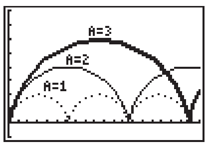
Chapter 11

Parametric, Vector, and Polar Functions

Section 11.1 Parametric Functions (pp. 543–549)

Exploration 1 Investigating Cycloids

1. 

[0, 20] by [−1, 8]

2. $x = 2na\pi$ for any integer n.

3. $a > 0$ and $1 - \cos t \geq 0$ so $y \geq 0$.

4. An arch is produced by one complete turn of the wheel. Thus, they are congruent.

5. The maximum value of y is $2a$ and occurs when $x = (2n+1)a\pi$ for any integer n.

6. The function represented by the cycloid is periodic with period $2a\pi$, and each arch represents one period of the graph. In each arch, the graph is concave down, has an absolute maximum of $2a$ at the midpoint, and an absolute minimum of 0 at the two endpoints.

Quick Review 11.1

1. $t = x - 1$
$y = 2t + 3 = 2(x-1) + 3 = 2x + 1$

2. $t = \frac{x}{3}$
$y = 54t^3 - 3 = 54\left(\frac{x}{3}\right)^3 - 3 = 2x^3 - 3$

3. $x^2 = \sin^2 t$
$y^2 = \cos^2 t$
$\cos^2 t + \sin^2 t = 1$
$x^2 + y^2 = 1$

4. $y = \sin 2t = 2\sin t \cos t$
$y = 2x$

5. $x^2 = \tan^2\theta$
$y^2 = \sec^2\theta$
$\sec^2\theta = 1 + \tan^2 x$
$y^2 = 1 + x^2$

6. $x^2 = \csc^2\theta = 1 + \cot^2\theta$
$y^2 = \cot^2\theta$
$x^2 = 1 + y^2$

7. $x^2 = \cos^2\theta$
$y = \cos 2\theta = 2\cos^2\theta - 1$
$y = 2x^2 - 1$

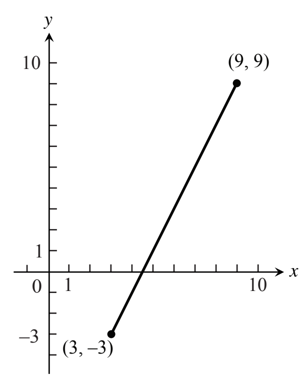
8. $x^2 = \sin^2\theta$
$y = \cos 2\theta = 1 - 2\sin^2\theta$
$y = 1 - 2x^2$

9. $\cos^2\theta + \sin^2\theta = 1$
$x^2 + y^2 = 1$
$y^2 = 1 - x^2$
$y = \sqrt{1 - x^2}$
since $y \geq 0$ for $0 \leq \theta \leq \pi$.

10. $\cos^2\theta + \sin^2\theta = 1$
$x^2 + y^2 = 1$
$y^2 = 1 - x^2$
$y = -\sqrt{1 - x^2}$
since $y \leq 0$ for $\pi \leq \theta \leq 2\pi$.

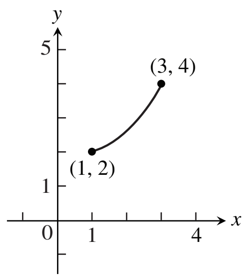
Section 11.1 Exercises

1. Yes, y is a function of x.
$y = 2x - 9$



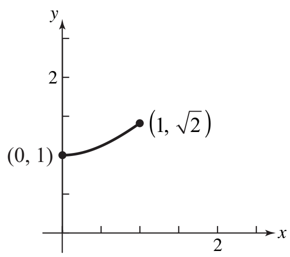
2. Yes, y is a function of x.

$$y = \frac{x^2 + 7}{4}$$



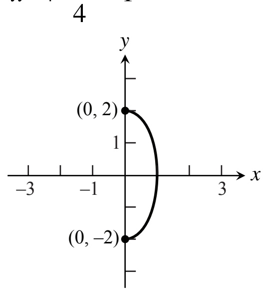
3. Yes, y is a function of x.

$$y = \sqrt{x^2 + 1}$$



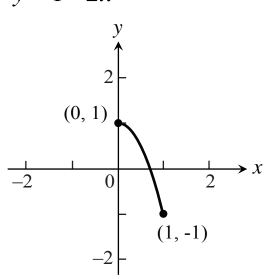
4. No, y is not a function of x.

$$x^2 + \frac{y^2}{4} = 1$$



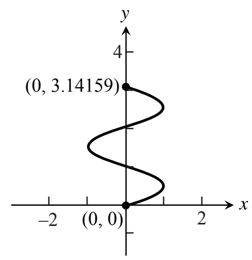
5. Yes, y is a function of x.

$$y = 1 - 2x^2$$



6. No, y is not a function of x.

$$x = \sin(3y)$$



7. **(a)** $$\frac{dy}{dx} = \frac{-2\sin t}{4\cos t} = -\frac{1}{2}\tan t$$

(b) $$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(-\frac{1}{2}\tan t\right)}{4\cos t} = \frac{-\frac{1}{2}\sec^2 t}{4\cos t} = -\frac{1}{8}\sec^3 t$$

8. **(a)** $$\frac{dy}{dx} = \frac{-\sqrt{3}\sin t}{-\sin t} = \sqrt{3}$$

(b) $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\sqrt{3}\right)}{-\sin t} = 0$$

9. **(a)** $$\frac{dy}{dx} = \frac{\frac{3}{2}(3t)^{-1/2}}{-\left(\frac{1}{2}\right)(t+1)^{-1/2}} = -\sqrt{\frac{9t+9}{3t}} = -\sqrt{3 + \frac{3}{t}}$$

(b) $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(-\sqrt{3 + \frac{3}{t}}\right)}{-\left(\frac{1}{2}\right)(t+1)^{-1/2}} = -\frac{\sqrt{3}}{t^{3/2}}$$

10. **(a)** $$\frac{dy}{dx} = \frac{\frac{1}{t}}{-\frac{1}{t^2}} = -t$$

(b) $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(-t)}{-\frac{1}{t^2}} = t^2$$

11. **(a)** $\frac{dy}{dx} = \frac{3t^2}{2t-3}$

(b) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t^2}{2t-3}\right)}{2t-3} = \frac{6t^2-18t}{(2t-3)^3}$

12. **(a)** $\frac{dy}{dx} = \frac{2t-1}{2t+1}$

(b) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{2t-1}{2t+1}\right)}{2t+1} = \frac{4}{(2t+1)^3}$

13. **(a)** $\frac{dy}{dx} = \frac{\sec t \tan t}{\sec^2 t} = \sin t$

(b) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\sin t}{\sec^2 t} = \cos^3 t$

14. **(a)** $\frac{dy}{dx} = \frac{-2\sin(2t)}{-2\sin t} = 2\cos t$

(b) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(2\cos t)}{-2\sin t} = 1$

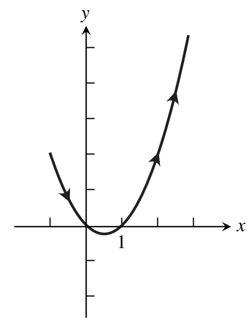
15. **(a)** $\frac{dy}{dx} = \frac{4\left(\frac{1}{t}\right)}{\frac{1}{t}} = 4$

(b) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(4)}{\frac{1}{t}} = 0$

16. **(a)** $\frac{dy}{dx} = \frac{5e^{5t}}{\frac{1}{t}} = 5te^{5t}$

(b) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(5te^{5t})}{\frac{1}{t}} = 25t^2e^{5t} + 5te^{5t}$

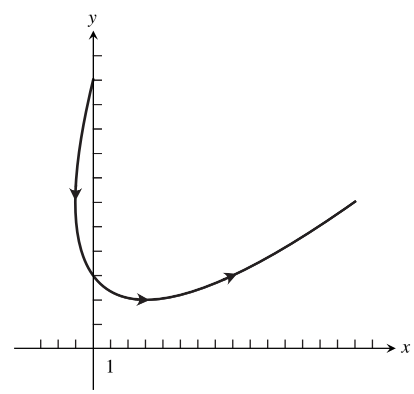
17. **(a)**



(b) (0.5, –0.25)

(c) We seek to minimize y as a function of t, so we compute $\frac{dy}{dt} = 2t+1$, which is negative for $-2 \le t < -0.5$ and positive for $-0.5 < t \le 2$. There is a relative minimum at $t = -0.5$, where $(x, y) = (0.5, -0.25)$.

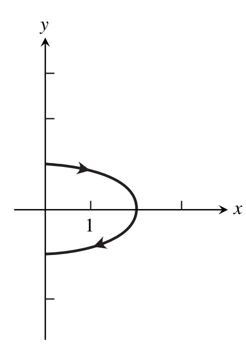
18. **(a)**



(b) (–1, 6)

(c) We seek to minimize x as a function of t, so we compute $\frac{dx}{dt} = 2t+2$, which is negative for $-2 \le t < -1$ and positive for $-1 < t \le 3$. There is a relative minimum at $t = -1$, where $(x, y) = (-1, 6)$.

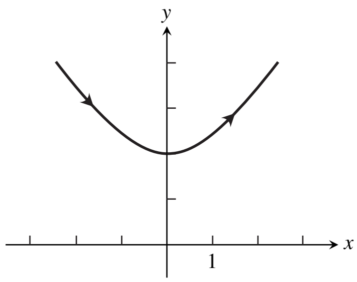
19. **(a)**



(b) (2, 0)

(c) We seek to maximize x as a function of t, so we compute $\frac{dx}{dt} = 2\cos t$, which is positive for $0 \le t < \frac{\pi}{2}$ and negative for $\frac{\pi}{2} < t \le \pi$. There is a relative maximum at $t = \frac{\pi}{2}$, where $(x, y) = (2, 0)$.

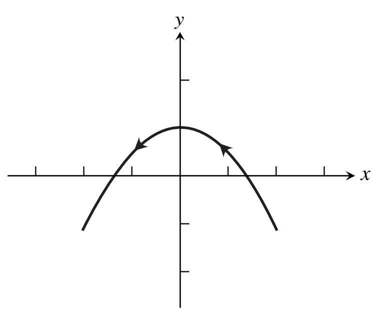
20. **(a)**



(b) (0, 2)

(c) We seek to minimize y as a function of t, so we compute $\frac{dy}{dt} = 2\sec t \tan t$, which is negative for $-1 \le t < 0$ and positive for $0 < t \le 1$. There is a relative minimum at $t = 0$, where $(x, y) = (0, 2)$.

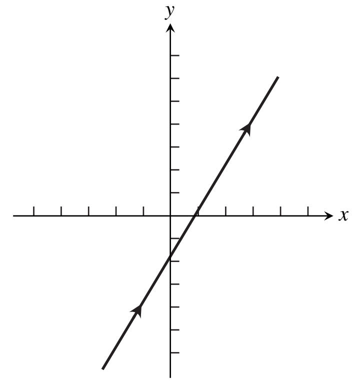
21. **(a)**



(b) (0, 1)

(c) We seek to maximize y as a function of t, so we compute $\frac{dy}{dt} = -2\sin(2t)$, which is positive for $1.5 \le t < \pi$ and negative for $\pi < t \le 4.5$. There is a relative maximum at $t = \pi$, where $(x, y) = (0, 1)$.

22. **(a)**



(b) $(\ln(50), \ln(400)) \approx (3.912, 5.991)$

(c) We seek to maximize x as a function of t, so we compute $\frac{dx}{dt} = \frac{1}{t}$, which is positive for all $t > 0$. There is an endpoint maximum at $t = 10$, where $(x, y) = (\ln(50), \ln(400))$.

23. $\frac{dy}{dt} = \frac{d}{dt}(-1 + \sin t) = \cos t$ and $\frac{dx}{dt} = \frac{d}{dt}(2 + \cos t) = -\sin t$.

(a) Tangent is horizontal when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
If $\frac{dy}{dt} = 0$, then $\cos t = 0$, and $x = 2 + \cos t = 2$.
If $\cos t = 0$, then $\sin t = \pm 1$, and $y = -1 + \sin t = 0$ or -2.
The points are $(2, 0)$ and $(2, -2)$.

(b) Tangent is vertical when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. If $\frac{dx}{dt} = 0$, then $\sin t = 0$, and $y = -1 + \sin t = -1$.
If $\sin t = 0$, then $\cos t = \pm 1$, and $x = 2 + \cos t = 1$ or 3.
The points are $(1, -1)$ and $(3, -1)$.

24. $\frac{dy}{dt} = \frac{d}{dt}(\tan t) = \sec^2 t$ and

$\frac{dx}{dt} = \frac{d}{dt}(\sec t) = \sec t \tan t.$

(a) Tangent is horizontal when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0.$ Since $\frac{dy}{dt} = \sec^2 t > 0$ for all t, there are no points where the tangent line is horizontal.

(b) Tangent is vertical when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0.$ If $\frac{dx}{dt} = 0,$ then $\sec t \tan t = 0,$ so $y = \tan t = 0.$
If $\tan t = 0,$ then $x = \sec t = \pm 1.$
The points are (1, 0) and (–1, 0).

25. $\frac{dy}{dt} = \frac{d}{dt}(t^3 - 4t) = 3t^2 - 4$ and

$\frac{dx}{dt} = \frac{d}{dt}(2 - t) = -1.$

(a) If $\frac{dy}{dt} = 0,$ then $3t^2 - 4 = 0$ and $t = \pm\frac{2}{\sqrt{3}}.$ Using the parametric formulas for x and y, the points are
$\left(2 - \frac{2}{\sqrt{3}}, -\frac{16\sqrt{3}}{9}\right)$ and $\left(2 + \frac{2}{\sqrt{3}}, \frac{16\sqrt{3}}{9}\right).$
(In decimal form, they are (0.845, –3.079) and (3.155, 3.079).)

(b) Tangent is vertical when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0.$ Since $\frac{dx}{dt} = -1 \neq 0$ for all x, there are no points where the tangent line is vertical.

26. $\frac{dy}{dt} = \frac{d}{dt}(1 + 3\sin t) = 3\cos t$ and

$\frac{dx}{dt} = \frac{d}{dt}(-2 + 3\cos t) = -3\sin t.$

(a) If $\frac{dy}{dt} = 0,$ then $\cos t = 0$ and
$x = -2 + 3\cos t = -2.$
If $\cos t = 0,$ then $\sin t = \pm 1,$ and
$y = 1 + 3\sin t = -2$ or 4.
The points are (–2, –2) and (–2, 4).

(b) Tangent is vertical when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0.$ If $\frac{dx}{dt} = 0,$ then $\sin t = 0,$ and
$y = -2 + 3\cos t = -2.$
If $\sin t = 0,$ then $\cos t = \pm 1,$ and
$x = -2 + 3\cos t = -5$ or 1.
The points are (–5, 1) and (1, 1).

