy}{dx} = \frac{d}{dx}e^{\tan^{-1}x} = e^{\tan^{-1}x}\frac{d}{dx}\tan^{-1}x = \frac{e^{\tan^{-1}x}}{1+x^2}$

24. $\frac{dy}{du} = \frac{d}{du}\sin^{-1}\sqrt{1-u^2}$

$$= \frac{1}{\sqrt{1-\left(\sqrt{1-u^2}\right)^2}}\frac{d}{du}\sqrt{1-u^2}$$

$$= \frac{1}{\sqrt{u^2}}\frac{1}{2\sqrt{1-u^2}}(-2u)$$

$$= \frac{-u}{|u|\sqrt{1-u^2}}$$

25. $\frac{dy}{dt} = \frac{d}{dt}\left(t\sec^{-1}t - \frac{1}{2}\ln t\right)$

$$= (t)\left(\frac{1}{|t|\sqrt{t^2-1}}\right) + (\sec^{-1}t)(1) - \frac{1}{2t}$$

$$= \frac{t}{|t|\sqrt{t^2-1}} + \sec^{-1}t - \frac{1}{2t}$$

26. $\frac{dy}{dt} = \frac{d}{dt}[(1+t^2)\cot^{-1}2t]$

$$= (1+t^2)\left(-\frac{1}{1+(2t)^2}\right)(2) + (\cot^{-1}2t)(2t)$$

$$= -\frac{2+2t^2}{1+4t^2} + 2t\cot^{-1}2t$$

27. $\frac{dy}{dz} = \frac{d}{dz}\left(z\cos^{-1} z - \sqrt{1-z^2}\right)$

$= (z)\left(-\frac{1}{\sqrt{1-z^2}}\right) + (\cos^{-1} z)(1) - \frac{1}{2\sqrt{1-z^2}}(-2z)$

$= -\frac{z}{\sqrt{1-z^2}} + \cos^{-1} z + \frac{z}{\sqrt{1-z^2}}$

$= \cos^{-1} z$

28. $\frac{dy}{dx} = \frac{d}{dx}\left(2\sqrt{x-1}\csc^{-1}\sqrt{x}\right)$

$= \left(2\sqrt{x-1}\right)\left(-\frac{1}{\left|\sqrt{x}\right|\sqrt{(\sqrt{x})^2-1}}\right)\left(\frac{1}{2\sqrt{x}}\right) + \left(2\csc^{-1}\sqrt{x}\right)\left(\frac{1}{2\sqrt{x-1}}\right)$

$= -\frac{\sqrt{x-1}}{(\sqrt{x})^2\sqrt{x-1}} + \frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$

$= -\frac{1}{x} + \frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$

29. $\frac{dy}{dx} = \frac{d}{dx}\csc^{-1}(\sec x)$

$= \left(-\frac{1}{\left|\sec x\right|\sqrt{\sec^2 x - 1}}\right)\frac{d}{dx}(\sec x)$

$= -\frac{1}{\left|\sec x\right|\sqrt{\tan^2 x}}\sec x \tan x$

$= -\frac{\sec x \tan x}{\left|\sec x \tan x\right|}$

$= -\frac{\frac{1}{\cos x}\frac{\sin x}{\cos x}}{\left|\frac{1}{\cos x}\frac{\sin x}{\cos x}\right|}$

$= -\frac{\sin x}{\left|\sin x\right|}$

$= -1 \text{ for } 0 \le x < \frac{\pi}{2}$

Alternate method:

On the domain $0 \le x < \frac{\pi}{2}$, we may rewrite the function as follows:

$y = \csc^{-1}(\sec x)$

$= \frac{\pi}{2} - \sec^{-1}(\sec x)$

$= \frac{\pi}{2} - \cos^{-1}(\cos x)$

$= \frac{\pi}{2} - x$

Therefore, $\frac{dy}{dx} = -1$ for $0 \le x < \frac{\pi}{2}$.

Note that the derivative exists at 0 only because this is an endpoint of the given domain; the two-sided derivative of $y = \csc^{-1}(\sec x)$ does not exist at this point.

30. $$\frac{dr}{d\theta} = \frac{d}{d\theta}\left(\frac{1+\sin\theta}{1-\cos\theta}\right)^2$$
$$= 2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{(1-\cos\theta)(\cos\theta)-(1+\sin\theta)(\sin\theta)}{(1-\cos\theta)^2}\right)$$
$$= 2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{\cos\theta-\cos^2\theta-\sin\theta-\sin^2\theta}{(1-\cos\theta)^2}\right)$$
$$= 2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{\cos\theta-\sin\theta-1}{(1-\cos\theta)^2}\right)$$

31. Since $y = \ln x^2$ is defined for all $x \neq 0$ and $\frac{dy}{dx} = \frac{1}{x^2}\frac{d}{dx}(x^2) = \frac{2x}{x^2} = \frac{2}{x}$, the function is differentiable for all $x \neq 0$.

32. Since $y = \sin(e^{2x})$ is defined for all real x and $\frac{dy}{dx} = 2e^{2x}\cos(e^{2x})$, the function is differentiable for all real x.

33. Since $y = \sqrt{\frac{1-x}{1+x^2}}$ is defined for all $x < 1$ and
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x^2}\right)^{-1/2}\frac{(1+x^2)(-1)-(1-x)(2x)}{(1+x^2)^2}$$
$$= \frac{x^2-2x-1}{2\sqrt{1-x}(1+x^2)^{3/2}},$$ which is defined only for $x < 1$, the function is differentiable for all $x < 1$.

