AP Calculus BC–3 2000

The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about $x = 5$.

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

(a) $f'(5) = \dfrac{-1!}{2(3)}$, $f''(5) = \dfrac{2!}{4(4)}$, $f'''(5) = \dfrac{-3!}{8(5)}$

$$P_3(f,5)(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$$

3 : $P_3(f,5)(x)$
<−1> each error or missing term

Note: <−1> max for improper use of extra terms, equality or +…

(b) $a_n = \dfrac{f^{(n)}(5)}{n!} = \dfrac{(-1)^n}{2^n (n+2)}$

$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)}}{\dfrac{(-1)^n (x-5)^n}{2^n (n+2)}} \right| = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+2}{n+3}\right)|x-5|$$

$$= \frac{|x-5|}{2} < 1$$

The radius of convergence is 2.

4 {
1 : general term
1 : sets up ratio test
1 : computes the limit
1 : applies ratio test to get radius of convergence

(c) The Taylor series about $x = 5$ for the function f, when evaluated at $x = 6$, is an alternating series with absolute value of terms decreasing to 0. The error in approximating $f(6)$ with the 6th degree Taylor polynomial at $x = 6$ is less than the first omitted term in the series.

$$|f(6) - P_6(f,5)(6)| \le \frac{1}{2^7(9)} = \frac{1}{1152} < \frac{1}{1000}$$

2 {
1 : error bound $< \dfrac{1}{1000}$
1 : refers to an alternating series and indicates the error bound is found from the next term

Question 6

The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

on its interval of convergence.

(a) Find the interval of convergence of the Maclaurin series for f. Justify your answer.

(b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

(c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

(a) $$\lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+2}}{n+2}}{\frac{(2x)^{n+1}}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)}{(n+2)} 2x \right| = \left| 2x \right|$$

$\left| 2x \right| < 1$ for $-\frac{1}{2} < x < \frac{1}{2}$

At $x = \frac{1}{2}$, the series is $\sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges since this is the harmonic series.

At $x = -\frac{1}{2}$, the series is $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n+1}$ which converges by the Alternating Series Test.

Hence, the interval of convergence is $-\frac{1}{2} \le x < \frac{1}{2}$.

5 points:
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies interior of interval of convergence
- 2 : analysis/conclusion at endpoints
  - 1 : right endpoint
  - 1 : left endpoint
- < −1 > if endpoints not $x = \pm\frac{1}{2}$
- < −1 > if multiple intervals

(b) $f'(x) = 2 + 4x + 8x^2 + 16x^3 + \ldots + 2(2x)^n + \ldots$

2 points:
- 1 : first 4 terms
- 1 : general term

(c) The series in (b) is a geometric series.

$$f'\left(-\frac{1}{3}\right) = 2 + 4\left(-\frac{1}{3}\right) + 8\left(-\frac{1}{3}\right)^2 + \ldots + 2\left(2 \cdot \left(-\frac{1}{3}\right)\right)^n + \ldots$$

$$= 2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots + 2\left(-\frac{2}{3}\right)^n + \ldots$$

$$= \frac{2}{1 + \frac{2}{3}} = \frac{6}{5}$$

OR

$f'(x) = \frac{2}{1-2x}$ for $-\frac{1}{2} < x < \frac{1}{2}$. Therefore,

$$f'\left(-\frac{1}{3}\right) = \frac{2}{1 + \frac{2}{3}} = \frac{6}{5}$$

2 points:
- 1 : substitutes $x = -\frac{1}{3}$ into infinite series from (b) or expresses series from (b) in closed form
- 1 : answer for student's series