43. $\lim_{n\to\infty}\left(\frac{1}{n!}\right)=0$, since $0\le\frac{1}{n!}\le\frac{1}{n}$ $n\ge 1$, and $\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$.

44. $\lim_{n\to\infty}\left(\frac{\sin^2 n}{2^n}\right)=0$, since $0\le\frac{\sin^2 n}{n}\le\frac{1}{n}$ for $n\ge 1$, and $\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$.

45. Graph (b)

46. Graph (c)

47. Table (d)

48. Table (a)

49. False; if the sequence is increasing, the terms will eventually become positive. Consider the sequence with nth term $a_n=-5+2(n-1)$. Here $a=-5$, $a_2=-3$, $a_3=-1$, and $a_4=1$.

50. True; $a_1>0$, $r=\frac{a_2}{a_1}>0$, and

$a_n=a_1 r^{n-1}>0$ for all $n\ge 2$.

51. C; $d=\frac{5-(-1)}{2}=3$

$a_6=-1+3(5)=14$

52. E; $r=\frac{1.25}{2.5}=\frac{1}{2}$

$a_1=\frac{2.5}{\frac{1}{2}}=5$

53. D; let $x=\frac{3\pi}{n}$. Then $x\to 0$ as $n\to\infty$, and

$$\lim_{n\to\infty}\left(n\sin\left(\frac{3\pi}{n}\right)\right)=\lim_{x\to 0}\left(\frac{3\pi}{x}\cdot\sin(x)\right)$$
$$=3\pi\cdot\lim_{x\to 0}\frac{\sin x}{x}$$
$$=3\pi\cdot 1$$
$$=3\pi$$

54. E; $n=2k$, $\lim_{n\to\infty}\left((-1)^n\frac{3n-1}{n+2}\right)=3$

$n=2k-1$, $\lim_{n\to\infty}\left((-1)^n\frac{3n-1}{n+2}\right)=-3$

55. **(a)** Draw a segment from the center of the circle to each vertex of the polygon, forming n isosceles triangles. The vertex angle in each triangle is $\frac{2\pi}{n}$. An altitude to the base divides the isosceles triangle into two right triangles with hypotenuse 1. In each of these triangles, the side opposite the angle of measure $\frac{\pi}{n}$ has length $\sin\left(\frac{\pi}{n}\right)$. It follows that each side of the polygon has length $2\sin\left(\frac{\pi}{n}\right)$ and the total perimeter is $2n\sin\left(\frac{\pi}{n}\right)$.

(b) Let $x=\frac{\pi}{n}$. Then $x\to 0$ as $n\to\infty$, and

$$\lim_{n\to\infty}\left(2n\sin\left(\frac{\pi}{n}\right)\right)=\lim_{x\to 0}\left(2\cdot\frac{\pi}{x}\cdot\sin(x)\right)$$
$$=2\pi\cdot\lim_{x\to 0}\frac{\sin x}{x}$$
$$=2\pi\cdot 1$$
$$=2\pi$$

56. **(a)** $a_n=a_{n-2}+a_{n-1}$ for $n\ge 3$

$a_1=1$; $a_2=1$

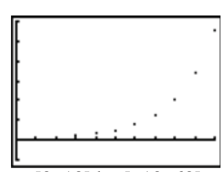
$a_3=a_1+a_2=1+1=2$

$a_4=a_2+a_3=1+2=3$

Continue in this fashion to get all of the first ten terms:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55

(b)



[0, 10] by [–10, 60]

(c) $\frac{a_2}{a_1}=\frac{1}{1}=1$

$\frac{a_3}{a_2}=\frac{1+1}{1}=\frac{2}{1}=2$

$\frac{a_4}{a_3}=\frac{2+1}{2}=\frac{3}{2}$

$\frac{a_5}{a_4}=\frac{3+2}{3}=\frac{5}{3}$

$\frac{a_6}{a_5} = \frac{5+3}{5} = \frac{8}{5}$

$\frac{a_7}{a_6} = \frac{8+5}{8} = \frac{13}{8}$

$\frac{a_8}{a_7} = \frac{13+8}{13} = \frac{21}{13}$

$\frac{a_9}{a_8} = \frac{21+13}{21} = \frac{34}{21}$

$\frac{a_{10}}{a_9} = \frac{34+21}{34} = \frac{55}{34}$

$\frac{a_{11}}{a_{10}} = \frac{55+34}{55} = \frac{89}{55}$

The 10th term is $\frac{89}{55} = 1.618181818$.