34. Since $y = \frac{1}{1-e^x}$ is defined for all $x \neq 0$ and
$$\frac{dy}{dx} = \frac{d}{dx}((1-e^x)^{-1})$$
$$= -(1-e^x)^{-2}(-e^x)$$
$$= \frac{e^x}{(1-e^x)^2},$$
the function is differentiable for all $x \neq 0$.

35. Use implicit differentiation.
$$xy + 2x + 3y = 1$$
$$\frac{d}{dx}(xy) + \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(1)$$
$$x\frac{dy}{dx} + (y)(1) + 2 + 3\frac{dy}{dx} = 0$$
$$(x+3)\frac{dy}{dx} = -(y+2)$$
$$\frac{dy}{dx} = -\frac{y+2}{x+3}$$

36. Use implicit differentiation.

$$5x^{4/5} + 10y^{6/5} = 15$$
$$\frac{d}{dx}(5x^{4/5}) + \frac{d}{dx}(10y^{6/5}) = \frac{d}{dx}(15)$$
$$4x^{-1/5} + 12y^{1/5}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{4x^{-1/5}}{12y^{1/5}}$$
$$= -\frac{1}{3(xy)^{1/5}}$$

37. Use implicit differentiation.

$$\sqrt{xy} = 1$$
$$\frac{d}{dx}\sqrt{xy} = \frac{d}{dx}(1)$$
$$\frac{1}{2\sqrt{xy}}\left[x\frac{dy}{dx} + (y)(1)\right] = 0$$
$$x\frac{dy}{dx} + y = 0$$
$$\frac{dy}{dx} = -\frac{y}{x}$$

Alternate method: $\ln\sqrt{xy} = \ln(1)$

$$\frac{1}{2}[\ln x + \ln y] = 0$$
$$\ln x + \ln y = 0$$
$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{y}{x}$$

38. Use implicit differentiation.

$$y^2 = \frac{x}{x+1}$$
$$\frac{d}{dx}y^2 = \frac{d}{dx}\frac{x}{x+1}$$
$$2y\frac{dy}{dx} = \frac{(x+1)(1) - (x)(1)}{(x+1)^2}$$
$$\frac{dy}{dx} = \frac{1}{2y(x+1)^2}$$

39. $x^3 + y^3 = 1$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(1)$$
$$3x^2 + 3y^2 y' = 0$$
$$y' = -\frac{x^2}{y^2}$$
$$y'' = \frac{d}{dx}\left(-\frac{x^2}{y^2}\right)$$
$$= -\frac{(y^2)(2x) - (x^2)(2y)(y')}{y^4}$$
$$= -\frac{(y^2)(2x) - (x^2)(2y)\left(-\frac{x^2}{y^2}\right)}{y^4}$$
$$= -\frac{2xy^3 + 2x^4}{y^5}$$
$$= -\frac{2x(x^3 + y^3)}{y^5}$$
$$= -\frac{2x}{y^5}$$

since $x^3 + y^3 = 1$.

40. $y^2 = 1 - \frac{2}{x}$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(1) - \frac{d}{dx}\left(\frac{2}{x}\right)$$
$$2yy' = \frac{2}{x^2}$$
$$y' = \frac{2}{x^{2(2y)}} = \frac{1}{x^2 y}$$
$$y'' = \frac{d}{dx}\left(\frac{1}{x^2 y}\right)$$
$$= -\frac{1}{(x^2 y)^2}\frac{d}{dx}(x^2 y)$$
$$= -\frac{1}{(x^2 y)^2}[(x^2)(y') + (y)(2x)]$$
$$= -\frac{1}{(x^2 y)^2}\left[(x^2)\left(\frac{1}{x^2 y}\right) + 2xy\right]$$
$$= -\frac{1}{x^4 y^2}\left(\frac{1}{y} + 2xy\right)$$
$$= -\frac{1 + 2xy^2}{x^4 y^3}$$

41.
$$y^3 + y = 2\cos x$$
$$\frac{d}{dx}(y^3) + \frac{d}{dx}(y) = \frac{d}{dx}(2\cos x)$$
$$3y^2y' + y' = -2\sin x$$
$$(3y^2+1)y' = -2\sin x$$
$$y' = -\frac{2\sin x}{3y^2+1}$$
$$y'' = \frac{d}{dx}\left(-\frac{2\sin x}{3y^2+1}\right)$$
$$= -\frac{(3y^2+1)(2\cos x) - (2\sin x)(6yy')}{(3y^2+1)^2}$$
$$= -\frac{(3y^2+1)(2\cos x) - (12y\sin x)\left(-\frac{2\sin x}{3y^2+1}\right)}{(3y^2+1)^2}$$
$$= -2\frac{(3y^2+1)^2\cos x + 12y\sin^2 x}{(3y^2+1)^3}$$

42.
$$x^{1/3} + y^{1/3} = 4$$
$$\frac{d}{dx}(x^{1/3}) + \frac{d}{dx}(y^{1/3}) = \frac{d}{dx}(4)$$
$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' = 0$$
$$y' = -\frac{x^{-2/3}}{y^{-2/3}} = -\left(\frac{y}{x}\right)^{2/3}$$
$$y'' = \frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{2/3}\right]$$
$$= -\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{xy' - (y)(1)}{x^2}\right)$$
$$= -\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{(x)\left[-\left(\frac{y}{x}\right)^{2/3}\right] - y}{x^2}\right)$$
$$= -\frac{2}{3}x^{1/3}y^{-1/3}(-x^{-5/3}y^{2/3} - x^{-2}y)$$
$$= \frac{2}{3}x^{-4/3}y^{1/3} + \frac{2}{3}x^{-5/3}y^{2/3}$$

