Taylor Polynomials and Lagrange Error Bound

1. The Taylor series for a function f about $x = 2$ is given by $\sum_{n=0}^{\infty} (-1)^n \frac{3n+1}{2^n}(x-2)^n$ and converges to f for $0 < x < 4$. If the third-degree Taylor polynomial for f about $x = 2$ is used to approximate $f\left(\frac{9}{4}\right)$, what is the alternating series error bound?

center

(A) $\frac{10}{8 \cdot 4^3}$

(B) $\frac{1}{24 \cdot 4^4}$

(C) $\frac{13}{16 \cdot 4^4}$ ✓

(D) $\frac{13 \cdot 9^4}{16 \cdot 4^4}$

$\frac{(-1)^4(3(4)+1)}{2^4} \cdot \left(\frac{9}{4}-2\right)^4 \quad \frac{13}{2^4} \cdot \left(\frac{1}{4}\right)^4 \quad \frac{13}{16 \cdot 4^4}$

2.

$\max\limits_{0 \le x \le 1.2} \left\lvert f^{(5)}(x) \right\rvert = 8.4$	$\max\limits_{0 \le x \le 1.2} \left\lvert f^{(6)}(x) \right\rvert = 58.8$	$\max\limits_{0 \le x \le 1.2} \left\lvert f^{(7)}(x) \right\rvert = 411.8$

Let $P(x)$ be the fifth-degree Taylor polynomial for a function f about $x = 0$. Information about the maximum of the absolute value of selected derivatives of f over the interval $0 \le x \le 1.2$ is given in the table above. Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that $|f(1.2) - P(1.2)| \le k$?

$\frac{58.8(1.2)^6}{6!}$

(A) 0.082

(B) 0.174

(C) 0.244 ✓

(D) 0.293

3. Let f be a polynomial function with nonzero coefficients such that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$. $T_3(x)$ is the third-degree Taylor polynomial for f about $x = c$ such that $T_3(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3$. Based on use of the Lagrange error bound, $f(x) - T_3(x)$ must equal which of the following?

$C\ a_1 + 2a_2$

$f' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$

$f'' \ 2a_2 + 6a_3 x + 12a_4 x^2$

$f''' \ 6a_3 +$

$f^{4} = \frac{24a_4 x}{4!}$

(A) 0

(B) $(x-c)^4$

(C) $a_4(x-c)^4$ ✓

(D) $4! \cdot a_4(x-c)^4$

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4. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
3	6	-10	16	-24	32

The function f has derivatives of all orders for all real numbers. Values of f and its first four derivatives at $x = 3$ are given in the table above.

(a) Write an equation for the line tangent to the graph of f at $x = 3$, and use it to approximate $f(2.5)$.

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Please respond on separate paper, following directions from your teacher.

(b) Write the third-degree Taylor polynomial for f about $x = 3$, and use it to approximate $f(2.5)$. Is there enough information to determine whether f has a critical point at $x = 2.5$? If not, explain why not. If so, determine whether $f(2.5)$ is a relative maximum, relative minimum, or neither, and give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

(c) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \leq 48$ for all $x > 2$. Use the Lagrange error bound to show that the approximation found in part (b) differs from $f(2.5)$ by no more than $\frac{1}{8}$.

Please respond on separate paper, following directions from your teacher.

(d) What is the coefficient of the $(x-3)^3$ term of the Taylor series for f', the derivative of f, about $x = 3$?

Please respond on separate paper, following directions from your teacher.

solution for FRQ starts here

Part A

Numerical simplification is not required for the second point. The second point may be earned with use of a tangent line equation that has the correct slope and a maximum of one computational error.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2

The student response accurately includes both of the criteria below.

- ☐ equation for tangent line
- ☐ approximation

Solution:

An equation for the line tangent to the graph of f at $x = 3$ is $y = 6 - 10(x - 3)$.

$f(2.5) \approx 6 - 10(2.5 - 3) = 11$

Part B

To be eligible for any points, the Taylor polynomial must be about $x = 3$.

The first and second points may be earned without algebraic or numerical simplification. A maximum of 1 point out of the first 2 points may be earned if all terms are correct AND the response includes extra terms and/or the use of $+\cdots$.

Numerical simplification is not required for the third point. However, use of the equals sign rather than approximately equals does not earn the point (e.g., $f(2.5) = 13.5$)

The fourth point requires a conclusion equivalent to "The Taylor polynomial for f about $x = 3$ only gives exact information for at $x = 3$. "

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3	4 ✔

The student response accurately includes all four of the criteria below.

- ☐ first two terms
- ☐ remaining terms
- ☐ approximation
- ☐ explanation

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Solution:

The third-degree Taylor polynomial for f about $x = 3$ is

$$6 - 10(x-3) + \frac{16}{2!}(x-3)^2 - \frac{24}{3!}(x-3)^3 = 6 - 10(x-3) + 8(x-3)^2 - 4(x-3)^3.$$

$$f(2.5) \approx 6 - 10(2.5-3) + 8(2.5-3)^2 - 4(2.5-3)^3$$
$$= 6 + 5 + 2 + 0.5 = 13.5$$

In this case, there is not enough information to determine whether f has a critical point at $x = 2.5$. The Taylor polynomial for f about $x = 3$ only gives exact information for f at $x = 3$ (Note that a Taylor polynomial for f can give exact information in the interval of convergence if it is known that the series converges to f over the interval.)

Part C

Both points may be earned with $\text{Error} \le \frac{\max\left|f^{(4)}(x)\right|}{4!} \cdot (2.5-3)^4 \le \frac{48}{4!} \cdot \left(\frac{1}{2}\right)^4 = \frac{1}{8}$ or equivalent. The first point requires $\text{Error} \le \frac{\max\left|f^{(4)}(x)\right|}{4!} \cdot (2.5-3)^4$ or equivalent. The second point requires an explicit connection to the error and indicates the numeric value is clearly less than or equal to $\frac{1}{8}$.

The response is not eligible for either point with use of the alternating series error bound.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2 ✔

The student response accurately includes both of the criteria below.

- ☐ form of the error bound
- ☐ analysis

Solution:

$$\text{Error} \le \frac{\max\limits_{2.5 \le x \le 3}\left|f^{(4)}(x)\right|}{4!} \cdot (2.5-3)^4 \le \frac{48}{4!} \cdot \left(\frac{1}{2}\right)^4 = \frac{1}{8}$$

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Part D

Numerical simplification is not required for the point. $\frac{f^{(4)}(3)}{4!} \cdot 4$ is not sufficient to earn the point; substitution of the value from the table is required.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1

The student response accurately includes a correct coefficient.

Solution:

The coefficient of the $(x-3)^4$ term of the Taylor series for f about $x=3$ is $\frac{f^{(4)}(3)}{4!}$.

Therefore, the coefficient of the $(x-3)^3$ term of the Taylor series for f' about $x=3$ is $\frac{f^{(4)}(3)}{4!} \cdot 4 = \frac{32}{3!} = \frac{16}{3}$.