(d) Since $\frac{34}{21} \approx 1.619047619$, the term $\frac{55}{34} \approx 1.6176470589$ is the first term that approximates $\frac{1+\sqrt{5}}{2}$ to within 0.001.

57. $a_n = ar^{n-1}$ implies that $\log a_n = \log a + (n-1)\log r$. Thus $\{\log a_n\}$ is an arithmetic sequence with first term log *a* and common difference log *r*.

58. $a_n = a + (n-1)d$ implies that $10^{a_n} = 10^{a+(n-1)d} = 10^a (10^d)^{n-1}$. Thus $\{10^{a_n}\}$ is a geometric sequence with first term 10^a and common ratio 10^d.

59. Given $\varepsilon > 0$ choose $M = \frac{1}{\varepsilon}$. Then $\left|\frac{1}{n} - 0\right| < \varepsilon$ if $n > M$.

Section 9.2 L'Hospital's Rule (pp. 452–460)

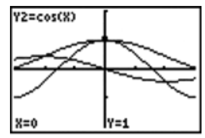
Exploration 1 Exploring L'Hospital's Rule Graphically

1. $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

2. The two graphs suggest that

$$\lim_{x\to 0} \frac{y_1}{y_2} = \lim_{x\to 0} \frac{y_1'}{y_2'}.$$

3. $y_5 = \frac{x\cos x - \sin x}{x^2}$. The graphs of y_3 and y_5 clearly show that L'Hospital's Rule does not say that $\lim_{x\to 0} \frac{y_1}{y_2}$ is equal to $\lim_{x\to 0}\left(\frac{y_1}{y_2}\right)'$.



[–3, 3] by [–2, 2]

Quick Review 9.2

1.

x	$\left(1+\frac{0.1}{x}\right)^x$
1	1.1000
10	1.1046
100	1.1051
1000	1.1052
10,000	1.1052
1,000,000	1.1052

As $x \to \infty$, $\left(1+\frac{0.1}{x}\right)^x$ approaches 1.1052.

2.

x	$x^{1/(\ln x)}$
0.1	2.7183
0.01	2.7183
0.001	2.7183
0.0001	2.7183
0.00001	2.7183

As $x \to 0^+$, $x^{1/(\ln x)}$ approaches 2.7183.

3.

x	$\left(1-\frac{1}{x}\right)^x$
– 1	0.5
– 0.1	0.78679
– 0.01	0.95490
– 0.001	0.99312
– 0.0001	0.99908
– 0.00001	0.99988
– 0.000001	0.99999

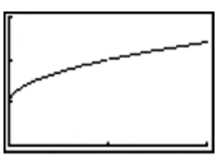
As $x \to 0^-$, $\left(1-\frac{1}{x}\right)^x$ approaches 1.

4.

x	$\left(1+\frac{1}{x}\right)^x$
–1.1	13.981
–1.01	105.77
–1.001	1007.9
–1.0001	10010

As $x \to -1^-$, $\left(1+\frac{1}{x}\right)^x$ goes to ∞.

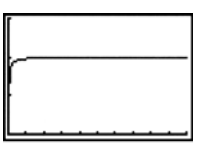
5.



[0, 2] by [0, 3]

As $t \to 1$, $\frac{t-1}{\sqrt{t}-1}$ approaches 2.

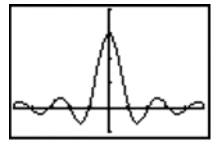
6.



[0, 500] by [0, 3]

As $x \to \infty$, $\frac{\sqrt{4x^2+1}}{x+1}$ approaches 2.

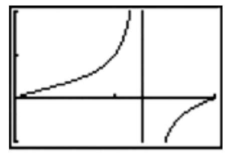
7.



[–5, 5] by [–1, 4]

As $x \to 0$, $\frac{\sin 3x}{x}$ approaches 3.

8.



[0, π] by [–1, 2]

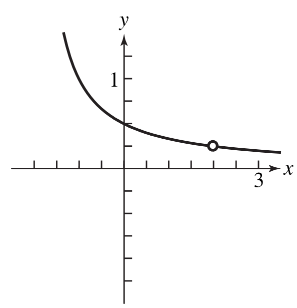
As $\theta \to \frac{\pi}{2}$, $\frac{\tan\theta}{2+\tan\theta}$ approaches 1.