43. $\frac{d^{40}y}{dx^{40}} = e^{x\sqrt[8]{2}}\left(\sqrt[8]{2}\right)^{40} = 2^5 e^{x\sqrt[8]{2}} = 32e^{x\sqrt[8]{2}}$

44. Note that the 4th, 8th, ... and 40th derivatives of $\sin x$ cycle back to $\sin x$. By the chain rule, each derivative generates another factor of $\sqrt[8]{2}$. Thus

$$\frac{d^{40}y}{dx^{40}} = \sin\left(x\sqrt[8]{2}\right)\left(\sqrt[8]{2}\right)^{40} = 32\sin\left(x\sqrt[8]{2}\right).$$

45.
$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{x^2-2x}$$
$$= \frac{1}{2\sqrt{x^2-2x}}(2x-2)$$
$$= \frac{x-1}{\sqrt{x^2-2x}}$$

At $x = 3$, $y = \sqrt{3^2 - 2(3)} = \sqrt{3}$

and $\frac{dy}{dx} = \frac{3-1}{\sqrt{3^2-2(3)}} = \frac{2}{\sqrt{3}}$.

(a) Tangent: $y - \sqrt{3} = \frac{2}{\sqrt{3}}(x-3)$

(b) Normal: $y - \sqrt{3} = -\frac{\sqrt{3}}{2}(x-3)$

46. $\frac{dy}{dx} = \frac{d}{dx}(\tan 2x) = 2\sec^2 2x$

At $x = \frac{\pi}{3}$, $y = \tan\frac{2\pi}{3} = -\sqrt{3}$ and

$\frac{dy}{dx} = 2\sec^2\frac{2\pi}{3} = 2(-2)^2 = 8.$

(a) Tangent: $y + \sqrt{3} = 8\left(x - \frac{\pi}{3}\right)$

(b) Normal: $y + \sqrt{3} = -\frac{1}{8}\left(x - \frac{\pi}{3}\right)$

47. Use implicit differentiation.
$$x^2 + 2y^2 = 9$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2y^2) = \frac{d}{dx}(9)$$
$$2x + 4y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{2x}{4y} = -\frac{x}{2y}$$

Slope at (1, 2): $-\frac{1}{2(2)} = -\frac{1}{4}$

(a) Tangent: $y-2=-\frac{1}{4}(x-1)$

(b) Normal: $y-2=4(x-1)$

48. Use implicit differentiation.

$$x+\sqrt{xy}=6$$

$$\frac{d}{dx}(x)+\frac{d}{dx}\left(\sqrt{xy}\right)=\frac{d}{dx}(6)$$

$$1+\frac{1}{2\sqrt{xy}}\left[(x)\left(\frac{dy}{dx}\right)+(y)(1)\right]=0$$

$$\frac{x}{2\sqrt{xy}}\frac{dy}{dx}=-1-\frac{y}{2\sqrt{xy}}$$

$$\frac{dy}{dx}=\frac{2\sqrt{xy}}{x}\left(-1-\frac{y}{2\sqrt{xy}}\right)$$

$$=-2\sqrt{\frac{y}{x}}-\frac{y}{x}$$

Slope at (4, 1): $-2\sqrt{\frac{1}{4}}-\frac{1}{4}=-\frac{2}{2}-\frac{1}{4}=-\frac{5}{4}$

(a) Tangent: $y-1=-\frac{5}{4}(x-4)$

(b) Normal: $y-1=\frac{4}{5}(x-4)$

49. $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-2\sin t}{2\cos t}=-\tan t$

At $t=\frac{3\pi}{4}$, we have $x=2\sin\frac{3\pi}{4}=\sqrt{2}$,

$y=2\cos\frac{3\pi}{4}=-\sqrt{2}$, and $\frac{dy}{dx}=-\tan\frac{3\pi}{4}=1.$

The equation of the tangent line is

$y+\sqrt{2}=1\left(x-\sqrt{2}\right).$

50. $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{4\cos t}{-3\sin t}=-\frac{4}{3}\cot t$

At $t=\frac{3\pi}{4}$, we have $x=3\cos\frac{3\pi}{4}=-\frac{3\sqrt{2}}{2}$,

$y=4\sin\frac{3\pi}{4}=2\sqrt{2}$, and $\frac{dy}{dx}=-\frac{4}{3}\cot\frac{3\pi}{4}=\frac{4}{3}.$

The equation of the tangent line is

$y-2\sqrt{2}=\frac{4}{3}\left(x+\frac{3\sqrt{2}}{2}\right).$

51. $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{5\sec^2 t}{3\sec t\tan t}=\frac{5\sec t}{3\tan t}=\frac{5}{3\sin t}$

At $t=\frac{\pi}{6}$, we have $x=3\sec\frac{\pi}{6}=2\sqrt{3}$,

$y=5\tan\frac{\pi}{6}=\frac{5\sqrt{3}}{3}$, and $\frac{dy}{dx}=\frac{5}{3\sin\left(\frac{\pi}{6}\right)}=\frac{10}{3}.$

The equation of the tangent line is

$y-\frac{5\sqrt{3}}{3}=\frac{10}{3}\left(x-2\sqrt{3}\right).$

52. $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1+\cos t}{-\sin t}$

At $t=-\frac{\pi}{4}$, we have $x=\cos\left(-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$,

