9.2 Notes L'Hôpital's / L'Hôspital's Rule

Solutions

RECALL: the definition of a derivative at x = a:

$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ or $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ using limits to find derivatives

NOW: we will use derivatives to find limits that are in an indeterminate form.

> Suppose *f* and *g* are differentiable on an open interval containing a, and $g'(x)\neq 0$ (except possibly at *a*).
>
> **If the** $\lim_{x\to a}\frac{f(x)}{g(x)}\to\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ **Then** $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ **as long as** $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ **exists (or is ∞ or -∞).**

L'Hôpital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied.

You must verify the condition that $\lim_{x\to a}\frac{f(x)}{g(x)}\to\frac{0}{0}$ *or* $\lim_{x\to a}\frac{f(x)}{g(x)}\to\frac{\pm\infty}{\pm\infty}$ **before using l'Hôpital's Rule.**

L'Hôpital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity. That is, "$x\to a$" can be replaced by "$x\to a^+$", "$x\to a^-$", "$x\to\infty$", or "$x\to-\infty$".

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}\to\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then L'Hôpital's Rule can be applied again and again.

Just remember to be careful…. L'Hôpital's Rule cannot be applied to all limits, only ones that are in the indeterminate form.

Examples:

$\lim_{x\to 0}\frac{\sin x}{x} \to \frac{0}{0}$

$\lim_{x\to 0}\frac{\cos x}{1} = \frac{\cos 0}{1} = \frac{1}{1}$

$= 1$

$\lim_{x\to 0}\frac{\sin x - x}{x^3} \to \frac{0}{0}$

$\lim_{x\to 0}\frac{\cos x - 1}{3x^2} \to \frac{0}{0}$

$\lim_{x\to 0}\frac{-\sin x}{6x} \to \frac{0}{0}$

$\lim_{x\to 0}\frac{-\cos x}{6} = \frac{-1}{6}$

$\lim_{n\to\infty}\frac{2\sqrt{x}}{\ln x} \to \frac{\infty}{\infty}$

$\lim_{x\to\infty}\frac{x^{-1/2}}{\frac{1}{x}}$

$\lim_{x\to\infty} x \cdot x^{-1/2}$

$\lim_{x\to\infty} x^{1/2} = \infty$

$\lim_{x\to 3}\frac{x-3}{x^2-9}$

you could but NO NEED L'Hop

$\lim_{x\to 3}\frac{\cancel{x-3}\;1}{\cancel{(x-3)}(x+3)} = \frac{1}{3+3} = \frac{1}{6}$

Two-sided limits:

Examples: $\lim_{x\to 0^+} \frac{\sin x}{x^2} \to \frac{0}{0}$ $\qquad \lim_{x\to 0^-} \frac{\sin x}{x^2} \to \frac{\infty}{\infty}$ $\qquad \lim_{x\to 0^+} \frac{\cos x}{2x} \to \frac{1}{0}$ $\qquad \lim_{x\to 0^-} \frac{\cos x}{2x} \to \frac{-1}{0} = -\infty$

$\lim_{x\to 0^+} \frac{\cos x}{2x} = \frac{1}{0} = \infty$ $\qquad \lim_{x\to 0^-} \frac{\cos x}{2x} = \frac{1}{0} = \infty$ $\qquad = \infty$

b/c Approaches from the left

You try……

$\lim_{x\to 3} \frac{x}{x+3} = \frac{3}{3+3} = \frac{3}{6} = \frac{1}{2}$

NO L'Hop

$\lim_{x\to 1} \frac{\ln x}{x-1} \to \frac{0}{0}$

$\lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1$

$\lim_{x\to \frac{\pi}{2}} \frac{\cos x}{\sin 3x}$ NO L'Hop

$= \frac{\cos \frac{\pi}{2}}{\sin \frac{3\pi}{2}} = \frac{0}{-1} = 0$

$\lim_{x\to \infty} \frac{3x^2-1}{2x^2-x-3} = \frac{3}{2}$

end behavior or L'Hop

$\lim_{\theta\to \frac{\pi}{2}} \frac{1-\sin\theta}{1+\cos 2\theta} \to \frac{0}{0}$