9. $y = \frac{1}{h}\sin h = \frac{\sin h}{h}$

10. $y = (1+h)^{1/h}$

Section 9.2 Exercises

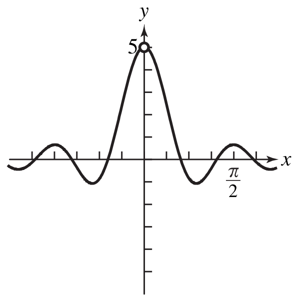
1. $\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right)$ appears to be about $\frac{1}{4}$;



By L'Hospital's Rule:

$$\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right) = \lim_{x\to 2}\frac{1}{2x} = \frac{1}{4}$$

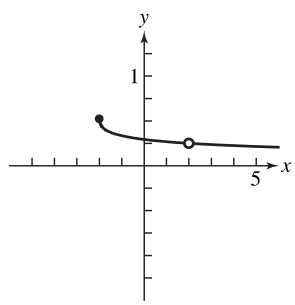
2. $\lim_{x\to 0}\left(\frac{\sin(5x)}{x}\right)$ appears to be about 5;



By L'Hospital's Rule:

$$\lim_{x\to 0}\left(\frac{\sin(5x)}{x}\right)=\lim_{x\to 0}\frac{5\cos(5x)}{1}=5\cos(0)=5$$

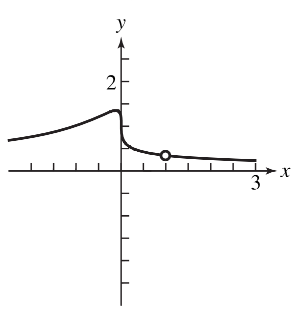
3. $\lim_{x\to 2}\left(\frac{\sqrt{2+x}-2}{x-2}\right)$ appears to be about $\frac{1}{4}$;



By L'Hospital's Rule:

$$\lim_{x\to 2}\left(\frac{\sqrt{2+x}-2}{x-2}\right)=\lim_{x\to 2}\left(\frac{\frac{1}{2}(2+x)^{-1/2}}{1}\right)=\frac{1}{2}(2+2)^{-1/2}=\frac{1}{2\sqrt{4}}=\frac{1}{4}$$

4. $\lim_{x\to 1}\left(\frac{\sqrt[3]{x}-1}{x-1}\right)$ appears to be about $\frac{1}{3}$;



By L'Hospital's Rule:

$$\lim_{x\to 1}\left(\frac{\sqrt[3]{x}-1}{x-1}\right)=\lim_{x\to 1}\left(\frac{\frac{1}{3}(x)^{-2/3}}{1}\right)=\frac{1}{3}(1)^{-2/3}=\frac{1}{3}$$

5. $$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)=\lim_{x\to 0}\left(\frac{\sin x}{2x}\right)=\lim_{x\to 0}\left(\frac{\cos x}{2}\right)=\frac{\cos(0)}{2}=\frac{1}{2}$$

6. $$\lim_{\theta\to\pi/2}\left(\frac{1-\sin\theta}{1+\cos(2\theta)}\right)=\lim_{\theta\to\pi/2}\left(\frac{-\cos\theta}{-2\sin(2\theta)}\right)=\lim_{\theta\to\pi/2}\left(\frac{\sin\theta}{-4\cos(2\theta)}\right)=\frac{\sin\frac{\pi}{2}}{-4\cos\pi}=\frac{1}{4}$$

7. $$\lim_{t\to 0}\left(\frac{\cos t-1}{e^t-t-1}\right)=\lim_{t\to 0}\left(\frac{-\sin t}{e^t-1}\right)=\lim_{t\to 0}\left(\frac{-\cos t}{e^t}\right)=\frac{-\cos(0)}{e^0}=-1$$

8. $$\lim_{x\to 2}\left(\frac{x^2-4x+4}{x^3-12x+16}\right)=\lim_{x\to 2}\left(\frac{2x-4}{3x^2-12}\right)=\lim_{x\to 2}\left(\frac{2}{6x}\right)=\frac{2}{6(2)}=\frac{1}{6}$$

9. **(a)** $\lim_{x\to 0^-}\left(\frac{\sin 4x}{\sin 2x}\right)=\lim_{x\to 0^-}\left(\frac{4\cos(4x)}{2\cos(2x)}\right)$
$=\frac{4\cos(0)}{2\cos(0)}$
$=2$