$y=-\frac{\pi}{4}+\sin\left(-\frac{\pi}{4}\right)=-\frac{\pi}{4}-\frac{\sqrt{2}}{2}$, and

$\frac{dy}{dx}=\frac{1+\cos\left(-\frac{\pi}{4}\right)}{-\sin\left(-\frac{\pi}{4}\right)}=\frac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=\sqrt{2}+1.$

The equation of the tangent line is

$y+\frac{\pi}{4}+\frac{\sqrt{2}}{2}=\left(1+\sqrt{2}\right)\left(x-\frac{\sqrt{2}}{2}\right).$

53. **(a)** $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(\sin ax+b\cos x)=b$ and $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(5x+3)=3.$ Thus $\lim_{x\to 0} f(x)=f(0)=3$ if and only if $b=3$.

(b) $f'(x)=\begin{cases} a\cos ax-b\sin x, & x<0 \\ 5, & x>0\end{cases}$

The slopes match at $x=0$ if and only if $a=5$.

(c) No; although the slopes match, the function is not continuous.

54. **(a)** The function is continuous for all values of m, because the right-hand limit as $x\to 0$ is equal to $f(0)=0$ for any value of m.

(b) The left-hand derivative at $x=0$ is $2\cos(2\cdot 0)=2$, and the right-hand derivative at $x=0$ is m, so in order for the function to be differentiable at $x=0$, m must be 2.

55. Note that $f(x)=\sqrt[7]{(x-1)^3}=(x-1)^{3/7}$, so $f'(x)=\frac{3}{7}(x-1)^{-4/7}$, which is defined if and only if $x \neq 1$. Thus the answers are

(a) For all $x \neq 1$

(b) At $x = 1$

(c) Nowhere

56. **(a)** For all x

(b) Nowhere

(c) Nowhere

57. The individual functions are continuous and differentiable on [–1, 3]. At $x = 1$ we have $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\sqrt{x^2+3}=2$ and $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x+1)=2=f(1)$, so f is continuous at $x = 1$. However,

$$f'(x)=\begin{cases}\dfrac{2x}{2\sqrt{x^2+3}}, & x<1\\ 1, & x>1\end{cases}$$

so the slope is $\frac{1}{2}$ coming from the left and 1 coming from the right. Thus f is not differentiable at $x = 1$. The answers are:

(a) $[-1, 1) \cup (1, 3]$

(b) At $x = 1$

(c) Nowhere

58. The individual functions are continuous and differentiable on [–3, 3]. At $x = 0$ we have $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\sin 2x=0$ and $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x^2+2x+1)=1=f(1)$, so f is not continuous at $x = 0$. Thus f is not differentiable at $x = 0$. The answers are

(a) $[-3, 0) \cup (0, 3]$

(b) Nowhere

(c) At $x = 0$

59. **(a)** Since $3 - \sin x > 0$ for all x,

$y=\left(\sqrt{(3-\sin x)}\right)^2=3-\sin x$, so

$\frac{dy}{dx}=-\cos x.$

(b)
$$\begin{aligned}y&=\ln(3e^{7x^2-13x+5})\\&=\ln 3+\ln(e^{7x^2-13x+5})\\&=\ln 3+7x^2-13x+5,\end{aligned}$$

so $\frac{dy}{dx}=14x-13.$

(c) $s=\tan(\tan^{-1}(t^2-3t))=t^2-3t$, so

$\frac{ds}{dt}=2t-3.$

(d) $s=\sqrt[3]{t^6}-5(\sin(\sin^{-1}t))^6=t^2-5t^6$, so

$\frac{ds}{dt}=2t-30t^5.$

60. **(a)** $y=\ln\left(\frac{2x+7}{3x+2}\right)=\ln(2x+7)-\ln(3x+2)$,

so $\frac{dy}{dx}=\frac{2}{2x+7}-\frac{3}{3x+2}.$

(b)
$$\begin{aligned}y&=\frac{(x^2-1)^2}{(x^2-2x+1)(x+1)}\\&=\frac{(x-1)^2(x+1)^2}{(x-1)^2(x+1)}\\&=x+1,\end{aligned}$$

so $\frac{dy}{dx}=1.$

(c)
$$\begin{aligned}s&=\sin^2(\cos^{-1}t)\\&=1-\cos^2(\cos^{-1}t)\\&=1-t^2,\end{aligned}$$

so $\frac{ds}{dt}=-2t.$

(d) $s=\left(\frac{2\sqrt{t}}{\sqrt[3]{t}}\right)^5$

$=\left(2\cdot t^{\frac{1}{2}-\frac{1}{3}}\right)^5$

$=(2t^{1/6})^5$

$=32t^{5/6}$,

so $\frac{ds}{dt}=\left(\frac{160}{6}\right)t^{-1/6}=\frac{80}{3\sqrt[6]{t}}$.

61. **(a)** The line passes through (2, 4), which must be the point of tangency. The slope of the tangent line is 3, so the slope of the normal line is $-\frac{1}{3}$. The equation of the normal line is $y-4=-\frac{1}{3}(x-2)$.

(b) The tangent line to f at (2, 4) has slope 3, so the tangent line to f^{-1} at (4, 2) has slope $\frac{1}{3}$. The equation of the line is $y-2=\frac{1}{3}(x-4)$.