27. $S = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$

$S = \int_0^{2\pi} \sqrt{1}\, dt$

$S = t\Big|_0^{2\pi} = 2\pi - 0$

$S = 2\pi$

28. $S = \int_0^{\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt$

$S = \int_0^{\pi} \sqrt{9}\, dt$

$S = 3t\Big|_0^{\pi} = 3\pi - 0$

$S = 3\pi$

29. $S=\int_0^{\pi/2}((-8\sin t+8\sin t+8t\cos t)^2+(8\cos t-8\cos t+8t\sin t)^2)^{1/2}\,dt$

$S=\int_0^{\pi/2}((8t\cos t)^2+(8t\sin t)^2)^{1/2}dt$

$S=\int_0^{\pi/2}8t\,dt$

$S=4t^2\Big|_0^{\pi/2}=\pi^2-0$

$S=\pi^2$

30. $\int_0^{2\pi}([-6\cos^2(t)\sin(t)]^2+[6\sin^2(t)\cos(t)]^2)^{1/2}\,dt$

$=\int_0^{2\pi}6\sqrt{\cos^4(t)\sin^2(t)+\sin^4(t)\cos^2(t)}\;dt$

$=\int_0^{2\pi}6\sqrt{[\cos^2(t)+\sin^2(t)][\cos^2(t)\sin^2(t)]}\,dt$

$=\int_0^{2\pi}6\sqrt{\cos^2(t)\sin^2(t)}\;dt$

$=4\cdot\int_0^{\pi/2}6\cos t\,\sin t\;dt$

$=4\cdot\int_0^{\pi/2}3\cdot\sin 2t\;dt$

$=12\left\{\frac{-\cos 2t}{2}\right\}\Big|_0^{\pi/2}$

$=6\cdot\;[1-(-1)]$

$=12$

31. $S=\int_0^3\left(\left(\sqrt{2t+3}\right)^2+(1+t)^2\right)^{1/2}dt$

$S=\int_0^3(t^2+4t+4)^{1/2}dt$

$S=\frac{21}{2}$

32. $S=\int_0^2\left(\left(\sqrt{8t+8}\right)^2+(2t+1)^2\right)^{1/2}dt$

$S=\int_0^2(4t^2+12t+9)^{1/2}\,dt$

$S=10$

33. $S=\int_0^1((t^2)^2+(t)^2)^{1/2}dt$

$S=\int_0^1(t^4+t^2)^{1/2}\,dt$

$S=\frac{2\sqrt{2}-1}{3}$

34. $S = \int_0^{\pi/3} \sqrt{(\sec t - \cos t)^2 + (-\sin t)^2}\ dt$

$= \int_0^{\pi/3} \sqrt{\tan^2 t}\ dt$

$= \int_0^{\pi/3} \tan\ x\ dx$

$= \left[-\ln\left|\cos x\right|\right]_0^{\pi/3}$

$= \ln 2$

35. **(a)** $x' = -2\sin\ 2t,\ y' = 2\cos\ 2t,$ so

Length $= \int_0^{\pi/2} \sqrt{(-2\sin\ 2t)^2 + (2\cos\ 2t)^2}$

$= \int_0^{\pi/2} 2\ dt$

$= \pi.$

(b) $x' = \pi\cos\pi t,\ y' = -\pi\sin\pi t,$ so

Length $= \int_{-1/2}^{1/2} \sqrt{(\pi\cos\ \pi t)^2 + (-\pi\ \sin\ \pi t)^2}\,dt$

$= \int_{-1/2}^{1/2} \pi\ dt$

$= \pi.$

36. $x' = -3\sin t,\ y' = 4\cos t,$ so Length $= \int_0^{2\pi} \sqrt{(-3\ \sin t)^2 + (4\ \cos t)^2}\,dt$ which using NINT evaluates to ≈ 22.103.

37. In the first integral, replace t with x. Then $\frac{dx}{dt}$ becomes $\frac{dx}{dx} = 1.$

38. Parameterize the curve as $x = g(y)$, $y = y$, $c \le y \le d$. The parameter is y itself, so replace t with y in the general formula. Then $\frac{dy}{dt}$ becomes $\frac{dy}{dy} = 1.$

39. $\frac{dx}{dt} = a(1 - \cos t)$

(Note: integrate with respect to x from 0 to $2a\pi$; integrate with respect to t from 0 to 2π.)

Area $= \int_0^{2a\pi} y\ dx$

$= \int_0^{2\pi} a(1 - \cos t)a(1 - \cos t)\ dt$

$= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\ dt$

$= a^2 \left[t - 2\sin t + \frac{t}{2} + \frac{1}{4}\sin 2t\right]_0^{2\pi}$

$= 3\pi a^2$

40. $\frac{dx}{dt} = a(1-\cos t)$, so

$$\text{Volume } = \int_0^{2\pi} \pi[a(1-\cos t)]^2 a(1-\cos t)dt$$
$$= \pi a^3 \int_0^{2\pi} (1-3\cos t + 3\cos^2 t - \cos^3 t)dt$$
$$= \pi a^3 \left[t - 3\sin t + \frac{3}{2}t + \frac{3}{4}\sin 2t - \left(\sin t - \frac{1}{3}\sin^3 t \right) \right]_0^{2\pi}$$
$$= 5\pi^2 a^3$$

41. $x = at - b\sin t$ and
$y = a - b\cos t \ (0 < b < a)$

42. $x = at - b\sin t$ and
$y = a - b\cos t \ (a < b < 2a)$

43. $S = \int_0^{2\pi} ((3-2\cos t)^2 + (2\sin t)^2)^{1/2} dt$
$S = \int_0^{2\pi} (9 - 12\cos t + 4\cos^2 t + 4\sin^2 t)^{1/2} dt$
$S = \int_0^{2\pi} (13 - 12\cos t)^{1/2} dt$
$S \approx 21.010$

44. $S = \int_0^{2\pi} ((2-3\cos t)^2 + (3\sin t))^{1/2} dt$
$S = \int_0^{2\pi} (4 - 12\cos t + 9\cos^2 t + 9\sin^2 t)^{1/2} dt$
$S = \int_0^{2\pi} (13 - 12\cos t)^{1/2} dt$
$S \approx 21.010$

45. False. Indeed, *y* may not even be a function of *x*. (See Example 1.)

46. True. The ordered pairs $(x, f(x))$ and $(t, f(t))$ are exactly the same.

47. B

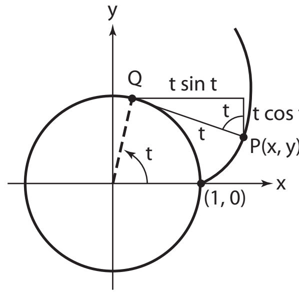
48. C; if $x = \sin t$ and $y = \csc t$, then we can eliminate the parameter to get $y = \frac{1}{x}$. Since sin *t* and csc *t* are both positive for $0 < t < \frac{\pi}{2}$, the path follows a portion of the curve in the first quadrant, where $y = \frac{1}{x}$ is decreasing and concave up.

49. C; $x = \ln(t) = \ln(y)$
$y = e^x$

50. D

51. **(a)** $\overline{QP}$ has length *t*, so *P* can be obtained by starting at *Q* and moving *t* sin *t* units right and *t* cos *t* units downward.
(If either quantity is negative, the corresponding direction is reversed.) Since $Q = (\cos t, \sin t)$, the

coordinates of P are $x = \cos t + t\sin t$ and $y = \sin t - t\cos t$.



(b) $x' = t\cos t,\ y' = t\sin t,$ so

$$\begin{aligned}\text{Length} &= \int_0^{2\pi} \sqrt{(t\cos t)^2 + (t\sin t)^2}\,dt \\ &= \int_0^{2\pi} t\ dt \\ &= 2\pi^2\end{aligned}$$

52. All distances are a times as big as before.

(a) $x = a(\cot t + t\sin t),\ y = a(\sin t - t\cos t)$

(b) $\text{Length} = 2a\pi^2$

For exercises 53–56, $x' = v_0\cos\theta$ and $y' = v_0\sin\theta - 32t,$ and $y = 0$ for $t = 0$ or $t = \dfrac{v_0\sin\theta}{16}$. The maximum height is attained in mid-flight at $t = \dfrac{v_0\sin\theta}{32}$. To find the path length, evaluate $\int_0^{v_0\sin\theta/16} \sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - 32t)^2}\,dt$ using NINT. To find the maximum height, calculate $y_{\text{max}} = (v_0\sin\theta)\left(\dfrac{v_0\sin\theta}{32}\right) - 16\left(\dfrac{v_0\sin\theta}{32}\right)^2$.

53. **(a)** The projectile hits the ground when $y = 0$.

$y = t(150\ \sin 20° - 16t) = 0$

$t = 0$ or $t = \dfrac{75}{8}\ \sin 20° \approx 3.206$

$x' = 150\ \cos 20°,\ y' = 150\ \sin 20° - 32t$

$\text{Length} = \int_0^{(75\sin 20°)/8} \sqrt{(150\ \cos 20°)^2 + (150\ \sin 20° - 32t)^2}\ dt$ which, using NINT, evaluates to ≈ 461.749 ft

(b) The maximum height of the projectile occurs when $y' = 0,$ so $t = \dfrac{75}{16}\sin 20°,\ \ y\left(\dfrac{75}{16}\sin 20°\right) \approx 41.125$ ft

54. **(a)** ≈ 641.236 ft

(b) $\dfrac{5625}{64} \approx 87.891$ ft

55. **(a)** ≈ 840.421 ft

(b) $\dfrac{16{,}875}{64} \approx 263.672$ ft

56. **(a)** It is not necessary to use NINT.

$$\text{Length} = \int_0^{75/8} (150 - 32t)\,dt$$
$$= [150t - 16t^2]_0^{75/8}$$
$$= \frac{5625}{8}$$
$$= 703.125 \text{ ft}$$

(b) $\dfrac{5625}{16} = 351.5625$ ft

57. $S = \int_0^{2\pi} 2\pi y\,((-\sin t)^2 + \cos t^2)^{1/2}\,dt$

$$S = \int_0^{2\pi} 2\pi y\,(1)\,dt$$
$$= \int_0^{2\pi} 2\pi(2 + \sin t)\,dt$$
$$= 2\pi(2t - \cos t)\Big|_0^{2\pi}$$
$$= 8\pi^2$$

58. $S = \int_0^2 2\pi y\,((t^{-1/2})^2 + (t^{1/2})^2)^{1/2}\,dt$

$$S = \int_0^2 2\pi y\left(\frac{1}{t} + t\right)^{1/2} dt$$
$$= \int_0^2 2\pi \cdot \frac{2}{3} t^{3/2}\left(\frac{1}{t} + t\right)^{1/2} dt$$
$$= \int_0^2 \frac{4\pi}{3}\left(t^3\left[\frac{1}{t} + t\right]\right)^{1/2} dt$$
$$= \int_0^2 \frac{4\pi}{3} t(1 + t^2)^{1/2}\,dt$$
$$\approx 14.214$$

59. $S = \int_0^3 2\pi y\,((2t)^2 + (1)^2)^{1/2}\,dt$

$$S = \int_0^3 2\pi y\,(4t^2 + 1)^{1/2}\,dt$$
$$= \int_0^3 2\pi(t + 1)(4t^2 + 1)^{1/2}\,dt$$
$$= 178.561$$

60. $S = \int_0^{\pi/3} 2\pi y((\sec t - \cos t)^2 + (-\sin t)^2)^{1/2}\,dt$

$$S = \int_0^{\pi/3} 2\pi \cos t(\sec^2 t - 1)^{1/2}\,dt$$
$$= \int_0^{\pi/3} 2\pi \cos t \tan t\,dt$$
$$= 2\pi \int_0^{\pi/3} \sin t\,dt$$
$$= 2\pi(-\cos t)\Big|_0^{\pi/3}$$
$$= 2\pi\left(-\frac{1}{2} + 1\right)$$
$$= \pi$$

Section 11.2 Vectors in the Plane (pp. 550–560)

Quick Review

1. $\sqrt{(5-1)^2 + (3-2)^2} = \sqrt{17}$

2. $\dfrac{3-2}{5-1} = \dfrac{1}{4}$

3. Solve $\dfrac{3-b}{5-3} = \dfrac{1}{4}$; $b = \dfrac{5}{2}$.

4. Slope of $\overline{PQ} = -\dfrac{1}{\overline{RQ}}$, so

$\dfrac{3-2}{5-1} = -\dfrac{5-3}{3-b}$ and $b = 11$.

5. Slope of $\overline{AB}$ = Slope of $\overline{CD}$, so $\dfrac{3-0}{1-0} = \dfrac{3-0}{5-a}$

and $a = 4$.

6. Slope of $\overline{AB}$ = Slope of $\overline{CD}$, so $\dfrac{5-1}{3-1} = \dfrac{2-b}{6-8}$

and $b = 6$.

7. $v(t) = \dfrac{d}{dt}(t \sin t)$

$$v(t) = \sin t + t\cos t$$
$$a(t) = \frac{d}{dt}(\sin t + t\cos t)$$
$$a(t) = 2\cos t - t\sin t$$

8. $x(t) = \int v(t)dt = \int (3t^2 - 12t)dt$

$$x(t) = t^3 - 6t^2 + C$$
$$x(0) = 0^3 - 6(0)^2 + C = 40$$
$$C = 40$$
$$x(4) = 4^3 - 6(4)^2 + 40 = 8$$

9. $\text{distance} = \int_0^4 |v(t)|\,dt$
$= \int_0^4 |3t^2 - 12t|\,dt$
$= \int_0^4 (12t - 3t^2)dt$
$= \left[6t^2 - t^3\right]_0^4$
$= 32$

10. $\int_0^{2\pi} \sqrt{(2\cos 2t)^2 + (-3\sin 3t)^2}\,dt$
$= \int_0^{2\pi} \sqrt{4\cos^2 2t + 9\sin^2 3t}\ dt$
$= 15.289$

Section 11.2 Exercises

1. $(2, 3) - (0, 0) = \langle 2, 3\rangle$

2. $(0, 0) - (2, 3) = \langle -2, -3\rangle$

3. $(2, -1) - (1, 3) = \langle 1, -4\rangle$

4. $P = \dfrac{-4+2}{2}, \dfrac{3-1}{2} = (-1, 1)$
$(-1, 1) - (0, 0) = \langle -1, 1\rangle$

5. $|v| = \sqrt{2^2 + 2^2} = \sqrt{8}$
$\tan\theta = \dfrac{\sqrt{2}}{2}, \theta = 45°$

6. $|v| = \sqrt{\left(-\sqrt{2}\right)^2 + \sqrt{2}^2} = 2$
$\tan\theta = -\dfrac{\sqrt{2}}{\sqrt{2}}, \ \theta = 135°$

7. $|v| = \sqrt{\sqrt{3}^2 + 1^2} = 2$
$\tan\theta = \dfrac{1}{\sqrt{3}}, \theta = 30°$

8. $|v| = \sqrt{-2^2 + \left(-2\sqrt{3}\right)^2} = 4$
$\tan\theta = \dfrac{-2\sqrt{3}}{2}, \theta = 240°$

9. $|v| = \sqrt{(-5)^2 + 0^2} = 5$
$\tan\theta = \dfrac{0}{-5}, \ \theta = 180°$

10. $|v| = \sqrt{0^2 + 4^2} = 4$
$\cos\theta = \dfrac{0}{4}, \ \theta = 90°$

11. $x = 4\cos(180°) = -4$
$y = 4\sin(180°) = 0$
$\langle -4, 0\rangle$

12. $x = 6\cos(270°) = 0$
$y = 6\sin(270°) = -6$
$\langle 0, -6\rangle$

13. $x = 5\cos(100°) \approx -0.868$
$y = 5\sin(100°) \approx 4.924$
$\langle -0.868, 4.924\rangle$

14. $x = 13\cos(200°) \approx -12.216$
$y = 13\sin(200°) \approx -4.446$
$\langle -12.216, -4.446\rangle$