(b) $\lim_{x\to 0^+}\left(\frac{\sin 4x}{\sin 2x}\right)=\lim_{x\to 0^+}\left(\frac{4\cos(4x)}{2\cos(2x)}\right)$
$=\frac{4\cos(0)}{2\cos(0)}$
$=2$

10. **(a)** $\lim_{x\to 0^-}\left(\frac{\tan x}{x}\right)=\lim_{x\to 0^-}\left(\frac{\sec^2 x}{1}\right)=1$

(b) $\lim_{x\to 0^+}\left(\frac{\tan x}{x}\right)=\lim_{x\to 0^+}\left(\frac{\sec^2 x}{1}\right)=1$

11. **(a)** $\lim_{x\to 0^-}\left(\frac{\sin x}{x^3}\right)=\lim_{x\to 0^-}\left(\frac{\cos x}{3x^2}\right)=\infty$

(b) $\lim_{x\to 0^+}\left(\frac{\sin x}{x^3}\right)=\lim_{x\to 0^+}\left(\frac{\cos x}{3x^2}\right)=\infty$

12. **(a)** $\lim_{x\to 0^-}\left(\frac{\tan x}{x^2}\right)=\lim_{x\to 0^-}\left(\frac{\sec^2 x}{2x}\right)=-\infty$

(b) $\lim_{x\to 0^+}\left(\frac{\tan x}{x^2}\right)=\lim_{x\to 0^+}\left(\frac{\sec^2 x}{2x}\right)=\infty$

13. Left:
$\lim_{x\to \pi^-}\left(\frac{\csc x}{1+\cot x}\right)=\frac{\infty}{-\infty}$
$\lim_{x\to \pi^-}\left(\frac{-\csc x\cot x}{-\csc^2 x}\right)=\lim_{x\to \pi^-}\left(\frac{\cot x}{\csc x}\right)$
$=\lim_{x\to \pi^-}\frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)}$
$=\lim_{x\to \pi^-}\cos x$
$=-1$

Right:
$\lim_{x\to \pi^+}\left(\frac{\csc x}{1+\cot x}\right)=\frac{-\infty}{\infty}$
$\lim_{x\to \pi^+}\left(\frac{-\csc x\cot x}{-\csc^2 x}\right)=\lim_{x\to \pi^+}\left(\frac{\cot x}{\csc x}\right)$
$=\lim_{x\to \pi^+}\frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)}$
$=\lim_{x\to \pi^+}\cos x$
$=-1$

14. Left:
$\lim_{x\to \pi/2^-}\left(\frac{1+\sec x}{\tan x}\right)=\frac{\infty}{\infty}$
$\lim_{x\to \pi/2^-}\left(\frac{\sec x\tan x}{\sec^2 x}\right)=\lim_{x\to \pi/2^-}\left(\frac{\tan x}{\sec x}\right)$
$=\lim_{x\to \pi/2^-}\frac{\left(\frac{\sin x}{\cos x}\right)}{\left(\frac{1}{\cos x}\right)}$
$=\lim_{x\to \pi/2^-}\sin x$
$=1$

Right:
$\lim_{x\to \pi/2^+}\left(\frac{1+\sec x}{\tan x}\right)=\frac{-\infty}{-\infty}$
$\lim_{x\to \pi/2^+}\left(\frac{\sec x\tan x}{\sec^2 x}\right)=\lim_{x\to \pi/2^+}\left(\frac{\tan x}{\sec x}\right)$
$=\lim_{x\to \pi/2^+}\frac{\left(\frac{\sin x}{\cos x}\right)}{\left(\frac{1}{\cos x}\right)}$
$=\lim_{x\to \pi/2^+}\sin x$
$=1$

15. $\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}=\frac{\infty}{\infty}$
$\lim_{x\to\infty}\left(\frac{\frac{1}{x+1}}{\frac{1}{x\ln 2}}\right)=\lim_{x\to\infty}\left(\frac{x\ln 2}{x+1}\right)=\frac{\infty}{\infty}$
$\lim_{x\to\infty}\left(\frac{\ln 2}{1}\right)=\ln 2$