(c) When $x = 2$, $y=\frac{f(2)}{2}=\frac{4}{2}=2$ and $\frac{dy}{dx}=\left.\frac{f'(x)\cdot x-1\cdot f(x)}{x^2}\right|_{x=2}=\frac{3\cdot 2-4}{4}=\frac{1}{2}$.

The line has equation $y-2=\frac{1}{2}(x-2)$.

62. **(a)** The line passes through (1, 3), which must be the point of tangency. The slope of the tangent line is –2, so the slope of the normal line is $\frac{1}{2}$. The equation of the normal line is $y-3=\frac{1}{2}(x-1)$.

(b) The tangent line to g at (1, 3) has slope –2, so the tangent line to g^{-1} at (3, 1) has slope $-\frac{1}{2}$. The equation of the line is $y-1=-\frac{1}{2}(x-3)$.

(c) When $x = 1$, $y=g(1^2)=g(1)=3$ and $\frac{dy}{dx}=g'(x^2)\cdot 2x\big|_{x=1}=g'(1)\cdot 2=-4$. The line has equation $y - 3 = -4(x - 1)$.

63. First, note that $\ln y = 5\ln(x + 2) + 4\ln(2x - 3) - 2\ln(x + 17)$. Then $\frac{1}{y}\frac{dy}{dx}=\frac{5}{x+2}+\frac{8}{2x-3}-\frac{2}{x+17}$, and so

$$\frac{dy}{dx}=\frac{(x+2)^5(2x-3)^4}{(x+17)^2}\left(\frac{5}{x+2}+\frac{8}{2x-3}-\frac{2}{x+17}\right)$$

64. First, note that $\ln y=(x+5)\ln(x^2+2)$. Then $\frac{1}{y}\frac{dy}{dx}=1\cdot\ln(x^2+2)+(x+5)\left(\frac{2x}{x^2+2}\right)$, and so $\frac{dy}{dx}=(x^2+2)^{x+5}\left(\ln(x^2+2)+\frac{2x^2+10x}{x^2+2}\right)$.

65. **(a)** $f(x)=\frac{x^2}{2}$ or $f(x)=\frac{x^2}{2}+C$

(b) $f(x)=e^x$ or $f(x)=Ce^x$

(c) $f(x)=e^{-x}$ or $f(x)=Ce^{-x}$

(d) $f(x)=e^x$ or $f(x)=e^{-x}$ or $f(x)=Ce^x+De^{-x}$

(e) $f(x) = \sin x$ or $f(x) = \cos x$ or $f(x) = C\sin x + D\cos x$

66. **(a)** $\frac{d}{dx}\left[\sqrt{x}f(x)\right]=\sqrt{x}f'(x)+\frac{1}{2\sqrt{x}}f(x)$

At $x = 1$, the derivative is

$$\sqrt{1}f'(1)+\frac{1}{2\sqrt{1}}f(1)=1\left(\frac{1}{5}\right)+\left(\frac{1}{2}\right)(-3)=-\frac{13}{10}.$$

(b) $\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

At $x = 0$, the derivative is

$\frac{f'(0)}{2\sqrt{f(0)}} = -\frac{2}{2\sqrt{9}} = -\frac{1}{3}.$

(c) $\frac{d}{dx} f(\sqrt{x}) = f'(\sqrt{x})\frac{d}{dx}\sqrt{x} = \frac{f'(\sqrt{x})}{2\sqrt{x}}$

At $x = 1$, the derivative is

$\frac{f'(\sqrt{1})}{2\sqrt{1}} = \frac{f'(1)}{2} = \frac{\frac{1}{5}}{2} = \frac{1}{10}.$

(d) $\frac{d}{dx} f(1 - 5\tan x)$

$= f'(1 - 5\tan x)(-5\sec^2 x)$

At $x = 0$, the derivative is

$f'(1 - 5\tan 0)(-5\sec^2 0) = f'(1)(-5)$

$= \left(\frac{1}{5}\right)(-5)$

$= -1.$

(e) $\frac{d}{dx}\frac{f(x)}{2 + \cos x}$

$= \frac{(2 + \cos x)(f'(x)) - (f(x))(-\sin x)}{(2 + \cos x)^2}$

At $x = 0$, the derivative is

$\frac{(2 + \cos 0)(f'(0)) - (f(0))(-\sin 0)}{(2 + \cos 0)^2}$

$= \frac{3f'(0)}{3^2}$

$= -\frac{2}{3}.$

(f) $\frac{d}{dx}[10\sin\left(\frac{\pi x}{2}\right) f^2(x)]$

$= 10\left(\sin\frac{\pi x}{2}\right)(2f(x)f'(x)) + 10f^2(x)\left(\cos\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)$

$= 20f(x)f'(x)\sin\frac{\pi x}{2} + 5\pi f^2(x)\cos\frac{\pi x}{2}$

At $x = 1$, the derivative is

$20f(1)f'(1)\ \sin\frac{\pi}{2} + 5\pi f^2(1)\cos\frac{\pi}{2}$

$= 20(-3)\left(\frac{1}{5}\right)(1) + 5\pi(-3)^2(0)$

$= -12.$

67. **(a)** $\frac{d}{dx}\left(\frac{f(2x)}{x - 1}\right) = \frac{f'(2x)\cdot 2\cdot(x - 1) - 1\cdot f(2x)}{(x - 1)^2}$