15. $x = 3\sqrt{2}\cos\left(\dfrac{180}{\pi}\dfrac{\pi}{4}\right) = 3$
$y = 3\sqrt{2}\sin\left(\dfrac{180}{\pi}\dfrac{\pi}{4}\right) = 3$
$\langle 3, 3\rangle$

16. $x = 2\sqrt{3}\cos\left(\dfrac{180}{\pi}\dfrac{\pi}{6}\right) = 3$
$y = 2\sqrt{3}\sin\left(\dfrac{180}{\pi}\dfrac{\pi}{6}\right) = \sqrt{3}$
$\langle 3, \sqrt{3}\rangle$

17. **(a)** $\langle 3(3), \ 3(-2)\rangle = \langle 9, -6\rangle$

(b) $\sqrt{9^2 + (-6)^2} = \sqrt{117} = 3\sqrt{13}$

18. **(a)** $\langle -2(-2), -2(5)\rangle = \langle 4, -10\rangle$

(b) $\sqrt{4^2 + (-10)^2} = \sqrt{116} = 2\sqrt{29}$

19. **(a)** $\langle 3 + (-2), -2 + 5\rangle = \langle 1, 3\rangle$

(b) $\sqrt{1^2+3^2}=\sqrt{10}$

20. **(a)** $\langle 3-(-2), -2-5\rangle = \langle 5, -7\rangle$

(b) $\sqrt{5^2+(-7)^2}=\sqrt{74}$

21. **(a)** $2\mathbf{u}=\langle 2(3), 2(-2)\rangle = \langle 6, -4\rangle$
$3\mathbf{v}=\langle 3\,(-2),\ 3(5)\rangle = \langle -6, 15\rangle$
$2\mathbf{u}-3\mathbf{v}=\langle 6-(-6), -4-15\rangle = \langle 12, -19\rangle$

(b) $\sqrt{12^2+(-19)^2}=\sqrt{505}$

22. **(a)** $-2\mathbf{u}=\langle -2(3), -2(-2)\rangle = \langle -6, 4\rangle$
$5\mathbf{v}=\langle 5\,(-2),\ 5\,(5)\rangle = \langle -10,\ 25\rangle$
$-2\mathbf{u}+5\mathbf{v}=\langle -6+(-10),\ 4+25\rangle$
$=\langle -16,\ 29\rangle$

(b) $\sqrt{(-16)^2+29^2}=\sqrt{1097}$

23. **(a)** $\frac{3}{5}\mathbf{u}=\left\langle \frac{3}{5}(3),\ \frac{3}{5}(-2)\right\rangle = \left\langle \frac{9}{5}, -\frac{6}{5}\right\rangle$
$\frac{4}{5}\mathbf{v}=\left\langle \frac{4}{5}(-2),\ \frac{4}{5}(5)\right\rangle = \left\langle -\frac{8}{5},\ 4\right\rangle$
$\frac{3}{5}\mathbf{u}+\frac{4}{5}\mathbf{v}=\left\langle \frac{9}{5}+\left(-\frac{8}{5}\right), -\frac{6}{5}+4\right\rangle$
$=\left\langle \frac{1}{5}, \frac{14}{5}\right\rangle$

(b) $\sqrt{\left(\frac{1}{5}\right)^2+\left(\frac{14}{5}\right)^2}=\frac{\sqrt{197}}{5}$

24. **(a)** $-\frac{5}{13}\mathbf{u}=\left\langle -\frac{5}{13}(3),\ -\frac{5}{13}(-2)\right\rangle$
$=\left\langle -\frac{15}{13}, \frac{10}{13}\right\rangle$
$\frac{12}{13}\mathbf{v}=\left\langle \frac{12}{13}(-2),\ \frac{12}{13}(5)\right\rangle = \left\langle -\frac{24}{13},\ \frac{60}{13}\right\rangle$
$-\frac{5}{13}\mathbf{u}+\frac{12}{13}\mathbf{v}=\left\langle -\frac{15}{13}+\left(-\frac{24}{13}\right), \frac{10}{13}+\frac{60}{13}\right\rangle$
$=\left\langle -3,\ \frac{70}{13}\right\rangle$

(b) $\sqrt{(-3)^2+\left(\frac{70}{13}\right)^2}=\frac{\sqrt{6421}}{13}$

25. Initial velocity is 70° north of east:
$325\langle \cos 70°,\ \sin 70°\rangle \approx \langle 111.157,\ 305.400\rangle$.
Wind velocity is 130° north of east:
$40\langle \cos 130°,\ \sin 130°\rangle \approx \langle -25.712,\ 30.642\rangle$.
Add the two vectors to get
$\approx \langle 85.445,\ 336.042\rangle$.
The speed is the magnitude, ≈ 346.735 mph.
The direction is $\tan^{-1}\left(\frac{336.042}{85.445}\right)\approx 75.734°$ north of east, or $\approx 14.266°$ east of north.

26. $x=4\cos 135=-2\sqrt{2}$
$y=4\sin 135=2\sqrt{2}$
The true velocity is $\left\langle 2-2\sqrt{2},\ 2\sqrt{2}\right\rangle$, so the true angle is $\theta=\cos^{-1}\left(\frac{2-2\sqrt{2}}{4}\right)\approx 106.3°$ and the true speed is
$\sqrt{(2-2\sqrt{2})^2+(2\sqrt{2})^2}\approx 2.95$ mph

27. $\mathbf{v}(t)=\frac{d\mathbf{r}(t)}{dt}=\frac{d}{dt}\left\langle 3t^2,\ 2t^3\right\rangle = \left\langle 6t,\ 6t^2\right\rangle$
$\mathbf{a}(t)=\frac{d\mathbf{v}(t)}{dt}=\frac{d}{dt}\left\langle 6t, 6t^2\right\rangle = \langle 6, 12t\rangle$

28. $\mathbf{v}(t)=\frac{d\mathbf{r}(t)}{dt}$
$=\frac{d}{dt}\langle \sin 2t, 2\cos t\rangle$
$=\langle 2\cos 2t, -2\sin t\rangle$
$\mathbf{a}(t)=\frac{d\mathbf{v}(t)}{dt}$
$=\frac{d}{dt}\langle 2\cos 2t, -2\sin t\rangle$
$=\langle -4\sin 2t, -2\cos t\rangle$

29. $\mathbf{v}(t)=\frac{d\mathbf{r}(t)}{dt}$
$=\frac{d}{dt}\left\langle te^{-t},\ e^{-t}\right\rangle$
$=\left\langle e^{-t}-te^{-t}, -e^{-t}\right\rangle$
$\mathbf{a}(t)=\frac{d\mathbf{v}(t)}{dt}$
$=\frac{d}{dt}\left\langle e^{-t}-te^{-t},\ -e^{-t}\right\rangle$
$=\left\langle -2e^{-t}+te^{-t},\ e^{-t}\right\rangle$

30. $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$

$$= \frac{d}{dt}\langle 2\cos 3t,\ 2\sin 4t\rangle$$
$$= \langle -6\sin 3t, 8\cos 4t\rangle$$

$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt}$

$$= \frac{d}{dt}\langle -6\sin 3t, 8\cos 4t\rangle$$
$$= \langle -18\cos 3t, -32\sin 4t\rangle$$

31. $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$

$$= \frac{d}{dt}\left\langle t^2 + \sin 2t, t^2 - \cos 2t\right\rangle$$
$$= \langle 2t + 2\cos 2t,\ 2t + 2\sin 2t\rangle$$

$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt}$

$$= \frac{d}{dt}\langle 2t + 2\cos 2t,\ 2t + 2\sin 2t\rangle$$
$$= \langle 2 - 4\sin 2t,\ 2 + 4\cos 2t\rangle$$

32. $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$

$$= \frac{d}{dt}\langle t\sin t, t\cos t\rangle$$
$$= \langle \sin t + t\cos t, \cos t - t\sin t\rangle$$

$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt}$

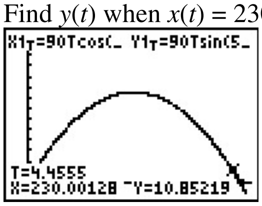
$$= \frac{d}{dt}\langle \sin t + t\cos t, \cos t - t\sin t\rangle$$
$$= \langle 2\cos t - t\sin t, -2\sin t - t\cos t\rangle$$

33. (a) The position vector of the ball at any time t is $\mathbf{r}(t) = \langle x(t),\ y(t)\rangle$, where $x(t) = 90t\cos 55°$ and $y(t) = 90t\sin 55° - 16t^2$.

(b) $\mathbf{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle$

$$= \langle 90\cos 55°, 90\sin 55° - 32t\rangle$$

(c) Find $y(t)$ when $x(t) = 230$.



[–20, 250] by [–10, 150]
$0 \le t \le 5$

No, the hit does not clear the 20-foot fence.

(d) See the graph in part (c). The ball hits the fence after about 4.4555 seconds.

(e) Evaluate $|\mathbf{v}(t)|$ at $t = 4.4555$.

$$|\mathbf{v}(t)| = \sqrt{(90\cos 55°)^2 + (90\sin 55° - 32t)^2}$$

$$|\mathbf{v}(4.4555)| = \sqrt{(90\cos 55°)^2 + (90\sin 55° - 32\cdot 4.4555)^2} \approx 86.055 \text{ ft/sec}$$

34. **(a)** The position vector of the ball at any time t is $\mathbf{r}(t) = \langle x(t),\ y(t)\rangle$, where $x(t) = 81t\cos 57°$ and $y(t) = 81t\sin 57° - 16t^2$.

Note that $\langle 0, 0\rangle$ is on the ground at the punter's position, 30 yards = 90 feet from the punter's goal line.

(b) $$\mathbf{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle = \langle 81\cos 57°, 81\sin 57° - 32t\rangle$$

(c) Find t when $x(t) = 270 - 90 = 180$.

$$81t\cos 57° = 180$$

$$t = \frac{180}{81\cos 57°}$$

The ball is "over" the player after about 4.08 seconds.

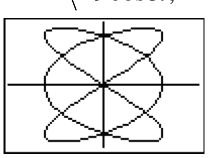
(d) Find $y(t)$ when $t = 4.08$.

$$y(4.08) = 81(4.08)\sin 57° - 16(4.08)^2 \approx 10.821$$

It is unlikely that the player will be able to catch the ball without backing up.

35. $$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \frac{d}{dt}\langle \cos 3t, \sin 2t\rangle = \langle -3\sin 3t, 2\cos 2t\rangle$$

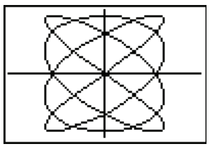
$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d}{dt}\langle -3\sin 3t,\ 2\cos 2t\rangle = \langle -9\cos 3t, -4\sin 2t\rangle$$



[–1.6, 1.6] by [–1.1, 1.1]
$0 \le t \le 2\pi$

36. $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$
$= \frac{d}{dt}\langle \sin 4t, \cos 3t \rangle$
$= \langle 4\cos 4t, -3\sin 3t \rangle$

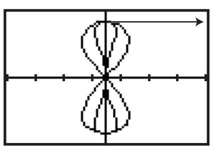
$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt}$
$= \frac{d}{dt}\langle 4\cos 4t, -3\sin 3t \rangle$
$= \langle -16\sin 4t, -9\cos 3t \rangle$



[−1.6, 1.6] by [−1.1, 1.1]
$0 \le t \le 2\pi$

37. **(a)** $\mathbf{v}(t) = \frac{d}{dt}\langle \sin 4t \cos t, \sin 2t \rangle$
$= \langle 4\cos 4t \cos t - \sin t \sin 4t, 2\cos 2t \rangle \Big|_{t=(5\pi/4)}$
$= \langle 2\sqrt{2}, 0 \rangle$
Speed: $2\sqrt{2}$

(b)



[−4, 4] by [−1.2, 1.2]
$0 \le t \le 2\pi$

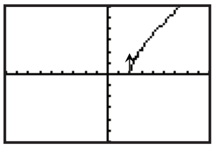
(c) To the right

38. **(a)** $\mathbf{v}(t) = \frac{d}{dt}\left\langle e^t + e^{-t},\ e^t - e^{-t} \right\rangle$
$= \left\langle e^t - e^{-t},\ e^t + e^{-t} \right\rangle$

(b) $\lim_{t\to\infty} \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \lim_{t\to\infty} \frac{e^t + e^{-t}}{e^t - e^{-t}} = \lim_{t\to\infty} \frac{e^{2t}+1}{e^{2t}-1} = 1$

(c) For any t,
$x^2 - y^2 = (e^t + e^{-t})^2 - (e^t - e^{-t})^2$
$= e^{2t} + 2 + e^{-2t} - (e^{2t} - 2 + e^{-2t})$
$= 4$

(d) The velocity at $t=0$ is $\langle 0,\ 2\rangle$.



[–9, 9] by [–6, 6]
$0 \le t \le 3$

39. (a) $\left\langle \int_0^3 3t^2 - 2t\ dt,\ \int_0^3 1+\cos \pi t\ dt \right\rangle + \langle 2, 6\rangle$

$= \left\langle \left(t^3 - t^2\right)\Big|_0^3, \left(t + \dfrac{1}{\pi}\sin \pi t\right)\Big|_0^3 \right\rangle + \langle 2, 6\rangle$

$= \langle 2+18, 6+3\rangle$

$= \langle 20, 9\rangle$

(b) $\int_0^3 \sqrt{(3t^2 - 2t)^2 + (1+\cos \pi t)^2}\, dt$

≈ 19.343

40. (a) $\left\langle \int_0^3 2\pi \cos 4\pi t dt, \int_0^3 4\pi \sin 2\pi t dt \right\rangle + \langle 7, 2\rangle$

$= \left\langle \dfrac{1}{2}\sin 4\pi t\Big|_0^3, -2\cos 2\pi t\Big|_0^3 \right\rangle + \langle 7, 2\rangle$

$= \langle 7+0,\ 2+0\rangle$

$= \langle 7,\ 2\rangle$

(b) $\int_0^3 \sqrt{(2\pi \cos 4\pi t)^2 + (4\pi \sin 2\pi t)^2}\, dt$

≈ 28.523

41. (a) $\left\langle \int_0^3 (t+1)^{-1} dt, \int_0^3 (t+2)^{-2} dt \right\rangle + \langle 3, -2\rangle$

$= \left\langle \ln(t+1)\Big|_0^3, -(t+2)^{-1}\Big|_0^3 \right\rangle + \langle 3, -2\rangle$

$= \langle 3+\ln 4,\ -1.7\rangle$

(b) $\int_0^3 \sqrt{((t+1)^{-1})^2 + ((t+2)^{-2})^2}\, dt$

≈ 1.419

42. (a) $\left\langle \int_0^3 e^t - t\ dt,\ \int_0^3 e^t + t\ dt \right\rangle + \langle 1, 1\rangle$

$= \left\langle \left(e^t - \dfrac{t^2}{2}\right)\Big|_0^3, \left(e^t + \dfrac{t^2}{2}\right)\Big|_0^3 \right\rangle + \langle 1, 1\rangle$

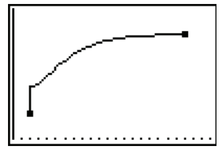
$= \langle 1+14.586, 1+23.586\rangle$

$= \langle 15.586, 24.586\rangle$

(b) $\int_0^3 \sqrt{(e^t - t)^2 + (e^t + t)^2}\, dt$

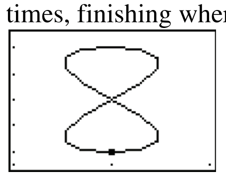
≈ 27.791

43. The parametric equations are $x = t^3 - t^2 + 2$ and $y = t + \dfrac{1}{\pi}\sin(\pi t) + 6.$



[0, 23] by [5, 10]
$0 \le t \le 3$

44. The parametric equations are
$x = \dfrac{1}{2}\sin(4\pi t) + 7$ and $y = -2\cos(2\pi t) + 4.$
(Note: The particle traverses the Figure-8 three times, finishing where it started.)