16. $\lim_{x\to\infty}\left(\frac{5x^2-3x}{7x^2+1}\right)=\frac{\infty}{\infty}$

$\lim_{x\to\infty}\left(\frac{10x-3}{14x}\right)=\frac{\infty}{\infty}$

$\lim_{x\to\infty}\left(\frac{10}{14}\right)=\frac{5}{7}$

$\text{Y1}=\frac{(5x^2-3x)}{(7x^2+1)}$

$\text{limit}=\frac{5}{7}$

17. $\lim_{x\to 0^+}(x\ln x)=0\cdot\infty$

$\lim_{x\to 0^+}\left(\frac{\ln x}{1/x}\right)=\lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)=\lim_{x\to 0^+}(-x)=0$

18. $\lim_{x\to\infty}\left(x\tan\left(\frac{1}{x}\right)\right)=\infty\cdot 0$

Let $h=\frac{1}{x}$. Then $h\to 0$ as $x\to\infty$, and

$$\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)=\lim_{h\to 0}\left(\frac{1}{h}\tan(h)\right)=\lim_{h\to 0}\left(\frac{\tan(h)}{h}\right)=\frac{\infty}{\infty}$$

$\lim_{h\to 0}\frac{\sec^2 h}{1}=\sec^2(0)=1$

19. $\lim_{x\to 0^+}(\csc x-\cot x+\cos x)$

$$=\lim_{x\to 0^+}(\csc x-(\cot x-\cos x))=\infty-\infty=\lim_{x\to 0^+}\frac{1-\cos x+\cos x\sin x}{\sin x}=\lim_{x\to 0^+}\frac{\sin x-\sin^2 x+\cos^2 x}{\cos x}=\frac{0-0+1}{1}=1.$$

20. $\lim_{x\to\infty}(\ln(2x)-\ln(x+1))=\infty-\infty$

$\ln(2x)-\ln(x+1)=\ln\left(\frac{2x}{x+1}\right)$

$\lim_{x\to\infty}\left(\frac{2x}{x+1}\right)=\lim_{x\to\infty}\frac{2}{1}=2$

Therefore:

$$\lim_{x\to\infty}(\ln(2x)-\ln(x+1))=\lim_{x\to\infty}\left(\ln\left(\frac{2x}{x+1}\right)\right)=\ln(2)$$

21. $\lim_{x\to 0}(e^x+x)^{1/x}=(1+0)^\infty=1^\infty$

$\ln f(x)=\ln[(e^x+x)^{1/x}]=\frac{\ln(e^x+x)}{x}$

$$\lim_{x\to 0}\left(\frac{\ln(e^x+x)}{x}\right)=\lim_{x\to 0}\frac{\frac{1}{e^x+x}\cdot(e^x+1)}{1}=\lim_{x\to 0}\frac{e^x+1}{e^x+x}=2$$

Therefore:

$$\lim_{x\to 0}(e^x+x)^{1/x}=\lim_{x\to 0}f(x)=\lim_{x\to 0}e^{\ln f(x)}=e^2$$

22. $\lim_{x\to 1}x^{1/(x-1)}=1^\infty$

$\ln f(x)=\ln[x^{1/(x-1)}]=\frac{\ln x}{x-1}$

$\lim_{x\to 1}\frac{\ln x}{x-1}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}=1$

Therefore:

$\lim_{x\to 1}x^{1/(x-1)}=\lim_{x\to 1}f(x)=\lim_{x\to 1}e^{\ln f(x)}=e^1=e$

23. $\lim_{x\to 1}(x^2-2x+1)^{x-1}=(1^2-2(1)+1)^{1-1}=0^0$

$$\ln f(x)=(x-1)\ln(x^2-2x+1)=\frac{\ln(x^2-2x+1)}{1/(x-1)}$$

$$\lim_{x\to 1}\frac{\ln(x^2-2x+1)}{\frac{1}{x-1}}=\lim_{x\to 1}\frac{\frac{2}{x-1}}{-\frac{1}{(x-1)^2}}=\lim_{x\to 1}\left(\frac{-2(x-1)^2}{x-1}\right)=0$$

Therefore:

$$\lim_{x\to 1}(x^2-2x+1)^{x-1}=\lim_{x\to 1}f(x)=\lim_{x\to 1}e^{\ln f(x)}=e^0=1$$