At $x = 0$, the derivative is

$\frac{f'(0)(-2) - f(0)}{1} = \frac{(-2)(-2) - (-1)}{1} = 5.$

(b) $\frac{d}{dx}\left[f^2(x)g^3(x)\right]$

$= f^2(x)\cdot 3g^2(x)g'(x) + g^3(x)\cdot 2f(x)f'(x)$

$= f(x)g^2(x)\left[3f(x)g'(x) + 2g(x)f'(x)\right]$

At $x = 0$, the derivative is

$f(0)g^2(0)\left[3f(0)g'(0) + 2g(0)f'(0)\right]$

$= (-1)(-3)^2\left[3(-1)(4) + 2(-3)(-2)\right]$

$= -9\left[-12 + 12\right] = 0.$

(c) $\frac{d}{dx} g(f(x)) = g'(f(x))f'(x)$

At $x = -1$, the derivative is

$g'(f(-1))f'(-1) = g'(0)f'(-1)$

$= (4)(2)$

$= 8.$

(d) $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

At $x = -1$, the derivative is

$f'(g(-1))g'(-1) = f'(-1)g'(-1)$

$= (2)(1)$

$= 2.$

(e) $\frac{d}{dx} f(g(2x - 1)) = 2f'(g(2x - 1))g'(2x - 1)$

At $x = 0$, the derivative is

$2f'(g(-1))g'(-1) = 2f'(-1)g'(-1)$

$= 2(2)(1)$

$= 4.$

(f) $\frac{d}{dx} g(x + f(x))$

$= g'(x + f(x))\frac{d}{dx}(x + f(x))$

$= g'(x + f(x))(1 + f'(x))$

At $x = 0$, the derivative is

$g'(0 + f(0))[1 + f'(0)]$

$= g'(0 - 1)[1 + (-2)]$

$= (1)(-1)$

$= -1$

68. $\frac{dw}{ds} = \frac{dw}{dr}\frac{dr}{ds}$

$= \frac{d}{dr}\left[\sin\left(\sqrt{r}-2\right)\right]\frac{d}{ds}\left[8\sin\left(s+\frac{\pi}{6}\right)\right]$

$= \left[\cos\left(\sqrt{r}-2\right)\frac{1}{2\sqrt{r}}\right]\left[8\cos\left(s+\frac{\pi}{6}\right)\right]$

At $s = 0$, we have $r = 8\sin\left(0+\frac{\pi}{6}\right) = 4$ and so

$\frac{dw}{ds} = \left[\cos\left(\sqrt{4}-2\right)\frac{1}{2\sqrt{4}}\right]\left[8\cos\left(0+\frac{\pi}{6}\right)\right]$

$= \left(\frac{\cos 0}{4}\right)\left(8\cos\frac{\pi}{6}\right)$

$= \left(\frac{1}{4}\right)\left(\frac{8\sqrt{3}}{2}\right)$

$= \sqrt{3}$

69. Solving $\theta^2 t + \theta = 1$ for t, we have $t = \frac{1-\theta}{\theta^2} = \theta^{-2} - \theta^{-1}$, and we may write:

$$\frac{dr}{d\theta} = \frac{dr}{dt}\frac{dt}{d\theta}$$

$$\frac{d}{d\theta}(\theta^2+7)^{1/3} = \frac{dr}{dt}\frac{d}{d\theta}(\theta^{-2}-\theta^{-1})$$

$$\frac{1}{3}(\theta^2+7)^{-2/3}(2\theta) = \left(\frac{dr}{dt}\right)(-2\theta^{-3}+\theta^{-2})$$

$$\frac{dr}{dt} = \frac{2\theta(\theta^2+7)^{-2/3}}{3(-2\theta^{-3}+\theta^{-2})} = \frac{2\theta^4(\theta^2+7)^{-2/3}}{3(\theta-2)}$$

At $t = 0$, we may solve $\theta^2 t + \theta = 1$ to obtain $\theta = 1$, and so

$$\frac{dr}{dt} = \frac{2(1)^4(1^2+7)^{-2/3}}{3(1-2)} = \frac{2(8)^{-2/3}}{-3} = -\frac{1}{6}.$$

70. **(a)** One possible answer:

$x(t) = 10\cos\left(t+\frac{\pi}{4}\right),\ y(t) = 0$

(b) $s(0) = 10\cos\frac{\pi}{4} = 5\sqrt{2}$

(c) Farthest left:
When $\cos\left(t+\frac{\pi}{4}\right) = -1$, we have $s(t) = -10$.
Farthest right:
When $\cos\left(t+\frac{\pi}{4}\right) = 1$, we have $s(t) = 10$.

(d) Since $\cos\frac{\pi}{2} = 0$, the particle first reaches the origin at $t = \frac{\pi}{4}$. The velocity is given by $v(t) = -10\sin\left(t+\frac{\pi}{4}\right)$, so the velocity at $t = \frac{\pi}{4}$ is $-10\sin\frac{\pi}{2} = -10$, and the speed at $t = \frac{\pi}{4}$ is $|-10| = 10$. The acceleration is given by $a(t) = -10\cos\left(t+\frac{\pi}{4}\right)$, so the acceleration at $t = \frac{\pi}{4}$ is $-10\cos\frac{\pi}{2} = 0$.

71. **(a)** $8x + 8(xy' + y) + 2yy' = 0$, so $y' = -\frac{4x+4y}{4x+y}$. Tangent lines are horizontal where $y' = 0$, which is where $4x + 4y = 0$, in which case $y = -x$. Letting $y = -x$ in the equation of the hyperbola, we obtain $4x^2 + 8x(-x) + (-x)^2 + 3 = 0$. Solving yields $x = \pm 1$. Recalling that $y = -x$, we get the points $A(-1, 1)$; $B(1, -1)$.