[6, 8] by [1.5, 6.5]
$0 \le t \le 3$

45. (a) $\mathbf{v}(t) = \dfrac{d}{dt}\left\langle 5\cos\dfrac{\pi}{6}t, 3\sin\dfrac{\pi}{6}t \right\rangle$

$\mathbf{v}(t) = \left\langle -\dfrac{5}{6}\pi \sin\dfrac{\pi}{6}t,\ \dfrac{1}{2}\pi\cos\dfrac{\pi}{6}t \right\rangle$

$|\mathbf{v}(t)| = \sqrt{\left(-\dfrac{5}{6}\pi\sin\dfrac{\pi}{6}t\right)^2 + \left(\dfrac{1}{2}\pi\cos\dfrac{\pi}{6}t\right)^2}$

$|\mathbf{v}(2)| = \pi\sqrt{\dfrac{7}{12}} \approx 2.399$

(b) $\mathbf{a}(t) = \dfrac{d}{dt}\left\langle -\dfrac{5}{6}\pi\sin\dfrac{\pi}{6}t,\ \dfrac{1}{2}\pi\cos\dfrac{\pi}{6}t \right\rangle$

$\mathbf{a}(2) = \left\langle -\dfrac{5\pi^2}{36}\cos\dfrac{\pi}{6}t, -\dfrac{\pi^2}{12}\sin\dfrac{\pi}{6}t \right\rangle\Bigg|_{t=2}$

$\mathbf{a}(2) = \left\langle -\dfrac{5\pi^2}{72}, -\dfrac{\pi^2\sqrt{3}}{24} \right\rangle$

(c) $x^2 = \left(5\cos\frac{\pi}{6}t\right)^2$

$y^2 = \left(3\sin\frac{\pi}{6}t\right)^2$

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

46. **(a)** $\mathbf{v}(t) = \frac{d}{dt}\langle \sec \pi t, \tan \pi t\rangle$

$\mathbf{v}\left(\frac{1}{4}\right) = \left\langle \pi \sec \pi t \tan \pi t, \pi \sec^2 \pi t \right\rangle\Big|_{t=1/4}$

$\mathbf{v}\left(\frac{1}{4}\right) = \left\langle \sqrt{2}\pi,\ 2\pi \right\rangle$

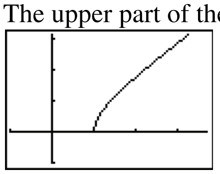
speed: $\sqrt{\left(\sqrt{2}\pi\right)^2 + (2\pi)^2} = \sqrt{6}\pi$

(b) $x^2 = (\sec \pi t)^2$

$y^2 = (\tan \pi t)^2$

$x^2 - y^2 = 1$

(c) The upper part of the right branch:



[−1, 3.7] by [−1, 3.1]
$0 \le t < 1/2$

47. **(a)** $\mathbf{v}(t) = \frac{d}{dt}\left\langle \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right\rangle$

$\mathbf{v}(t) = \left\langle -\frac{4t}{(1+t^2)^2}, \frac{2-2t^2}{(1+t^2)^2} \right\rangle$

(b) No; the x-component of velocity is zero only if $t = 0$, while the y-component of velocity is zero only if $t = 1$. At no time will the velocity be $\langle 0, 0\rangle$.

(c) $\lim_{t\to\infty} \left\langle \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right\rangle = \langle -1,\ 0\rangle$

48. **(a)** $\mathbf{v}(t) = \frac{d}{dt}\langle \sin t, \cos 2t\rangle$

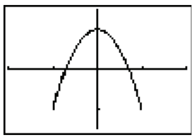
$\mathbf{v}(t) = \langle \cos t, -2\sin 2t\rangle$

(b) $\mathbf{v}(t) = \langle 0, 0\rangle$ where $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

(c) $\cos 2t = 1 - 2\sin^2 t$

$y = 1 - 2x^2$

(d)



[−2, 2] by [−1.5, 1.5]
$0 \le t \le 2\pi$

49. (a) $\frac{d}{dt}\left\langle e^t \sin t,\ e^t \cos t\right\rangle = \left\langle e^t \sin t + e^t \cos t,\ e^t \cos t - e^t \sin t\right\rangle$

$m = \left.\frac{e^t \cos t - e^t \sin t}{e^t \sin t + e^t \cos t}\right|_{t=\frac{\pi}{2}} = -1$

(b) $\left\langle e^1 \sin(1) + e^1 \cos(1),\ e^1 \cos(1) - e^1 \sin(1)\right\rangle = \langle 3.756, -\ 0.817\rangle$

$|\mathbf{v}(t)| = \sqrt{(3.756)^2 + (-0.817)^2} \approx 3.844$

(c) $\int_0^1 \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2}\, dt \approx 2.430$

50. (a) $\frac{d}{dt}\left\langle t^2 - 3,\ \frac{2}{3}t^3\right\rangle = \langle 2t,\ 2t^2\rangle$

$|\mathbf{v}(4)| = \sqrt{(2(4))^2 + (2(4)^2)^2}$

$|\mathbf{v}(4)| = \sqrt{1088} \approx 32.985$

(b) $\int_0^4 ((2t)^2 + (2t^2)^2)^{1/2}\, dt \approx 46.062$

(c) $t = \sqrt{3 + x}$

$y = \frac{2}{3}(3 + x)^{3/2}$

$\frac{dy}{dx} = (3 + x)^{1/2}$

51. (a) $3 + \int_2^4 (2 + \sin(t^2))\, dt \approx 3.942$

(b) $y - y_1 = m(x - x_1)$

$y - 5 = \frac{-6}{2 + \sin 4}(x - 3)$

(c) Speed = $\sqrt{(2 + \sin 4)^2 + (-6)^2} \approx 6.127$

(d) $\langle 8\cos 16,\ 2(2 + \sin 16) + 7(8)\cos 16\rangle \approx \langle -7.661, -50.205\rangle$

52. **(a)** $y - y_1 = m(x - x_1)$

$y - 5 = \frac{3\cos 4}{\sin 8}(x - 4)$

(b) Speed $= \sqrt{(3\cos 4)^2 + (\sin 8)^2} \approx 2.196$

(c) $\int_0^1 \sqrt{(3\cos(t^2))^2 + (\sin(t^3))^2}\,dt \approx 2.741$

(d) $\left(4 + \int_2^3 \sin(t^3)\,dt,\ 5 + \int_2^3 3\cos(t^2)\,dt\right)$

$\approx (4.004,\ 5.724)$

53. False; for example, **u** and –1(**u**) have opposite directions.

54. False; for example, $\left\langle \sqrt{3},\ 0\right\rangle + \langle 0,\ 1\rangle = \left\langle \sqrt{3},\ 1\right\rangle$, which has a direction angle of 30°.

55. E; $\frac{d}{dt}\left\langle t^2 + 1,\ \ln(2t+3)\right\rangle = \left\langle 2t,\ \frac{1}{2t+3}\right\rangle$

$\frac{d}{dt}\left\langle 2t,\ \frac{1}{2t+3}\right\rangle = \left\langle 2,\ -\frac{4}{(2t+3)^2}\right\rangle$

56. D; $\left\langle 4 + \int_0^2 \cos(t^2)\,dt,\ 7 + \int_0^2 \sin(t^3)\,dt\right\rangle$

$\langle 4 + 0.461,\ 7 + 0.452\rangle$

$= \langle 4.461,\ 7.452\rangle$

57. B; $x_1 = 7\cos 40 = 5.36$

$y_1 = 7\sin 40 = 4.50$

$x_2 = 4\cos 140 = -3.06$

$y_2 = 4\sin 140 = 2.57$

$\sqrt{(5.36 - 3.06)^2 + (4.50 + 2.57)^2}$

$= 7.435$

58. B; $\frac{dx}{dt} = 2\cos 2t$

$\frac{dy}{dt} = -5\sin 5t$

Speed $= \sqrt{(2\cos 4)^2 + (-5\sin 10)^2} = 3.018$

59. The velocity vector is $\left\langle -x, \sqrt{1-x^2}\right\rangle$, which has slope $-\frac{\sqrt{1-x^2}}{x}$. The acceleration vector is

$\left\langle \frac{d}{dt}(-x), \frac{d}{dt}\left(\sqrt{1-x^2}\right)\right\rangle = \left\langle -\frac{dx}{dt}, \frac{-x}{\sqrt{1-x^2}}\frac{dx}{dt}\right\rangle$

$= \left\langle x, \frac{x^2}{\sqrt{1-x^2}}\right\rangle$, which has slope $\frac{x}{\sqrt{1-x^2}}$.

Since the slopes are negative reciprocals of each other, the vectors are orthogonal.

60. The position vector is $\langle \cos t, \sin t\rangle$, which has slope tan *t*. The velocity vector is $\langle -\sin t, \cos t\rangle$, which has slope $-\frac{1}{\tan t}$. The acceleration vector is $\langle -\cos t, -\sin t\rangle$, which has slope tan *t*. The velocity slope is the negative reciprocal of the position and acceleration slopes, so velocity is orthogonal to position and to acceleration.

61. **(a)** $t - 3 = \frac{3t}{2} - 4$

$t - \frac{3t}{2} = -1$

$t = 2$

Since $t = 2$ also solves $(t-3)^2 = \frac{3t}{2} - 2$, the particles collide when $t = 2$.

(b) First particle: $\mathbf{v}_1(2) = \langle 1, -2\rangle$, so the direction unit vector is $\left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\rangle$.

Second particle: $\mathbf{v}_2(t) = \left\langle \frac{3}{2}, \frac{3}{2}\right\rangle$, so the direction unit vector is $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$.

62. **(a)** Referring to the figure, look at the circular arc from the point where $t = 0$ to the point "*m*". On the one hand, this arc has length given by $r_0\theta$, but it also has length given by *vt*. Setting these two quantities equal gives the result.

(b) $\mathbf{v}(t) = \left\langle -v\sin\frac{vt}{r_0},\ v\cos\frac{vt}{r_0}\right\rangle$ and

$\mathbf{a}(t) = \left\langle -\frac{v^2}{r_0}\cos\frac{vt}{r_0},\ -\frac{v^2}{r_0}\sin\frac{vt}{r_0}\right\rangle$

$= -\frac{v^2}{r_0}\left\langle \cos\frac{vt}{r_0},\ \sin\frac{vt}{r_0}\right\rangle$

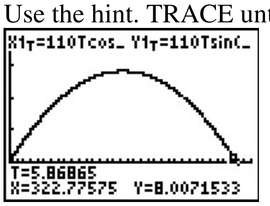
(c) From part (b), $\mathbf{a}(t) = -\left(\dfrac{v}{r_0}\right)^2 \mathbf{r}(t)$. So, by Newton's second law, $\mathbf{F} = -m\left(\dfrac{v}{r_0}\right)^2 \mathbf{r}$. Substituting for $\mathbf{F}$ in the law of gravitation gives the result.

(d) Set $\dfrac{vT}{r_0} = 2\pi$ and solve for vT.

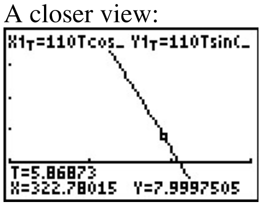
(e) Substitute $\dfrac{2\pi r_0}{T}$ for v in $v^2 = \dfrac{GM}{r_0}$ and solve for T^2.

$$\left(\frac{2\pi r_0}{T}\right)^2 = \frac{GM}{r_0}$$
$$\frac{4\pi^2 r_0^{\,2}}{T^2} = \frac{GM}{r_0}$$
$$\frac{1}{T^2} = \frac{GM}{4\pi^2 r_0^{\,3}}$$
$$T^2 = \frac{4\pi^2}{GM} r_0^{\,3}$$

63. Use the hint. TRACE until Y is approximately 8.



A closer view:



The screens confirm that the ball is at a height of 8 feet after about 5.869 seconds when it is about 323 feet from home plate.

64. Let $\mathbf{u} = \langle a, b \rangle$ be one of the vectors. It has slope $\dfrac{b}{a}$, so the perpendicular vector $\mathbf{v}$ must have slope $-\dfrac{a}{b}$.

Thus $\mathbf{v} = \langle kb, -ka \rangle$ for some nonzero scalar k, and the dot product is

$\mathbf{u} \cdot \mathbf{v} = \langle a, b \rangle, \ \langle kb, -ka \rangle = kab + (-kab) = 0.$

65. **(a)** The diagram shows, by vector addition, that $\mathbf{v} + \mathbf{w} = \mathbf{u}$, so $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

(b) This is just the Law of Cosines applied to the triangle, the sides of which are the magnitudes of the vectors.

(c) By the HMT Rule, $w = \langle u_1 - v_1,\ u_2 - v_2 \rangle$. So

$$\begin{aligned}|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 &= \left(u_1^2 + u_2^2\right) + \left(v_1^2 + v_2^2\right) - \left[(u_1 - v_1)^2 + (u_2 - v_2)^2\right] \\ &= u_1^2 + u_2^2 + v_1^2 + v_2^2 - \left[u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2\right] \\ &= 2u_1v_1 + 2u_2v_2 \\ &= 2(u_1v_1 + u_2v_2)\end{aligned}$$

(d) From part (b), $|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$, so $|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2|\mathbf{u}||\mathbf{v}|\cos\theta$.

From part (c), $|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2)$. Substituting, we get $2(u_1v_1 + u_2v_2) = 2|\mathbf{u}||\mathbf{v}|\cos\theta$, so $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 = |\mathbf{u}||\mathbf{v}|\cos\theta$.

Section 11.3 Polar Functions (pp. 561–574)

Quick Review 11.3

1. $x = 4\ \cos 30 = 2\sqrt{3}$
$y = 4\ \sin 30 = 2$
$\langle 2\sqrt{3},\ 2 \rangle$

2. $A = \pi r^2 \dfrac{30}{360} = \pi(6)^2 \dfrac{30}{360} = 3\pi$

3. $A = \pi r^2\ \dfrac{\pi}{8}\left(\dfrac{1}{2\pi}\right) = \dfrac{1}{16}\pi\ (8)^2 = 4\pi$

4. $x^2 + y^2 = 25$

5. Graph $y = \left(\dfrac{4 - x^2}{3}\right)^{1/2}$ and $y = -\left(\dfrac{4 - x^2}{3}\right)^{1/2}$.

6. $$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt}(5\sin t)}{\frac{d}{dt}(3\cos t)} \\ &= \frac{5\cos t}{-3\sin t} \\ &= -\frac{5}{3}\cot t\end{aligned}$$

7. $-\dfrac{5}{3}\cot(2) = 0.763$

8. $-\frac{5}{3}\cot t = 0,\quad$ so $\quad t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$x = 3\cos\frac{\pi}{2} = 0 \quad x = 3\cos\frac{3\pi}{2} = 0$

$y = 5\sin\frac{\pi}{2} = 5 \quad y = 5\sin\frac{3\pi}{2} = -5$

(0, 5) and (0, −5)

9. $-3\sin t = 0$, s $t = 0$ or $t = \pi$.

