Precalculus H **Name______________________________**

9.2 Day 1 Binomial Theorem

Expand the following binomials

$(x+y)^0$ = 1

$(x+y)^1$ = $x+y$

$(x+y)^2$ = $x^2+2xy+y^2$

$(x+y)^3$ = $x^3+3x^2y+3xy^2+y^3$

$(x+y)^4$ = $x^4+4x^3y+6x^2y^2+4xy^3+y^4$

Pascal's Triangle

If we eliminate the plus signs and the powers of the variables a and b in the "triangular" array of binomial coefficients with which we began this section, we get:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \therefore & & & & \vdots & & & & \ddots &
\end{array}
$$

This is called **Pascal's triangle** in honor of Blaise Pascal (1623–1662), who used it in his work but certainly did not discover it. It appeared in 1303 in a Chinese text, the *Precious Mirror*, by Chu Shih-chieh, who referred to it even then as a "diagram of the old method for finding eighth and lower powers."

For convenience, we refer to the top "1" in Pascal's triangle as row 0. That allows us to associate the numbers along row n with the expansion of $(a+b)^n$.

Pascal's triangle is so rich in patterns that people still write about them today. One of the simplest patterns is the one that we use for getting from one row to the next, as in the following example.

If you expand $(a+b)^n$ for $n = 0, 1, 2, 3, 4$, and 5, here is what you get:

$$\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= 1a^1b^0 + 1a^0b^1\\
(a+b)^2 &= 1a^2b^0 + 2a^1b^1 + 1a^0b^2\\
(a+b)^3 &= 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3\\
(a+b)^4 &= 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4\\
(a+b)^5 &= 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5
\end{aligned}$$

Expand:

$(x+y)^7$ = ${}_7C_0\, x^7 y^0 + {}_7C_1\, x^6 y^1 + {}_7C_2\, x^5 y^2 + {}_7C_3\, x^4 y^3 + {}_7C_4\, x^3 y^4 + {}_7C_5\, x^2 y^5 + {}_7C_6\, x y^6 + {}_7C_7\, x^0 y^7$

DEFINITION Binomial Coefficient

The binomial coefficients that appear in the expansion of $(a+b)^n$ are the values of ${}_nC_r$ for $r = 0, 1, 2, 3, \ldots, n$.
A classical notation for ${}_nC_r$, especially in the context of binomial coefficients, is $\binom{n}{r}$. Both notations are read "n choose r."

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\frac{5!}{3!2!}$$

Expand.

$\binom{5}{0}$	$\binom{5}{1}$	$\binom{5}{2}$	$\binom{5}{3}$	$\binom{5}{4}$	$\binom{5}{5}$
${}_5C_0$	${}_5C_1$	${}_5C_2$	${}_5C_3$	${}_5C_4$	${}_5C_5$
1	5	10	10	5	1

Do you notice anything interesting? Are there any common terms? Why or why not?

Binomial Theorem

For any positive integer n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n,$$

where

$$\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n-r)!}.$$

Expand: $\left(2x - y^2\right)^4$ $\quad (a+b)^4 \quad a = 2x \quad b = -y^2$

${}_4C_0\, a^4 + {}_4C_1\, a^3 b + {}_4C_2\, a^2 b^2 + {}_4C_3\, a b^3 + {}_4C_4\, b^4$

$1 \cdot (2x)^4 + 4(2x)^3(-y^2) + 6(2x)^2(-y^2)^2 + 4(2x)(-y^2)^3 + (-y^2)^4$

$\boxed{16x^4 - 32x^3y^2 + 24x^2y^4 - 8xy^6 + y^8}$

You try….. Expand $\left(x^{-3} + 3y\right)^5$ $\quad (a+b)^5 \quad a = x^{-3} \quad b = 3y$

${}_5C_0\, a^5 b^0 + {}_5C_1\, a^4 b^1 + {}_5C_2\, a^3 b^2 + {}_5C_3\, a^2 b^3 + {}_5C_4\, a b^4 + {}_5C_5\, a^0 b^5$

$(x^{-3})^5 + 5(x^{-3})^4(3y) + 10(x^{-3})^3(3y)^2 + 10(x^{-3})^2(3y)^3 + 5(x^{-3})(3y)^4 + (3y)^5$

$x^{-15} + 15x^{-12}y + 90x^{-9}y^2 + 270x^{-6}y^3 + 405x^{-3}y^4 + 243y^5$

Find the coefficient of the given term in the binomial expansion.

a. $x^5 y^{\boxed{3}}$ **term** $(x+y)^8$

$\binom{8}{5} = \binom{8}{\boxed{3}} \quad {}_8C_3 = 56$

b. x^7 **term** $(x-3)^{11}$ $\quad (a+b)^{11} \quad a = x \quad b = -3$

${}_{11}C_4\, a^7 \cdot b^4$

$330x^7(-3)^4$

$\boxed{26730}x^7$

coefficient