(b) Again, $y' = -\frac{4x+4y}{4x+y}$. Tangent lines are vertical where y' is undefined, which is where $4x + y = 0$, in which case $y = -4x$. Letting $y = -4x$ in the equation of the hyperbola, we obtain $4x^2 + 8x(-4x) + (-4x)^2 + 3 = 0$. Solving yields $x = \pm 0.5$. Recalling that $y = -4x$, we get the points $C(-0.5, 2)$; $D(0.5, -2)$.

72. **(a)** $4x - 2(xy' + y) + 2yy' = 0$, so $y' = \frac{y-2x}{y-x}$. Tangent lines are horizontal where $y' = 0$, which is where $y - 2x = 0$, in which case $y = 2x$. Letting $y = 2x$ in the equation of the ellipse, we obtain $2x^2 - 2x(2x) + (2x)^2 - 4 = 0$. Solving yields $x = \pm\sqrt{2}$. Recalling that $y = 2x$, we get the points $A\left(-\sqrt{2}, -2\sqrt{2}\right)$; $B\left(\sqrt{2}, 2\sqrt{2}\right)$.

(b) Again, $y' = \frac{y-2x}{y-x}$. Tangent lines are vertical where y' is undefined, which is where $y - x = 0$, in which case $y = x$. Letting $y = x$ in the equation of the ellipse, we obtain $2x^2 - 2x(x) + (x)^2 - 4 = 0$. Solving yields $x = \pm 2$. Recalling that $y = x$, we get the points $C(-2, -2)$; $D(2, 2)$.

73. (a) $2x - 2(xy' + y) + 2yy' - 4 = 0$, so $y' = \frac{y-x+2}{y-x}$. The tangent line is vertical where y' is undefined, which is where $y - x = 0$, in which case $y = x$. Letting $y = x$ in the equation of the parabola, we obtain $x^2 - 2x(x) + x^2 - 4x = 8$. Solving yields $x = -2$. Recalling that $y = x$, we get the point $A(-2, -2)$.

(b) Again, $y' = \frac{y-x+2}{y-x}$. The tangent line is horizontal where $y' = 0$, which is where $y - x + 2 = 0$, in which case $y = x - 2$. Letting $y = x - 2$ in the equation of the parabola, we obtain $x^2 - 2x(x-2) + (x-2)^2 - 4x = 8$. Solving yields $x = -1$. Recalling that $y = x - 2$, we get the point $B(-1, -3)$.

74. Letting $x = 0$ in the equation of the parabola, we get $y^2 = 8$, which has solutions $y = \pm 2\sqrt{2}$. Letting $y = 0$ in the equation of the parabola, we get $x^2 - 4x = 8$, which has solutions $x = 2 \pm 2\sqrt{3}$.
Implicitly differentiating, $2x - 2(xy' + y) + 2yy' - 4 = 0$, so
$y' = \frac{y-x+2}{y-x}$.
At y-intercept $\left(0, 2\sqrt{2}\right)$ the slope is
$\frac{2\sqrt{2}-0+2}{2\sqrt{2}-0} = \frac{2+\sqrt{2}}{2}$.
At y-intercept $\left(0, -2\sqrt{2}\right)$ the slope is
$\frac{-2\sqrt{2}-0+2}{-2\sqrt{2}-0} = \frac{2-\sqrt{2}}{2}$.
At x-intercept $\left(2+2\sqrt{3}, 0\right)$ the slope is
$\frac{0-\left(2+2\sqrt{3}\right)+2}{0-\left(2+2\sqrt{3}\right)} = \frac{\sqrt{3}}{\sqrt{3}+1}$.
At x-intercept $\left(2-2\sqrt{3}, 0\right)$ the slope is
$\frac{0-\left(2-2\sqrt{3}\right)+2}{0-\left(2-2\sqrt{3}\right)} = \frac{\sqrt{3}}{\sqrt{3}-1}$.

75. Every sinusoid with amplitude 3 and period π is the graph of some equation of the form $y = 3\sin(2x + C) + D$. The slope at any x is $\frac{dy}{dx} = 6\cos(2x + C)$. Since the maximum value of cosine is 1, the maximum slope is 6.

76. Every sinusoid with amplitude A and period p is the graph of some equation of the form $y = A\sin\left(\frac{2\pi}{p}x + k\right) + D$. The slope at any x is $\frac{dy}{dx} = A \cdot \frac{2\pi}{p}\cos\left(\frac{2\pi}{p}x + k\right)$. Since the maximum value of cosine is 1, the maximum slope is $\frac{2\pi A}{p}$.

77. Let $f(x) = \sin(x - \sin x)$. Then

$$f'(x) = \cos(x - \sin x)\frac{d}{dx}(x - \sin x)$$
$$= \cos(x - \sin x)(1 - \cos x).$$

This derivative is zero when $\cos(x - \sin x) = 0$ (which we need not solve) or when $\cos x = 1$, which occurs at $x = 2k\pi$ for integers k. For each of these values,

$$f(x) = f(2k\pi)$$
$$= \sin(2k\pi - \sin 2k\pi)$$
$$= \sin(2k\pi - 0)$$
$$= 0.$$

Thus, $f(x) = f'(x) = 0$ for $x = 2k\pi$, which means that the graph has a horizontal tangent at each of these values of x.