$x = 3\cos 0 = 3 \quad x = 3\cos\pi = -3$

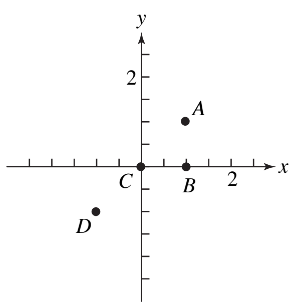
$y = 5\sin 0 = 0 \quad y = 5\sin\pi = 0$

(3, 0) and (−3, 0)

10. $\int_0^{\pi}\sqrt{(3\cos t)^2 + (5\sin t)^2}\,dt \approx 12.763$

Section 11.3 Exercises

1.



(a) $x = \sqrt{2}\cos\left(\frac{\pi}{4}\right) = 1$

$y = \sqrt{2}\sin\left(\frac{\pi}{4}\right) = 1$

(1, 1)

(b) $x = 1\cos(0) = 1$

$y = 1\sin(0) = 0$

(1, 0)

(c) $x = 0\cos\left(\frac{\pi}{2}\right) = 0$

$y = 0\sin\left(\frac{\pi}{2}\right) = 0$

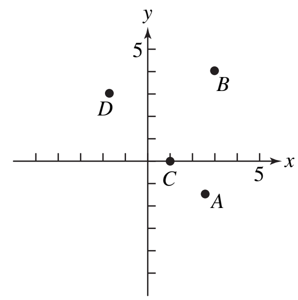
(0, 0)

(d) $x = -\sqrt{2}\cos\left(\frac{\pi}{4}\right) = -1$

$y = -\sqrt{2}\sin\left(\frac{\pi}{4}\right) = -1$

(−1, −1)

2.



(a) $x = -3\cos\left(\frac{5\pi}{6}\right) = \frac{3\sqrt{3}}{2}$

$y = -3\sin\left(\frac{5\pi}{6}\right) = \frac{-3}{2}$

$\left(\frac{3\sqrt{3}}{2}, \frac{-3}{2}\right)$

(b) $x = 5\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right) = 3$

$y = 5\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right) = 4$

(3, 4)

(c) $x = -1\cos 7\pi = 1$

$y = -1\sin 7\pi = 0$

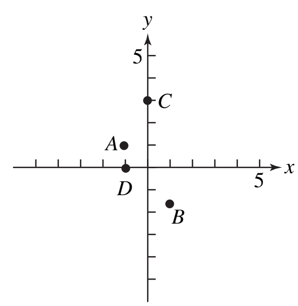
(1, 0)

(d) $x = 2\sqrt{3}\cos\left(\frac{2\pi}{3}\right) = -\sqrt{3}$

$y = 2\sqrt{3}\sin\left(\frac{2\pi}{3}\right) = 3$

$\left(-\sqrt{3}, 3\right)$

3.



(a) $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{5\pi}{4}, \frac{3\pi}{4}$

$\left(\sqrt{2}, \frac{-5\pi}{4}\right)$ and $\left(\sqrt{2}, \frac{3\pi}{4}\right)$

(b) $r = \sqrt{1+\left(-\sqrt{3}\right)^2} = \pm 2$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}, \frac{2\pi}{3}$

$\left(-2, \frac{2\pi}{3}\right)$ and $\left(2, -\frac{\pi}{3}\right)$

(c) $r = \sqrt{0^2 + 3^2} = \pm 3$

$\theta = \tan^{-1}\left(\frac{3}{0}\right) = \frac{\pi}{2}, \frac{5\pi}{2}$

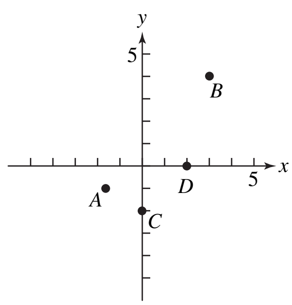
$\left(3, \frac{\pi}{2}\right)$ and $\left(3, \frac{5\pi}{2}\right)$

(d) $r = \sqrt{(-1)^2 + 0^2} = \pm 1$

$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0, \pi$

$(-1, 0)$ and $(1, \pi)$

4.



(a) $r = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = \pm 2$

$\theta = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = \frac{\pi}{6}, \frac{7\pi}{6}$

$\left(-2, \frac{\pi}{6}\right)$ and $\left(2, \frac{7\pi}{6}\right)$

(b) $r = \sqrt{3^2 + 4^2} = \pm 5$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$\left(-5, \pi + \tan^{-1}\left(\frac{4}{3}\right)\right)$ and $\left(5, \tan^{-1}\left(\frac{4}{3}\right)\right)$

(c) $r = \sqrt{0^2 + (-2)^2} = \pm 2$

$\theta = \tan^{-1}\left(\frac{-2}{0}\right)$

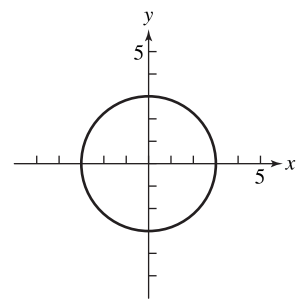
$= -\frac{\pi}{2}, \frac{3\pi}{2}$, $\left(2, -\frac{\pi}{2}\right)$ and $\left(2, \frac{3\pi}{2}\right)$

(d) $r = \sqrt{2^2 + 0^2} = \pm 2$

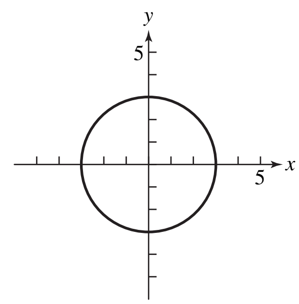
$\theta = \tan^{-1}\left(\frac{0}{2}\right) = 0, 2\pi$

$(2, 0)$ and $(2, 2\pi)$

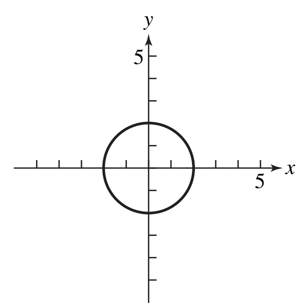
5.



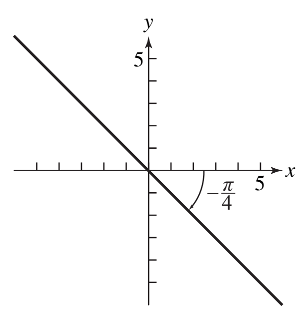
6.



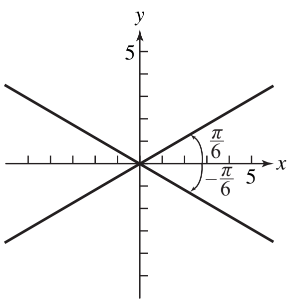
7.



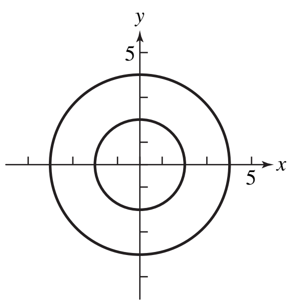
8.



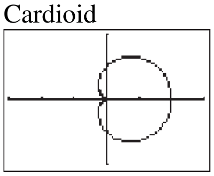
9.



10.

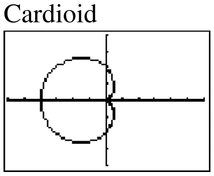


11. Cardioid



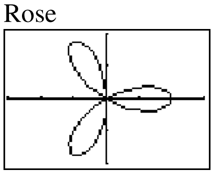
[–3, 3] by [–2, 2]
$0 \le \theta \le 2\pi$

12. Cardioid



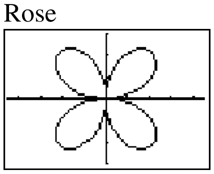
[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

13. Rose



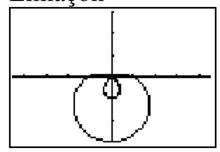
[–3, 3] by [–2, 2]
$0 \le \theta \le \pi$

14. Rose



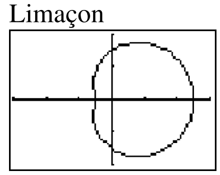
[–4.5, 4.5] by [–3, 3]
$0 \le \theta \le 2\pi$

15. Limaçon



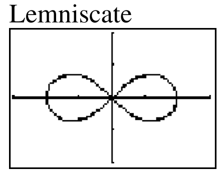
[–4.5, 4.5] by [–3, 3]
$0 \le \theta \le 2\pi$

16. Limaçon



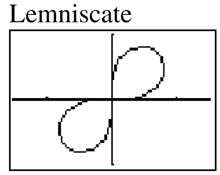
[–3, 3] by [–2, 2]
$0 \le \theta \le 2\pi$

17. Lemniscate



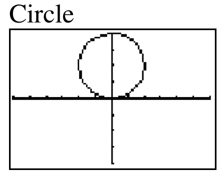
[–3, 3] by [–2, 2]
$-\pi/4 \le \theta \le \pi/4$

18. Lemniscate



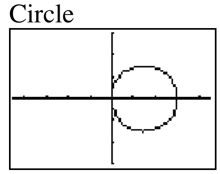
[–1.5, 1.5] by [–1, 1]
$0 \le \theta \le \pi/2$

19. Circle



[–6, 6] by [–4, 4]
$0 \le \theta \le \pi$

20. Circle



[–4.5, 4.5] by [–3, 3]
$0 \le \theta \le \pi$

21. $r = 4\csc\theta$

$1 = \dfrac{4}{r\sin\theta}$

$y = 4$, a horizontal line

22. $r = -3\sec\theta$

$1 = \dfrac{-3}{r\cos\theta}$

$x = -3$, a vertical line

23. $x + y = 1$, a line

(slope = −1, y-intercept = 1)

24. $r^2 = x^2 + y^2 = 1$,

a circle (center = (0, 0), radius = 1)

25. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

$1 = \dfrac{5}{r\sin\theta - 2r\cos\theta}$

$y - 2x = 5$, a line (slope = 2, y-intercept = 5)

26.

$$r^2 \sin 2\theta = 2$$
$$r^2(2\sin\theta\cos\theta) = 2$$
$$2xy = 2$$
$$xy = 1, \text{ a hyperbola}$$

27. $r^2\cos^2\theta = r^2\sin\theta$

$x^2 = y^2$, the union of two lines: $y = \pm x$

28.

$$r^2 = -4r\cos\theta$$
$$x^2 + y^2 = -4x$$
$$x^2 + 4x + 4 - 4 + y^2 = 0$$
$$(x+2)^2 + y^2 = 4, \text{ a circle}$$

(center = (−2, 0), radius = 2)

29.

$$r^2 = 8r\sin\theta$$
$$x^2 + y^2 - 8r\sin\theta + 16 - 16 = 0$$
$$x^2 + (y-4)^2 = 16, \text{ a circle}$$

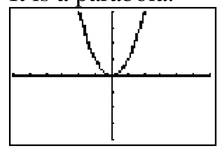
(center = (0, 4), radius = 4)

30.

$$r = 2\cos\theta + 2\sin\theta$$
$$r^2 = 2r\cos\theta + 2r\sin\theta$$
$$x^2 + y^2 = 2x + 2y$$
$$(x-1)^2 + (y-1)^2 = 2, \text{ a circle}$$

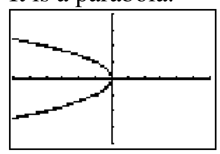
$\left(\text{center} = (1, 1), \text{radius} = \sqrt{2}\right)$

31. It is a parabola.



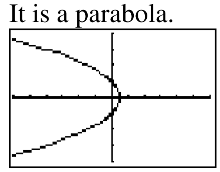
[−6, 6] by [−4, 4]
$0 \le \theta \le 2\pi$

32. It is a parabola.



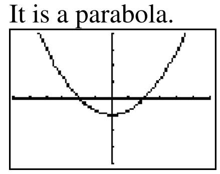
[−6, 6] by [−4, 4]
$0 \le \theta \le 2\pi$

33. It is a parabola.



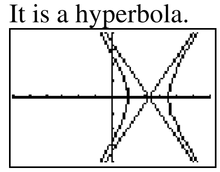
[−6, 6] by [−4, 4]
$0 \le \theta \le 2\pi$

34. It is a parabola.



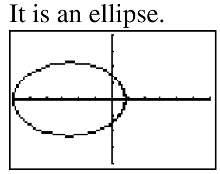
[−6, 6] by [−4, 4]
$0 \le \theta \le 2\pi$

35. It is a hyperbola.



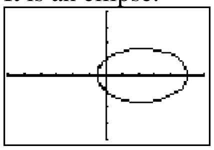
[−6, 6] by [−4, 4]
$0 \le \theta \le 2\pi$

36. It is an ellipse.



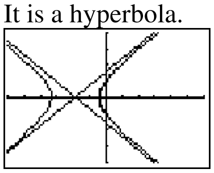
[−6, 6] by [−4, 4]
$0 \le \theta \le 2\pi$

37. It is an ellipse.



[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

38. It is a hyperbola.



[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

39. $r = -1 + \sin\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}((-1+\sin\theta)\sin\theta)}{\frac{d}{d\theta}((-1+\sin\theta)\cos\theta)}$$

$$\frac{dy}{dx} = \frac{(2\sin\theta - 1)\cos\theta}{\cos^2\theta - \sin^2\theta - \sin\theta}$$

At $\theta = 0$: -1

At $\theta = \pi$: 1

40. $r = \cos 2\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(\cos 2\theta \sin\theta)}{\frac{d}{d\theta}(\cos 2\theta \cos\theta)}$$

$$\frac{dy}{dx} = \frac{\cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta}{-\sin\theta\cos 2\theta - 2\cos\theta\sin 2\theta}$$

At $\theta = 0$: undefined

At $\theta = -\frac{\pi}{2}$: 0

At $\theta = \frac{\pi}{2}$: 0

At $\theta = \pi$: undefined

41. $r = 2 - 3\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(2 - 3\sin\theta)\sin\theta}{\frac{d}{d\theta}(2 - 3\sin\theta)\cos\theta}$$

$$\frac{dy}{dx} = \frac{(2 - 6\sin\theta)\cos\theta}{\sin\theta(3\sin\theta - 2) - 3\cos^2\theta}$$

At $(2, 0)$: $-\frac{2}{3}$

At $\left(-1, \frac{\pi}{2}\right)$: 0

At $(2, \pi)$: $\frac{2}{3}$

At $\left(5, \frac{3\pi}{2}\right)$: 0

42. $r = 3(1 - \cos\theta)$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}\left(3(1-\cos\theta)\sin\theta\right)}{\frac{d}{d\theta}\left(3(1-\cos\theta)\cos\theta\right)}$$

$$\frac{dy}{dx} = \frac{-6\cos^2\theta + 3\cos\theta + 3}{3\sin\theta(2\cos\theta - 1)}$$

At $\left(1.5, \frac{\pi}{3}\right)$: undefined

At $\left(4.5, \frac{2\pi}{3}\right)$: 0

At $(6, \pi)$: undefined

At $\left(3, \frac{3\pi}{2}\right)$: 1

43. $\int_0^{2\pi} \frac{1}{2}(4 + 2\cos\theta)^2 \, d\theta$

$$= \int_0^{2\pi} \frac{1}{2}(16 + 16\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (8 + 8\cos\theta + 1 + \cos 2\theta)\, d\theta$$

$$= \left[9\theta + 8\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= 18\pi$$