24. $\lim_{x\to 0^+} (\sin x)^x = 0^0$

$\ln f(x) = \ln[(\sin x)^x] = x\ln(\sin x) = \dfrac{\ln(\sin x)}{\frac{1}{x}}$

$$\begin{aligned}\lim_{x\to 0^+} \frac{\ln(\sin x)}{\frac{1}{x}} &= \lim_{x\to 0^+}\left(\frac{\frac{1}{\tan x}}{-\frac{1}{x^2}}\right)\\ &= \lim_{x\to 0^+}\left(-\frac{x^2}{\tan x}\right)\\ &= \lim_{x\to 0^+}\left(-\frac{2x}{\sec^2 x}\right)\\ &= -\frac{0}{1}\\ &= 0\end{aligned}$$

Therefore:

$$\begin{aligned}\lim_{x\to 0^+} (\sin x)^x &= \lim_{x\to 0^+} f(x)\\ &= \lim_{x\to 0^+} e^{\ln f(x)}\\ &= e^0\\ &= 1\end{aligned}$$

25. $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x = (1+\infty)^0 = \infty^0$

$\ln f(x) = x\ln\left(1+\dfrac{1}{x}\right) = \dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

$$\begin{aligned}\lim_{x\to 0^+} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} &= \lim_{x\to 0^+}\left(\frac{\frac{-1}{x(x+1)}}{-\frac{1}{x^2}}\right)\\ &= \lim_{x\to 0^+}\frac{x^2}{x(x+1)}\\ &= \lim_{x\to 0^+}\frac{x}{x+1}\\ &= 0.\end{aligned}$$

Therefore:

$$\begin{aligned}\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x &= \lim_{x\to 0^+} f(x)\\ &= \lim_{x\to 0^+} e^{\ln f(x)}\\ &= e^0\\ &= 1\end{aligned}$$

26. $\lim_{x\to\infty} (\ln x)^{1/x} = \infty^0$

$\ln f(x) = \ln[(\ln x)^{1/x}] = \frac{\ln(\ln x)}{x}$

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{x} = \lim_{x\to\infty}\left(\frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1}\right) = \lim_{x\to\infty}\left(\frac{1}{x\ln x}\right) = 0.$$

Therefore:

$$\lim_{x\to\infty} (\ln x)^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$$

27. (a)

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.1513	0.2303	0.0345	0.00461	0.00058

Estimate the limit to be 0.

(b) $\lim_{x\to\infty} \frac{\ln x^5}{x} = \lim_{x\to\infty} \frac{5\ln x}{x} = \lim_{x\to\infty} \frac{\frac{5}{x}}{1} = \frac{0}{1} = 0$

28. (a)

x	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$f(x)$	0.1585	0.1666	0.1667	0.1667	0.1667

Estimate the limit to be $\frac{1}{6}$.

(b) $$\lim_{x\to 0^+} \frac{x-\sin x}{x^3} = \lim_{x\to 0^+} \frac{1-\cos x}{3x^2} = \lim_{x\to 0^+} \frac{\sin x}{6x} = \lim_{x\to 0^+} \frac{\cos x}{6} = \frac{1}{6}$$

29. Let $f(\theta) = \frac{\sin 3\theta}{\sin 4\theta}$.

θ	$\pm 10^0$	$\pm 10^{-1}$	$\pm 10^{-2}$	$\pm 10^{-3}$	$\pm 10^{-4}$
$f(\theta)$	−0.1865	0.7589	0.7501	0.7500	0.7500

Estimate the limit to be $=\frac{3}{4}$.

$$\lim_{\theta\to 0}\frac{\sin 3\theta}{\sin 4\theta}=\lim_{\theta\to 0}\frac{3\cos 3\theta}{4\cos 4\theta}=\frac{3}{4}$$

30. Let $f(t)=\frac{1}{\sin t}-\frac{1}{t}=\frac{t-\sin t}{t\sin t}$.

t	$\pm 10^0$	$\pm 10^{-1}$	$\pm 10^{-2}$	$\pm 10^{-3}$
$f(t)$	±0.1884	±0.0167	±0.0017	±0.00017

Estimate the limit to be 0.

$$\begin{aligned}\lim_{t\to 0}\left(\frac{1}{\sin t}-\frac{1}{t}\right)&=\lim_{t\to 0}\frac{t-\sin t}{t\sin t}\\&=\lim_{t\to 0}\frac{1-\cos t}{t\cos t+\sin t}\\&=\lim_{t\to 0}\frac{\sin t}{-t\sin t+\cos t+\cos t}\\&=0\end{aligned}$$