78. (a) $P(0) = \frac{200}{1+e^5} \approx 1$ student

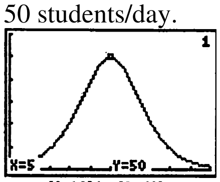
(b) $\lim_{h\to\infty} P(t) = \lim_{h\to\infty} \frac{200}{1+e^{5-t}}$
$= \frac{200}{1}$
$= 200$ students

(c) $P'(t) = \frac{d}{dt} 200(1+e^{5-t})^{-1}$

$= -200(1+e^{5-t})^{-2}(e^{5-t})(-1)$

$= \frac{200e^{5-t}}{(1+e^{5-t})^2}$

A graph of the derivative $y = P'(t)$ shows a maximum value at $t = 5$, at which point $P'(t) = 50$. The spread of the disease is greatest at $t = 5$, when the rate is 50 students/day.



[0, 10] by [0, 60]

The maximum rate occurs at $t = 5$, and this rate is

$P'(5) = \frac{200e^0}{(1+e^0)^2} = \frac{200}{2^2} = 50$ students per day.

79. Differentiating implicitly,

$2x + 2(y + xy') + 4yy' = 0$, so $y' = -\frac{x+y}{x+2y}$.

(a) At (1, 1) we get $y' = \frac{dy}{dx} = -\frac{2}{3}$.

(b) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dx}\left(-\frac{x+y}{x+2y}\right)$

$= -\frac{(1+y')(x+2y)-(1+2y')(x+y)}{(x+2y)^2}$

At (1, 1) and recalling that $y' = \frac{dy}{dx} = -\frac{2}{3}$, we get

$\frac{d^2y}{dx^2} = -\frac{\left(1-\frac{2}{3}\right)(1+2)-\left(1-\frac{4}{3}\right)(1+1)}{(1+2)^2}$

$= -\frac{5}{27}$.

80. Use implicit differentiation.

$x^2 - y^2 = 1$

$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$

$2x - 2yy' = 0$

$y' = \frac{2x}{2y} = \frac{x}{y}$

$y'' = \frac{d}{dx}\frac{x}{y}$

$= \frac{(y)(1)-(x)(y')}{y^2}$

$= \frac{y - x\left(\frac{x}{y}\right)}{y^2}$

$= \frac{y^2 - x^2}{y^3}$

$= -\frac{1}{y^3}$

(since the given equation is $x^2 - y^2 = 1$)

At $\left(2, \sqrt{3}\right)$, $\frac{d^2y}{dx^2} = -\frac{1}{y^3} = -\frac{1}{(\sqrt{3})^3} = -\frac{1}{3\sqrt{3}}$.

81. (a) $g'(x) = k \cdot e^{kx} + f'(x)$, so $g'(0) = k + 3$.

$g''(x) = k^2 \cdot e^{kx} + f''(x)$, so

$g''(0) = k^2 - 1$.

(b) $h'(x) = -b\sin(bx)f(x) + f'(x)\cos(bx)$, so $h'(0) = -b \cdot \sin(0) + 3 \cdot \cos(0) = 3$. Note that $h(0) = \cos(0) \cdot f(0) = 1 \cdot 2 = 2$, so the tangent line has equation $y - 2 = 3(x - 0)$.

82. (a) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}(e^x + e^{-x})\right)$

$= \frac{1}{2}(e^x + e^{-x}(-1))$

$= \frac{e^x - e^{-x}}{2}$

(b) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{2}(e^x - e^{-x})\right)$

$= \frac{1}{2}(e^x - e^{-x}(-1))$

$= \frac{e^x + e^{-x}}{2}$

(c) $y(1) = \frac{e + e^{-1}}{2} \approx 1.543;$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{e - e^{-1}}{2} \approx 1.175;$$

$$y - 1.543 = 1.175(x - 1)$$
$$y = 1.175x + 0.368$$

(d)
$$m = -\frac{1}{dy/dx} = -\frac{2}{e - e^{-1}} \approx -\frac{1}{1.175} \approx -0.851$$
$$y - 1.543 = -0.851(x - 1)$$
$$y = -0.851x + 2.394$$

(e) $\frac{e^x - e^{-x}}{2} = 0$ $\left(\text{set } \frac{dy}{dx} \text{ equal to } 0\right)$

$e^x - e^{-x} = 0$ (multiply both sides by 2)

$e^x = e^{-x}$

$e^{2x} = e^0$ (multiply both sides by e^x)

$\ln e^{2x} = \ln e^0$

$2x = 0$

$x = 0$

The tangent line is horizontal at $x = 0$.

83. **(a)** $1 - x^2 > 0 \to x^2 < 1 \to \sqrt{x^2} < \sqrt{1} \to |x| < 1$
$\to -1 < x < 1$
Domain of $f = (-1, 1)$

(b) $f'(x) = \frac{1}{1 - x^2}(-2x) = -\frac{2x}{1 - x^2}$

(c) Domain of $f' = \{x \mid x^2 \neq 1$ and $x \in$ Domain of $f\}$
Domain of $f' = (-1, 1)$

(d)
$$f''(x) = -\frac{(1 - x^2)(2) - (-2x)(2x)}{(1 - x^2)^2} = -\frac{2 - 2x^2 + 4x^2}{(1 - x^2)^2} = -\frac{2 + 2x^2}{(1 - x^2)^2} = -\frac{2(x^2 + 1)}{(x^2 - 1)^2} < 0 \text{ for } x \neq \pm 1$$

(The numerator and denominator are clearly both positive.) Therefore, $f''(x) < 0$ for all $x \in (-1, 1)$.