44. $\int_0^{2\pi} \frac{1}{2}(2 + 2\sin\theta)^2 \, d\theta$

$$= \int_0^{2\pi} \frac{1}{2}(4 + 8\sin\theta + 4\sin^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (2 + 4\sin\theta + 1 - \cos 2\theta)\, d\theta$$

$$= \left[3\theta - 4\cos\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= 6\pi - 4 + 4$$

$$= 6\pi$$

45. $\int_{-\pi/4}^{\pi/4} \frac{1}{2}(\cos 2\theta)^2\, d\theta = \int_0^{\pi/4} \cos^2(2\theta)\, d\theta$

$= \int_0^{\pi/4} \left(\frac{1}{2} + \frac{\cos 4\theta}{2}\right) d\theta$

$= \left[\frac{\theta}{2} + \frac{\sin 4\theta}{8}\right]_0^{\pi/4}$

$= \frac{\pi}{8}$

46. $\int_0^{2\pi} \frac{1}{2}(2\sin 4\theta)^2\, d\theta = \int_0^{2\pi} 2\sin^2(4\theta)\, d\theta$

$= \int_0^{2\pi} (1 - \cos 8\theta)\, d\theta$

$= \left[\theta - \frac{\sin 8\theta}{8}\right]_0^{2\pi}$

$= 2\pi$

47. $\int_{-\pi/4}^{\pi/4} \frac{1}{2}(4\cos 2\theta)\, d\theta = \sin 2\theta\Big|_{-\pi/4}^{\pi/4}$

$= 2$

48. $6\int_0^{\pi/3} \frac{1}{2}(2\sin 3\theta)\, d\theta = -2\cos 3\theta\Big|_0^{\pi/3}$

$= 2 - (-2) = 4$

49. $\int_0^{2\pi} \frac{1}{2}(3 - 2\cos\theta)^2\, d\theta$

$= \int_0^{2\pi} \frac{1}{2}(9 - 12\cos\theta + 4\cos^2\theta)\, d\theta$

$= \int_0^{2\pi} \left(\frac{9}{2} - 6\cos\theta + 1 + \cos 2\theta\right) d\theta$

$= \left[\frac{11\theta}{2} - 6\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$

$= 11\pi - 0 = 11\pi$

50. $\int_{\pi/6}^{5\pi/6} \frac{1}{2}(2\sin\theta - 1)^2\, d\theta$

$= \int_{\pi/6}^{5\pi/6} \left(2\sin^2\theta - 2\sin\theta + \frac{1}{2}\right) d\theta$

$= \int_{\pi/6}^{5\pi/6} \left(1 - \cos 2\theta - 2\sin\theta + \frac{1}{2}\right) d\theta$

$= \left[\frac{3\theta}{2} - \frac{\sin 2\theta}{2} + 2\cos\theta\right]_{\pi/6}^{5\pi/6}$

$= \pi - \frac{3\sqrt{3}}{2} = 0.544$

51. $2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2 d\theta = \int_0^{\pi/4} 4\sin^2\theta d\theta$

$= \int_0^{\pi/4} 2(1 - \cos 2\theta)\, d\theta$

$= [2\theta - \sin 2\theta]_0^{\pi/4}$

$= \frac{\pi}{2} - 1$

52. $2\left(\int_0^{\pi/6} \frac{1}{2}(2\sin\theta)^2 d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}(1)^2 d\theta\right)$

$= \int_0^{\pi/6} 4\sin^2\theta d\theta + \int_{\pi/6}^{\pi/2} 1 d\theta$

$= \int_0^{\pi/6} (2 - 2\cos 2\theta)\, d\theta + \frac{\pi}{3}$

$= [2\theta - \sin 2\theta]_0^{\pi/6} + \frac{\pi}{3}$

$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

53. $2\left(\int_0^{\pi/2} \frac{1}{2}[2(1 - \cos\theta)]^2 d\theta + \int_{\pi/2}^{\pi} \frac{1}{2}(2)^2 d\theta\right)$

$= \int_0^{\pi/2} 4(1 - 2\cos\theta + \cos^2\theta) d\theta + \int_{\pi/2}^{\pi} 4 d\theta$

$= \int_0^{\pi/2} (4 - 8\cos\theta + 2(1 + \cos 2\theta))\, d\theta + 2\pi$

$= [6\theta - 8\sin\theta + \sin 2\theta]_0^{\pi/2} + 2\pi$

$= 5\pi - 8$

54. $4\int_0^{\pi/2} \frac{1}{2}(2(1 - \cos\theta))^2 d\theta$

$= 4\int_0^{\pi/2} (2 - 4\cos\theta + 2\cos^2\theta)\, d\theta$

$= \int_0^{\pi/2} (8 - 16\cos\theta + 4(1 + \cos 2\theta))\, d\theta$

$= [12\theta - 16\sin\theta + 2\sin 2\theta]_0^{\pi/2}$

$= 6\pi - 16$

55. The requested area is inside of the upper semicircle and outside of the portion of the cardioid that is in Quadrants I and II.

$2\pi - \int_0^{\pi} \frac{1}{2}(2(1 - \sin\theta))^2 d\theta$

$= 2\pi - \int_0^{\pi} (2 - 4\sin\theta + 2\sin^2\theta) d\theta$

$= 2\pi - \int_0^{\pi} (2 - 4\sin\theta + 1 - \cos 2\theta) d\theta$

$= 2\pi - \left[3\theta + 4\cos\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$

$= 8 - \pi$

56. $4\int_{-\pi/6}^{\pi/6}\frac{1}{2}((4\cos 2\theta)^2-2^2)\,d\theta$

$=4\int_0^{\pi/6}(16\cos^2 2\theta-4)\,d\theta$

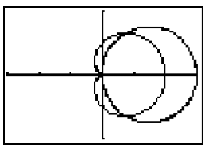
$=4\int_0^{\pi/6}(8(1+\cos 4\theta)-4)\,d\theta$

$=\int_0^{\pi/6}(16+32\cos 4\theta)\,d\theta$

$=[16\theta+8\sin 4\theta]_0^{\pi/6}$

$=\frac{8\pi}{3}+4\sqrt{3}$

57.



[–3, 3] by [–2, 2]

$0\le\theta\le\pi$ for the circle
$0\le\theta\le 2\pi$ for the cardioid

$2\int_0^{\pi/3}\frac{1}{2}((3\cos\theta)^2-(1+\cos\theta)^2)d\theta$

$=\int_0^{\pi/3}(8\cos^2\theta-1-2\cos\theta)\,d\theta$

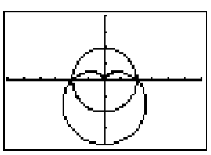
$=\int_0^{\pi/3}(4(1+\cos 2\theta)-1-2\cos\theta)d\theta$

$=[3\theta+2\sin 2\theta-2\sin\theta]_0^{\pi/3}$

$=\pi+\sqrt{3}-\sqrt{3}-0$

$=\pi$

58.



[–6, 6] by [–4, 4]

$0\le\theta\le 2\pi$

$\int_0^{\pi}\frac{1}{2}(2^2-(2(1-\sin\theta))^2)\,d\theta$

$=\int_0^{\pi}(2-2+4\sin\theta-2\sin^2\theta)\,d\theta$

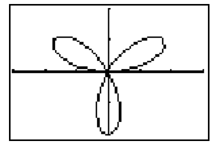
$=\int_0^{\pi}(4\sin\theta-1+\cos 2\theta)\,d\theta$

$=\left[-4\cos\theta-\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi}$

$=4-\pi-(-4)$

$=8-\pi$

59.



[–3, 3] by [–2, 2]

$0\le\theta\le\pi$

$\int_0^{\pi}\frac{1}{2}(2\sin 3\theta)^2\,d\theta=\int_0^{\pi}2\sin^2 3\theta d\theta$

$=\int_0^{\pi}(1-\cos 6\theta)d\theta$

$=\left[\theta-\frac{1}{6}\sin 6\theta\right]_0^{\pi}$

$=\pi$

$$\text{Slope}=\left.\frac{\frac{d}{d\theta}(2\sin 3\theta\,\sin\theta)}{\frac{d}{d\theta}(2\sin 3\theta\,\cos\theta)}\right|_{\theta=\pi/4}$$

$$=\left[\frac{6\sin\theta\cos 3\theta+2\cos\theta\sin 3\theta}{6\cos\theta\cos 3\theta-2\sin\theta\sin 3\theta}\right]_{\theta=\pi/4}$$

$$=\frac{1}{2}$$

60. (a) $\int_0^{3/4}\left(\sqrt{1+y^2}-\frac{5y}{3}\right]dy$

$$=\left[\frac{\ln\left(\left|\sqrt{y^2+1}+y\right|\right)}{2}+\frac{y\sqrt{y^2+1}}{2}-\frac{5y^2}{6}\right]_0^{3/4}$$

$=0.347$

(b) $x=r\cos\theta$
$y=r\sin\theta$

$$x^2=1+y^2$$
$$x^2-y^2=1$$
$$r^2(\cos^2\theta-\sin^2\theta)=1$$
$$r^2=\frac{1}{\cos^2\theta-\sin^2\theta}$$

(c) Let $\alpha=\tan^{-1}\left(\frac{3}{5}\right)$.

Then the area is

$\int_0^{\alpha}\frac{1}{2}\left(\frac{1}{\cos^2\theta-\sin^2\theta}\right)d\theta.$

61. True; polar coordinates determine a unique point.

62. False; integrating from 0 to 2π traverses the curve twice, giving twice the area. The correct upper limit of integration is π.

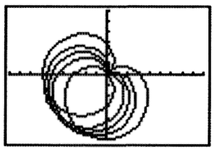
63. D

64. E

65. B

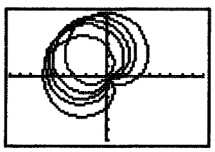
66. D

67. **(a)**



[–9, 9] by [–6, 6]

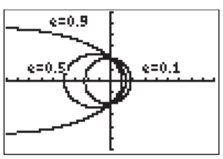
(b)



[–9, 9] by [–6, 6]

(c) The graph of r_2 is the graph of r_1 rotated counterclockwise about the origin by angle α.

68. **(a)**

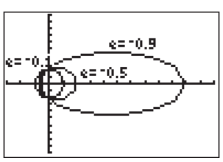


[–9, 9] by [–6, 6]

The graphs are ellipses.

(b) As $e \to 0^+$, the graph approaches the circle of radius 2 centered at the origin.

69. **(a)**

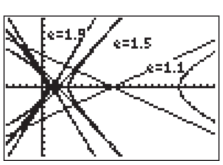


[–6, 24] by [–10, 10]

The graphs are ellipses that stretch out to the right.

(b) The ellipse with eccentricity $-e$ is the reflection across the y-axis of the ellipse with eccentricity e.

70. **(a)**

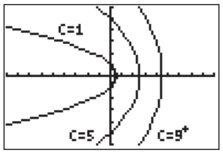


[–5, 25] by [–10, 10]

The graphs are hyperbolas.

(b) As $e \to 1^+$, the right branch of the hyperbola goes to infinity and “disappears.” The left branch approaches the parabola $y^2 = 4 - 4x$.

71. **(a)**



[–9, 9] by [–6, 6]

The graphs are parabolas.

(b) As $c \to 0^+$, the limit of the graph is the negative x-axis.

72. $d = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2}$

$d = [(r_2 \cos\theta_2 - r_1 \cos\theta_1)^2 + (r_2 \sin\theta_2 - r_1 \sin\theta_1)^2]^{1/2}$

$d = [r_2{}^2 \cos^2\theta_2 + r_1{}^2 \cos^2\theta_1 + r_2{}^2 \sin^2\theta_2 + r_1{}^2 \sin^2\theta_1 + 2\, r_1 r_2 \cos\theta_1 \cos\theta_2 + 2\, r_1 r_2 \sin\theta_1 \sin\theta_2]^{1/2}$

$d = [r_1{}^2 + r_2{}^2 - 2\, r_1 r_2 \cos(\theta_1 - \theta_2)]^{1/2}$

73. **(a)** $\dfrac{1}{2\pi - 0}\displaystyle\int_0^{2\pi} a(1 - \cos\theta)\, d\theta = \dfrac{a}{2\pi}[\theta - \sin\theta]_0^{2\pi} = a$

(b) $\dfrac{1}{2\pi - 0}\displaystyle\int_0^{2\pi} a\, d\theta = \dfrac{a}{2\pi}[\theta]_0^{2\pi} = a$

(c) $\dfrac{1}{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)}\displaystyle\int_{-\pi/2}^{\pi/2} a\cos\theta\, d\theta = \dfrac{a}{\pi}[\sin\theta]_{-\pi/2}^{\pi/2} = \dfrac{2a}{\pi}$

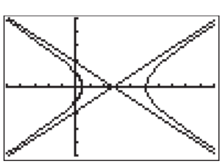
74.

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= (f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2 \\
&= (f'(\theta)\cos\theta)^2 + (f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta)^2 + (f(\theta)\cos\theta)^2 \\
&= (f(\theta))^2(\cos^2\theta + \sin^2\theta) + (f'(\theta))^2(\cos^2\theta + \sin^2\theta) \\
&= (f(\theta))^2 + (f'(\theta))^2 \\
&= r^2 + \left(\frac{dr}{d\theta}\right)^2
\end{aligned}$$

75.

$$\begin{aligned}
\int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta &= \int_0^{2\pi} \sqrt{2\cos\theta + 2}\, d\theta \\
&= 2\int_0^{\pi} \sqrt{2\cos\theta + 2}\, d\theta \\
&= 2\int_0^{\pi} 2\cos\left(\frac{\theta}{2}\right) d\theta \\
&= 8\sin\left(\frac{\theta}{2}\right)\Bigg|_0^{\pi} \\
&= 8
\end{aligned}$$

76. **(a)**



[–5, 10] by [–5, 5]

You see both branches of the hyperbola as well as the lines that appear to be the asymptotes.

(b) The branch on the left corresponds to $-\cos^{-1}\left(-\frac{1}{1.2}\right) < \theta < \cos^{-1}\left(-\frac{1}{1.2}\right)$.

The branch on the right corresponds to $\cos^{-1}\left(-\frac{1}{1.2}\right) < \theta < 2\pi - \cos^{-1}\left(-\frac{1}{1.2}\right)$.

(c) What appear to be asymptotes are actually lines put in by the graphing calculator to connect the last point it finds on one branch of the hyperbola to the first point it finds on the other branch.

77.

$$r = \frac{c}{1+e\cos\theta}$$

$$r + re\cos\theta = c$$

$$r = c - e(r\cos\theta)$$

Use $x = r\cos\theta$ and $r^2 = x^2 + y^2$.

$$r^2 = x^2 + y^2 = (c - ex)^2 = c^2 - 2cex + e^2x^2$$

$$x^2 + y^2 = c^2 - 2cex + e^2x^2$$

$$(1-e^2)x^2 + 2cex + y^2 = c^2$$

Complete the square on x by adding $\frac{c^2e^2}{1-e^2}$ to both sides.

$$(1-e^2)\left(x^2 + \frac{2ce}{1-e^2}x + \frac{c^2e^2}{(1-e^2)^2}\right) + y^2 = c^2 + \frac{c^2e^2}{1-e^2}$$

$$(1-e^2)\left(x + \frac{ce}{1-e^2}\right)^2 + y^2 = \frac{c^2}{1-e^2}$$

$$\frac{(1-e^2)^2}{c^2}\left(x + \frac{ce}{1-e^2}\right)^2 + \frac{1-e^2}{c^2}y^2 = 1$$

Let $a = \frac{c}{1-e^2}$ and $b = \frac{c}{\sqrt{1-e^2}}$.