$\lim_{\theta\to \frac{\pi}{2}} \frac{-\cos\theta}{-2\sin 2\theta} \to \frac{0}{0}$

$\lim_{\theta\to \frac{\pi}{2}} \frac{\sin\theta}{-4\cos 2\theta} = \frac{1}{4}$

$\lim_{x\to \infty} \frac{1}{x}$ NO L'Hop

$= 0$

More practice……. Find the following limits. Remember to check conditions before applying L'Hôpital's Rule, conditions matter.

$$\lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \to \frac{0}{0}$$

$$\therefore \lim_{x\to\infty} \frac{\sin}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\cancel{-\frac{1}{x^2}}\cos\left(\frac{1}{x}\right)}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$$

$$= 1$$

$$\lim_{x\to 0}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right) \to \frac{0}{0}$$

$$\therefore \lim_{x\to 0}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right) = \lim_{x\to 0}\left(\frac{-\sin x}{\cos x + 1\cdot\cos x - x\sin x}\right) = \frac{0}{1+1-0} = \frac{0}{2}$$

$$= 0$$

$$\lim_{x\to 4}\frac{x^3-64}{x^2+16} \to \frac{0}{32} = 0$$

$$\lim_{x\to 0^+}\left(\frac{-\ln x}{\frac{1}{x}}\right) \to \frac{0}{0} \quad \therefore \lim_{x\to 0^+}\left(\frac{-\ln x}{\frac{1}{x}}\right) = \lim_{x\to 0^+}\left(\frac{-\frac{1}{x}}{-\frac{1}{x^2}}\right) = \lim_{x\to 0^+} x = 0$$

$$\lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\right) \to \frac{0}{0} \quad \therefore \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-1/2}}{1}$$

$$= \frac{\frac{1}{2}(1)^{-1/2}}{1} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\lim_{x\to 3}\frac{2x^2-5x-3}{x-4} = \frac{18-15-3}{-1} = \frac{0}{-1} = 0$$

$$\lim_{x\to 0}\left(\frac{\sqrt{1+x}-1-x/2}{x^2}\right) \to \frac{0}{0} \quad \therefore \lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2} = \lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-1/2}-\frac{1}{2}}{2x}$$

$$\to \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot(1+x)^{-3/2}}{2} = \frac{-\frac{1}{4}\cdot 1}{2} = \frac{-1}{8}$$

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{\sec x}{1+\tan x}\right) \to \frac{\infty}{\infty} \quad \therefore \lim_{x\to\frac{\pi}{2}}\left(\frac{\sec x}{1+\tan x}\right) = \lim_{x\to\frac{\pi}{2}}\left(\frac{\sec x\tan x}{\sec^2 x}\right)$$

$$= \lim_{x\to\pi/2}\left(\frac{\tan x}{\sec x}\right) = \lim_{x\to\pi/2}\left(\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}\right) = \lim_{x\to\pi/2}\sin x = 1$$

$\lim_{x\to 0}\left(\frac{\sin x}{x^2}\right)$ DNE

$\lim_{x\to 0^+}\left(\frac{\sin x}{x^2}\right) \to \frac{0}{0}$

$\lim_{x\to 0^-}\left(\frac{\sin x}{x^2}\right) \to \frac{0}{0}$

$$\lim_{x \to 0^+} \frac{\cos x}{2x} \to \frac{1}{0} = +\infty \qquad \lim_{x \to 0^-} \frac{\cos x}{2x} \to \frac{1}{0} = -\infty$$

$$\lim_{x\to\infty}\left(\frac{\ln x}{2\sqrt{x}}\right) \to \frac{\infty}{\infty} \quad \therefore \lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{x^{-1/2}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{x^{1/2}}} = \lim_{x \to \infty} \frac{1}{x^{3/2}} = 0$$

$$\lim_{x\to 0}\frac{\sin 4x}{x^2+3x+1} \to \frac{0}{1} = 0$$

$$\lim_{x\to 0}\frac{\cos 2x-1}{\sin 5x} \to \frac{0}{0} \quad \therefore \lim_{x \to 0} \frac{\cos 2x - 1}{\sin 5x} = \lim_{x \to 0} \frac{-2\sin 2x}{5\cos 5x} = \frac{0}{5} = 0$$