The last equation becomes $\frac{1}{a^2}(x+ae)^2 + \frac{1}{b^2}y^2 = 1$ or $\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1$. This is an ellipse with center $(-ae, 0)$.

78. The distance from the center to one focus is

$$\sqrt{a^2 - b^2} = \sqrt{\frac{c^2}{(1-e^2)^2} - \frac{c^2}{1-e^2}}$$

$$= \sqrt{\frac{c^2e^2}{(1-e^2)^2}}$$

$$= \frac{ce}{1-e^2}$$

This is *ae*, which is the distance from the center of the ellipse in Exercise 77 to the origin.

79. The area swept out from time t_0 to time t is given by $A(t) = \int_{t_0}^{t} \frac{1}{2}r^2\frac{d\theta}{dt}dt$. Thus $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$. Since $\frac{dA}{dt} = K$, we have $\frac{1}{2}r^2\frac{d\theta}{dt} = K$ or $\frac{d\theta}{dt} = \frac{2K}{r^2}$.

80. From Exercise 79, $r^2 \frac{d\theta}{dt} = 2K$, and since $r^2 = \frac{c^2}{(1+e\cos\theta)^2}$, this gives $\frac{1}{(1+e\cos\theta)^2}\frac{d\theta}{dt} = \frac{2K}{c^2}$.

$$\begin{aligned}\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} &= \frac{-c\sin\theta(1+e\cos\theta) - c\cos\theta(-e\sin\theta)}{(1+e\cos\theta)^2}\frac{d\theta}{dt}\\ &= \frac{-c\sin\theta}{(1+e\cos\theta)^2}\frac{d\theta}{dt}\\ &= -c\sin\theta\cdot\frac{2K}{c^2}\\ &= \frac{-2K\sin\theta}{c}\end{aligned}$$

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{-2K}{c}\cos\theta\frac{d\theta}{dt} = \frac{-2K}{c}\cos\theta\cdot\frac{2K}{r^2}\\ &= \frac{-4K^2}{cr^2}\cos\theta\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} &= \frac{c\cos\theta(1+e\cos\theta) - c\sin\theta(-e\sin\theta)}{(1+e\cos\theta)^2}\frac{d\theta}{dt}\\ &= \frac{ce + c\cos\theta}{(1+e\cos\theta)^2}\frac{d\theta}{dt}\\ &= (ce + c\cos\theta)\frac{2K}{c^2}\\ &= \frac{2K}{c}(e+\cos\theta)\end{aligned}$$

$$\frac{d^2y}{dt^2} = \frac{-2K}{c}\sin\theta\frac{d\theta}{dt} = \frac{-4K^2}{cr^2}\sin\theta$$

Thus the acceleration vector is $\left\langle \frac{-4K^2}{cr^2}\cos\theta, \frac{-4K^2}{cr^2}\sin\theta\right\rangle = \frac{-4K^2}{cr^2}\langle\cos\theta, \sin\theta\rangle$ which has magnitude $\frac{4K^2}{cr^2}$ and it points toward the origin from the point $(x, y) = (r\cos\theta, r\sin\theta)$.

Quick Quiz Sections 11.1–11.3

1. A

2. C; $\frac{dx}{dt} = \frac{d}{dt}(t^3 - t^2 - 1) = 3t^2 - 2t$

$$3t^2 - 2t = 0$$
$$t(3t-2) = 0$$
$$t = 0, \frac{2}{3}$$

3. D

4. **(a)** Area $= \frac{1}{2}\int_0^{\pi}(\theta + \sin 2\theta)^2\, d\theta = 4.382$

(b) $-2 = r\cos\theta = (\theta + \sin 2\theta)\cos\theta$
$\Rightarrow \theta = 2.786$

(c) The graph is getting closer to the origin as θ increases from $\frac{\pi}{3}$ to $\frac{2\pi}{3}$.

(d) Maximize $r = \theta + \sin 2\theta$ for $0 \le \theta \le \frac{\pi}{2}$.

$$\frac{dr}{d\theta} = 1 + 2\cos 2\theta$$
$$1 + 2\cos 2\theta = 0$$
$$\cos 2\theta = -\frac{1}{2}$$
$$\theta = \frac{\pi}{3}$$

Since $\frac{dr}{d\theta} = 1 + 2\cos 2\theta > 0$ for $0 < \theta < \frac{\pi}{3}$ and $\frac{dr}{d\theta} = 1 + 2\cos 2\theta < 0$ for $\frac{\pi}{3} < \theta < \frac{\pi}{2}$, there is a maximum of r when $\theta = \frac{\pi}{3}$ by the First Derivative test. The curve is farthest from the origin when $\theta = \frac{\pi}{3}$.

Chapter 11 Review Exercises (pp. 575–576)

1. **(a)** $3\langle -3, 4\rangle - 4\langle 2, -5\rangle = \langle -17, 32\rangle$

(b) $\sqrt{(-17)^2 + (32)^2} = \sqrt{1313}$

2. **(a)** $\langle -3, 4\rangle + \langle 2, -5\rangle = \langle -1, -1\rangle$

(b) $\sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

3. **(a)** $-2\langle -3, 4\rangle = \langle 6, -8\rangle$

(b) $\sqrt{6^2 + (-8)^2} = 10$

4. **(a)** $5\langle 2, -5\rangle = \langle 10, -25\rangle$

(b) $\sqrt{10^2 + (-25)^2} = \sqrt{725} = 5\sqrt{29}$

5. $y = \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$x = \cos\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$

6. $y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$x = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$

7. $r = \sqrt{(4)^2 + (-1)^2} = \sqrt{17}$

$2\left\langle \frac{4}{\sqrt{17}}, \frac{-1}{\sqrt{17}}\right\rangle = \left\langle \frac{8}{\sqrt{17}}, \frac{-2}{\sqrt{17}}\right\rangle$

8. $-5\left\langle \frac{3}{5}, \frac{4}{5}\right\rangle = \langle -3, -4\rangle$

9. **(a)** $y = \frac{\sqrt{3}}{2}x + \frac{1}{4}$

(b) $\frac{dy}{dx} = \frac{\frac{d}{dt}\left(\frac{1}{2}\sec t\right)}{\frac{d}{dt}\left(\frac{1}{2}\tan t\right)}$

$\frac{dy}{dx} = \sin t$

$\frac{d^2y}{dx^2} = \left.\frac{\frac{d}{dt}(\sin t)}{\frac{1}{2}\sec^2 t}\right|_{t=\frac{\pi}{3}}$

$\frac{d^2y}{dx^2} = \frac{1}{4}$

10. **(a)** $y = -3x + \frac{13}{4}$

(b) $\frac{dy}{dx} = \frac{\frac{d}{dt}\left(1 - \frac{3}{t}\right)}{\frac{d}{dt}\left(1 + \frac{1}{t^2}\right)}$

$\frac{dy}{dx} = \frac{\frac{3}{t^2}}{-\frac{2}{t^3}}$

$\frac{dy}{dx} = -\frac{3}{2}t$

$\frac{d^2y}{dx^2} = \left.\frac{\frac{d}{dt}\left(-\frac{3}{2}t\right)}{-\frac{2}{t^3}}\right|_{t=2} = \left.\frac{3}{4}t^3\right|_{t=2}$

$\frac{d^2y}{dx^2} = 6$

11. **(a)** $\frac{dy}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sec t\right) = \frac{1}{2}\sec t \tan t$

Horizontal: $\frac{dy}{dt} = 0 \Rightarrow \tan t = 0 \Rightarrow x = 0$

Also, $\tan t = 0 \Rightarrow \sec t = \pm 1 \Rightarrow y = \pm\frac{1}{2}$

The point are $\left(0, -\frac{1}{2}\right)$ and $\left(0, \frac{1}{2}\right)$.

(b) $\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1}{2}\tan t\right) = \frac{1}{2}\sec^2 t$

Vertical: $\frac{dx}{dt} = 0 \Rightarrow \sec t = 0$ (impossible)

There are no points where the tangents are vertical.

12. **(a)** $\frac{d}{dt}(2\sin t) = 2\cos t$

$2\cos t = 0$

$t = \frac{\pi}{2}$ and $t = -\frac{\pi}{2}$

$y = 2\sin\left(\frac{\pi}{2}\right) = 2$ and $y = 2\sin\left(-\frac{\pi}{2}\right) = -2$

$x = -2\cos\left(\frac{\pi}{2}\right) = 0$ and $x = -2\cos\left(-\frac{\pi}{2}\right) = 0$

$(0, 2)$ and $(0, -2)$

(b) $\frac{d}{dt}(2\cos t) = -2\sin t$

$-2\sin t = 0$

$t = 0$ and $t = \pi$

$x = -2\cos 0 = -2$ and $x = -2\cos\pi = 2$

$y = 2\sin 0 = 0$ and $y = 2\sin\pi = 0$

$(-2, 0)$ and $(2, 0)$

13. **(a)** $\frac{d}{dt}(\cos^2 t) = -2\sin t\cos t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\cos t\sin t}{\sin t} = -2\cos t$

$\frac{dy}{dx} = 0$ at $(0, 0)$

(b) $\frac{dx}{dy} = -\frac{1}{2\cos t}$ is never zero. There are no vertical tangents.

14. **(a)** $\frac{d}{dt}(9\sin t) = 9\cos t$

$9\cos t = 0$

$t = \frac{\pi}{2}$ and $t = -\frac{\pi}{2}$

$x = 4\cos\frac{\pi}{2} = 0$ $\quad x = 4\cos\left(-\frac{\pi}{2}\right) = 0$

$y = 9\sin\frac{\pi}{2} = 9$ $\quad y = 9\sin\left(-\frac{\pi}{2}\right) = -9$

$(0, 9)$ and $(0, -9)$

(b) $\frac{d}{dt}(4\cos t) = -4\sin t$

$-4\sin t = 0$

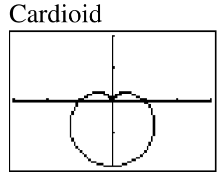
$t = 0$ or $t = \pi$

$x = 4\cos 0 = 4,\quad x = 4\cos\pi = -4$

$y = 9\sin 0 = 0,\quad y = 9\sin\pi = 0$

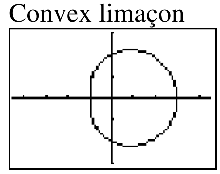
$(4, 0)$ and $(-4, 0)$

15. Cardioid



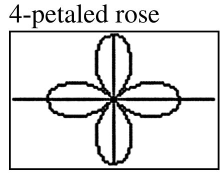
$[-3, 3]$ by $[-2, 2]$
$0 \le \theta \le 2\pi$

16. Convex limaçon



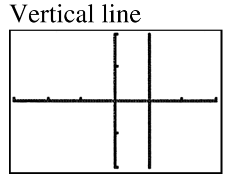
$[-4.5, 4.5]$ by $[-3, 3]$
$0 \le \theta \le 2\pi$

17. 4-petaled rose



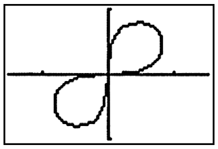
$[-1.5, 1.5]$ by $[-1, 1]$
$0 \le \theta \le 2\pi$

18. Vertical line



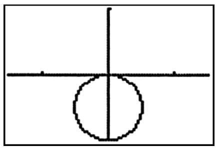
$[-3, 3]$ by $[-2, 2]$
$-\pi/2 \le \theta \le \pi/2$

19. Lemniscate



$[-1.5, 1.5]$ by $[-1, 1]$
$0 \le \theta \le \pi/2$

20. Circle



$[-1.5, 1.5]$ by $[-1, 1]$
$0 \le \theta \le \pi$

21. $$\frac{\frac{d}{d\theta}(\cos 2\theta)\sin\theta}{\frac{d}{d\theta}(\cos 2\theta)\cos\theta}$$
$$= \left[\frac{\cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta}{-\sin\theta\cos 2\theta - 2\cos\theta\sin 2\theta}\right]_{\theta=\frac{\pi}{3}}$$
$$= 4.041$$

22. $$\frac{\frac{d}{d\theta}(2+\cos 2\theta)\sin\theta}{\frac{d}{d\theta}(2+\cos 2\theta)\cos\theta}$$
$$= \left[\frac{\cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta + 2\cos\theta}{-\sin\theta\cos 2\theta - 2\cos\theta\sin 2\theta - 2\sin\theta}\right]_{\theta=\pi/3}$$
$$= 0.346$$

23. $$\frac{d}{d\theta}\left(1-\cos\left(\frac{\theta}{2}\right)\right)\sin\theta$$
$$= -\cos\theta\cos\left(\frac{\theta}{2}\right) + \frac{\sin\theta\sin\left(\frac{\theta}{2}\right)}{2} + \cos\theta$$
$$= 0$$
$y = 0, \quad y \approx \pm 0.443, \; y \approx \pm 1.739$
$$\frac{d}{d\theta}\left(1-\cos\left(\frac{\theta}{2}\right)\right)\cos\theta$$
$$= \sin\theta\cos\left(\frac{\theta}{2}\right) + \frac{\cos\theta\sin\left(\frac{\theta}{2}\right)}{2} - \sin\theta = 0$$
$x = 2, \; x \approx 0.067, \; x \approx -1.104$

24. $$\frac{d}{d\theta}(2(1-\sin\theta)\sin\theta) = (2-4\sin\theta)\cos\theta = 0$$
$y = \frac{1}{2}, \; y = -4$
$$\frac{d}{d\theta}(2(1-\sin\theta)\cos\theta)$$
$$= 2\sin\theta(\sin\theta - 1) - 2\cos^2\theta$$
$$= 0$$
$x = 0, \; x \approx \pm 2.598$

25. The tips of the petals are at the points where $r = \sin 2\theta = 1$, which are the points where $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$. The slope is
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dy}{d\theta}}$$
$$= \frac{\left(\frac{d}{d\theta}\right)(\sin 2\theta\sin\theta)}{\left(\frac{d}{d\theta}\right)(\sin 2\theta\cos\theta)}$$
$$= \frac{2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta}{2\cos 2\theta\cos\theta + \sin 2\theta\sin\theta}.$$

θ	(x, y)	$m = \frac{dy}{dx}$	Tangent line
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	-1	$y = -x + \sqrt{2}$
$\frac{3\pi}{4}$	$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	1	$y = x + \sqrt{2}$
$\frac{5\pi}{4}$	$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	-1	$y = -x - \sqrt{2}$
$\frac{7\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	1	$y = x - \sqrt{2}$

26. The cardioid crosses the x-axis at the points where $\theta = 0$ and $\theta = \pi$. The slope is
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$
$$= \frac{\left(\frac{d}{d\theta}\right)((1+\sin\theta)\sin\theta)}{\left(\frac{d}{d\theta}\right)((1+\sin\theta)\cos\theta)}$$
$$= \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta + (1+\sin\theta)\sin\theta}.$$

θ	(x, y)	$m = \frac{dy}{dx}$	Tangent line
0	(1, 0)	1	$y = x - 1$
π	(–1, 0)	–1	$y = -x - 1$

27. $x = y$, a line

28. $r^2 = 3r\cos\theta$
$x^2 + y^2 = 3x$, a circle
$\left(\text{center} = \left(\frac{3}{2}, 0\right), \text{radius} = \frac{3}{2}\right)$

29. $$r^2 = 4r\frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta}$$
$$r^2\cos^2\theta = 4r\sin\theta$$
$$x^2 = 4y, \text{ a parabola}$$

30. $$r\left(\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right) = 2\sqrt{3}$$
$$x\cos\frac{\pi}{3} - y\sin\frac{\pi}{3} = 2\sqrt{3}$$
$$\frac{1}{2}x - \frac{\sqrt{3}}{2}y = 2\sqrt{3}$$
$$x - \sqrt{3}y = 4\sqrt{3}$$
$$\text{or } y = \frac{x}{\sqrt{3}} - 4,$$
a line

31. $x^2 + y^2 + 5y = 0$
$r^2 + 5r\sin\theta = 0$
$r = -5\sin\theta$

32. $x^2 + y^2 - 2y = 0$
$r^2 - 2r\sin\theta = 0$
$r = 2\sin\theta$

33. $$x^2 + 4y^2 = 16$$
$$r^2\cos^2\theta + 4r^2\sin^2\theta = 16, \text{ or}$$
$$r^2 = \frac{16}{\cos^2\theta + 4\sin^2\theta}$$

34. $$(x+2)^2 + (y-5)^2 = 16$$
$$(r\cos\theta + 2)^2 + (r\sin\theta - 5)^2 = 16$$

35. $$\int_0^{2\pi}\left(\frac{1}{2}(2-\cos 2\theta)^2\right)d\theta$$
$$= \left[\frac{1}{8}(\sin 2\theta\cos 2\theta - 2(4\sin 2\theta - 9\theta))\right]_0^{2\pi}$$
$$= \frac{9\pi}{2} - 0$$
$$= \frac{9\pi}{2}$$

36. $$\frac{1}{3}\int_0^{\pi}\left(\frac{1}{2}(\sin 3\theta)^2\right)d\theta$$
$$= \left[-\frac{1}{36}(\sin 3\theta\cos 3\theta - 3\theta)\right]_0^{\pi}$$
$$= \frac{\pi}{12}$$

37. $$4\left[\int_0^{\pi/4}\frac{1}{2}(1+\cos 2\theta)^2 d\theta - \int_0^{\pi/4}\frac{1}{2}(1)^2 d\theta\right]$$
$$= 4\left[\int_0^{\pi/4}\left(\cos 2\theta + \frac{1+\cos 4\theta}{4}\right)d\theta\right]$$
$$= 4\left[\frac{\sin 2\theta}{2} + \frac{4\theta + \sin 4\theta}{16}\right]_0^{\pi/4}$$
$$= \left[2\sin 2\theta + \theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4}$$
$$= 2 + \frac{\pi}{4}$$

38. $$\int_0^{2\pi} \frac{1}{2}(2(1+\sin\theta))^2\, d\theta - \int_0^{\pi} \frac{1}{2}(2\sin\theta)^2\, d\theta = \int_0^{2\pi} 2(1+2\sin\theta+\sin^2\theta)\, d\theta - \int_0^{\pi} 2\sin^2\theta d\theta$$
$$= \int_0^{2\pi} (2+4\sin\theta+1-\cos 2\theta)\, d\theta - \int_0^{\pi} (1-\cos 2\theta)\, d\theta$$
$$= \left[3\theta - 4\cos\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} - \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$
$$= 6\pi - 4 - 0 + 4 - \pi + 0$$
$$= 5\pi$$

39. **(a)** $$\mathbf{v}(t) = \left\langle \frac{d}{dt} 4\cos t, \frac{d}{dt}\sqrt{2}\sin t \right\rangle$$
$$\mathbf{v}(t) = \left\langle -4\sin t, \sqrt{2}\cos t \right\rangle$$
$$\mathbf{a}(t) = \left\langle \frac{d}{dt}(-4\sin t), \frac{d}{dt}\sqrt{2}\cos t \right\rangle$$
$$\mathbf{a}(t) = \left\langle -4\cos t, -\sqrt{2}\sin t \right\rangle$$

(b) $$\left\langle -4\cos\frac{\pi}{4}, -\sqrt{2}\sin\frac{\pi}{4} \right\rangle$$
$$\left\langle -4\frac{\sqrt{2}}{2}, -\sqrt{2}\frac{\sqrt{2}}{2} \right\rangle$$
$$\text{Speed} = \sqrt{\left(-2\sqrt{2}\right)^2 + (-1)^2}$$
$$= 3$$

40. **(a)** $$\mathbf{v}(t) = \left\langle \frac{d}{dt}\sqrt{3}\sec t, \frac{d}{dt}\sqrt{3}\tan t \right\rangle$$
$$\mathbf{v}(t) = \left\langle \sqrt{3}\sec t\tan t, \sqrt{3}\sec^2 t \right\rangle$$
$$\mathbf{a}(t) = \left\langle \frac{d}{dt}\sqrt{3}\sec t\tan t, \frac{d}{dt}\sqrt{3}\sec^2 t \right\rangle$$
$$\mathbf{a}(t) = \left\langle \sqrt{3}(\sec t\tan^2 t + \sec^3 t), 2\sqrt{3}\sec^2 t\tan t \right\rangle$$

(b) $$\left\langle \sqrt{3}\sec(0)\tan(0), \sqrt{3}\sec^2(0) \right\rangle$$
$$\left\langle 0, \sqrt{3} \right\rangle$$
$$\text{Speed} = \sqrt{0^2 + \sqrt{3}^2}$$
$$= \sqrt{3}$$

41. $\mathbf{v}(t)=\left\langle \frac{d}{dt}\frac{1}{\sqrt{1+t^2}}, \frac{d}{dt}\frac{t}{\sqrt{1+t^2}} \right\rangle$

$\mathbf{v}(t)=\left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}} \right\rangle$

$\text{speed}=\sqrt{\left(\frac{-t}{(1+t^2)^{3/2}}\right)^2+\left(\frac{1}{(1+t^2)^{3/2}}\right)^2}$

$\text{speed}=\frac{1}{1+t^2}$

The maximum value of $\frac{1}{1+t^2}$ is 1, when $t=0$.

42. $\mathbf{r}(t)=\left\langle e^t\cos t, e^t\sin t\right\rangle$, with slope

$\frac{e^t\sin t}{e^t\cos t}=\tan t.$

$\mathbf{v}(t)=\left\langle e^t\cos t-e^t\sin t, e^t\sin t+e^t\cos t\right\rangle$

$\mathbf{a}(t)=\left\langle -2e^t\sin t, 2e^t\cos t\right\rangle$, with slope

$\frac{2e^t\cos t}{-2e^t\sin t}=-\frac{1}{\tan t}$. Since the slopes are negative reciprocals, the angle is always 90°.

43. $\mathbf{r}(t)=\left\langle \int -\sin t\,dt, \int\cos t\,dt\right\rangle$

$\mathbf{r}(t)=\langle \cos t+C_1, \sin t+C_2\rangle$

$\cos(0)+C_1=0, C_1=-1$

$\sin(0)+C_2=1, C_2=1$

$\mathbf{r}(t)=\langle \cos t-1, \sin t+1\rangle$

44. $\mathbf{r}(t)=\left\langle \int\frac{dt}{t^2+1}, \int\frac{t\,dt}{\left(t^2+1\right)^{1/2}}\right\rangle$

$\mathbf{r}(t)=\left\langle \tan^{-1}t+C_1, \sqrt{t^2+1}+C_2\right\rangle$

$\tan^{-1}0+C_1=1, C_1=1$

$\sqrt{0^2+1}+C_2=1, C_2=0$

$\mathbf{r}(t)=\left\langle \tan^{-1}t+1, \sqrt{t^2+1}\right\rangle$

45. $\mathbf{v}(t)=\left\langle \int 0dt, \int 2dt\right\rangle$

$\mathbf{v}(t)=\langle 0, 2t+C\rangle$

$2(0)+C=0$

$C=0$

So $\mathbf{v}(t)=\langle 0, 2t\rangle$

$\mathbf{r}(t)=\left\langle \int 0\,dt, \int 2t\,dt\right\rangle$

$\mathbf{r}(t)=\langle 1, t^2+C\rangle$

$0^2+C=0$

$C=0$

$\mathbf{r}(t)=\left\langle 1, t^2\right\rangle$

46. $\mathbf{v}(t)=\left\langle \int -2\,dt, \int -2\,dt\right\rangle$

$\mathbf{v}(t)=\left\langle -2t+C_1, -2t+C_2\right\rangle$

$-2(1)+C_1=4, C_1=6$

$-2(1)+C_2=0, C_2=2$

$\mathbf{r}(t)=\left\langle \int(-2t+6)dt, \int(-2t+2)dt\right\rangle$

$\mathbf{r}(t)=\left\langle -t^2+6t+C_1, -t^2+2t+C_2\right\rangle$

$-(1)^2+6(1)+C_1=3, C_1=-2$

$-(1)^2+2(1)+C_2=3, C_2=2$

$\mathbf{r}(t)=\left\langle -t^2+6t-2, -t^2+2t+2\right\rangle$

47. (a) $\frac{d}{dt}\left(3\cos\frac{\pi}{4}t\right)=\frac{-3\pi\sin\left(\frac{\pi}{4}t\right)}{4}$

$\frac{d}{dt}\left(5\sin\frac{\pi}{4}t\right)=\frac{5\pi\cos\left(\frac{\pi}{4}t\right)}{4}$

$\sqrt{\left(\frac{-3\pi\sin\left(\frac{\pi}{4}t\right)}{4}\right)^2+\left(\frac{5\pi\cos\left(\frac{\pi}{4}t\right)}{4}\right)^2}\Bigg|_{t=3}$

$=\frac{\pi\sqrt{17}}{4}$

(b) $\dfrac{d}{dt}\left(\dfrac{-3\pi\sin\left(\frac{\pi}{4}t\right)}{4}\right)\Bigg|_{t=3}$

$= \dfrac{-3\pi^2\cos\left(\frac{\pi}{4}t\right)}{16}\Bigg|_{t=3}$

$= \dfrac{3\pi^2}{16\sqrt{2}}$

$\dfrac{d}{dt}\left(\dfrac{5\pi\cos\left(\frac{\pi}{4}t\right)}{4}\right)\Bigg|_{t=3}$

$= -\dfrac{5\pi^2\sin\left(\frac{\pi}{4}t\right)}{16}\Bigg|_{t=3}$

$= -\dfrac{5\pi^2}{16\sqrt{2}}$

(c) $x^2 = \left(3\cos\dfrac{\pi}{4}t\right)^2$ and $y^2 = \left(5\sin\dfrac{\pi}{4}t\right)^2$

$\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

48. (a) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$= \dfrac{e^t\sin t + e^t\cos t}{e^t\cos t - e^t\sin t}$

$= \dfrac{\cos t + \sin t}{\cos t - \sin t}$

$\dfrac{dy}{dx}\Bigg|_{t=\pi} = \dfrac{-1}{-1} = 1$

(b) $\dfrac{dy}{dt} = e^t(\sin t + \cos t),\ \dfrac{dx}{dt} = e^t(\cos t - \sin t)$

$\left(\dfrac{dy}{dt}\right)^2 = e^{2t}(\sin^2 t + 2\sin t\cos t + \cos^2 t)$

$= e^{2t}(1 + 2\sin t\cos t)$

$\left(\dfrac{dx}{dt}\right)^2 = e^{2t}(\cos^2 t - 2\cos t\sin t + \sin^2 t)$

$= e^{2t}(1 - 2\cos t\sin t)$

$|\mathbf{v}(t)| = \sqrt{e^{2t}\cdot 2} = e^t\cdot\sqrt{2}$

$|\mathbf{v}(3)| = e^3\sqrt{2}$

(c) Distance $= \int_0^3 |\mathbf{v}(t)|\,dt$

$= \int_0^3 e^t\sqrt{2}\,dt$

$= \sqrt{2}[e^t]_0^3$

$= (e^3 - 1)\sqrt{2}$

49. (a) $\mathbf{v}(t) = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}\right\rangle = \left\langle 2t, \dfrac{6}{5}t^2\right\rangle$,

$\mathbf{v}(4) = \left\langle 8, \dfrac{96}{5}\right\rangle$, and

$|\mathbf{v}(4)| = \sqrt{8^2 + \left(\dfrac{96}{5}\right)^2} = \dfrac{104}{5}$

(b) Distance $= \int_0^4 \sqrt{(2t)^2 + \left(\dfrac{6}{5}t^2\right)^2}\,dt$

$= \int_0^4 \dfrac{2}{5}t\sqrt{25 + 9t^2}\,dt$

$= \left[\dfrac{2}{135}(25 + 9t^2)^{3/2}\right]_0^4$

$= \dfrac{4144}{135}$

(c) $t = \sqrt{x + 2}$, so

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dy}{dt}} = \dfrac{\frac{6t^2}{5}}{2t} = \dfrac{3}{5}t = \dfrac{3}{5}\sqrt{x + 2}.$

50. x degrees east of north is $(90 - x)$ degrees north of east.
Add the vectors:
$$\langle 540\cos 10°, 540\sin 10°\rangle + \langle 55\cos(-10°), 55\sin(-10°)\rangle = \langle 595\cos 10°, 485\sin 10°\rangle \approx \langle 585.961, 84.219\rangle.$$

Speed $\approx \sqrt{585.961^2 + 84.219^2} \approx 591.982$ mph.

Direction $\approx \tan^{-1}\left(\dfrac{585.961}{84.219}\right) \approx 81.821°$ east of north

51. **(a)** $\vec{a}_x = 2$

$\vec{v}_x = 2t + 0$

$x = t^2 + C \qquad x(0) = \pi \Rightarrow C = \pi$

$x = t^2 + \pi$

$y = \cos(t^2 + \pi)$

Position $= (t^2 + \pi, \cos(t^2 + \pi))$

(b) At this point $t^2 + \pi = 4$, so $t = \sqrt{4 - \pi}.$

$$\text{Speed} = \left.\sqrt{(2t)^2 + (2t(-\sin(t^2+\pi))^2}\right|_{t=\sqrt{4-\pi}} = 2.324$$

52. **(a)** $\mathbf{v}_A(t) = \langle 1, 2\rangle$ and $\mathbf{v}_B(t) = \left\langle \dfrac{3}{2}, \dfrac{3}{2}\right\rangle$

(b) $\int_0^3 \sqrt{1 + (2t-4)^2}\, dt \approx 6.126$

(c) Setting $x_A = x_B$, we find that $t = 4$. Plugging $t = 4$ into y_A and y_B, we find that both values are the same (4). Thus, the particles collide when $t = 4$. (*Note*: If you graph both paths, they will cross at (–1, 1). However, the particles are there at different times.)

53. **(a)** Area $= \int_0^{\pi} \dfrac{1}{2}\left(\dfrac{4}{1+\sin\theta}\right)^2 d\theta = \dfrac{32}{3}$

(b) The polar equation is equivalent to $r + r\sin\theta = 4$. Thus,

$$\begin{aligned} r &= 4 - r\sin\theta \\ r^2 &= (4 - r\sin\theta)^2 \\ x^2 + y^2 &= (4 - y)^2 \\ x^2 + y^2 &= 16 - 8y + y^2 \\ 8y &= 16 - x^2 \end{aligned}$$

(c) Area $= \int_{-4}^{4}\left(2 - \dfrac{x^2}{8}\right) dx$, which, indeed, is $\dfrac{32}{3}$.