31. Let $f(x)=(1+x)^{1/x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.271	1.0472	1.0069	1.0009	1.0001

Estimate the limit to be 1.

$$\ln f(x)=\frac{\ln(1+x)}{x}$$

$$\lim_{x\to\infty}\frac{\ln(1+x)}{x}=\lim_{x\to\infty}\frac{\frac{1}{1+x}}{1}=\frac{0}{1}=0$$

$$\lim_{x\to\infty}(1+x)^{1/x}=\lim_{x\to\infty}f(x)=\lim_{x\to\infty}e^{\ln f(x)}=e^0=1$$

32. Let $f(x)=\frac{x-2x^2}{3x^2+5x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	–0.5429	–0.6525	–0.6652	–0.6665	–0.6667

Estimate the limit to be $-\frac{2}{3}$.

$$\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5x}=\lim_{x\to\infty}\frac{1-4x}{6x+5}=\lim_{x\to\infty}-\frac{4}{6}=-\frac{2}{3}.$$

33. $\lim_{\theta\to 0}\frac{\sin\theta^2}{\theta}=\lim_{\theta\to 0}\frac{2\theta\cos\theta^2}{1}$
$=(2)(0)\cos(0)^2$
$=0$

34. $\lim_{t\to 1}\frac{t-1}{\ln t-\sin\pi t}=\lim_{t\to 1}\frac{1}{\frac{1}{t}-\pi\cos\pi t}$
$=\frac{1}{1-\pi(-1)}$
$=\frac{1}{\pi+1}$

35. $\lim_{x\to\infty}\frac{\log_2 x}{\log_3(x+3)}=\lim_{x\to\infty}\frac{\frac{1}{x\ln 2}}{\frac{1}{(x+3)\ln 3}}$
$=\lim_{x\to\infty}\frac{(x+3)\ln 3}{x\ln 2}$
$=\lim_{x\to\infty}\frac{x\ln 3+3\ln 3}{x\ln 2}$
$=\lim_{x\to\infty}\frac{\ln 3}{\ln 2}$
$=\frac{\ln 3}{\ln 2}$

36. $\lim_{y\to 0^+}\frac{\ln(y^2+2y)}{\ln y}=\lim_{y\to 0^+}\frac{\frac{2y+2}{y^2+2y}}{\frac{1}{y}}$
$=\lim_{y\to 0^+}\frac{y(2y+2)}{y^2+2y}$
$=\lim_{y\to 0^+}\frac{(2y^2+2y)}{y^2+2y}$
$=\lim_{y\to 0^+}\frac{4y+2}{2y+2}$
$=\frac{4(0)+2}{2(0)+2}$
$=\frac{2}{2}$
$=1$

37. $\lim_{y\to\pi/2}\left(\frac{\pi}{2}-y\right)\tan y$
$=\lim_{y\to\pi/2}\frac{\left(\frac{\pi}{2}-y\right)\sin y}{\cos y}$
$=\lim_{y\to\pi/2}\frac{\left(\frac{\pi}{2}-y\right)\cos y+(-1)\sin y}{-\sin y}$
$=\frac{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)\cos\frac{\pi}{2}+(-1)\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}}$
$=\frac{(-1)(1)}{-(1)}$
$=1$

38. $\lim_{x\to 0^+}(\ln x-\ln\sin x)=\lim_{x\to 0^+}\ln\frac{x}{\sin x}$

Let $f(x)=\frac{x}{\sin x}$.

$\lim_{x\to 0^+}\frac{x}{\sin x}=\lim_{x\to 0^+}\frac{1}{\cos x}=1$. Therefore,
$\lim_{x\to 0^+}(\ln x-\ln\sin x)=\lim_{x\to 0^+}\ln f(x)=\ln 1=0$

39. This problem does not require L'Hospital's Rule.

$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sqrt{x}}\right)=\lim_{x\to 0^+}\frac{1-\sqrt{x}}{x}=\infty$

40. The limit leads to the indeterminate form ∞^0.

Let $f(x)=\left(\frac{1}{x^2}\right)^x$.

$\ln\left(\frac{1}{x^2}\right)^x=x\ln\left(\frac{1}{x^2}\right)=\frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}}$

$\lim_{x\to 0}\frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}}=\lim_{x\to 0}\frac{\frac{\frac{-2}{x^3}}{\frac{1}{x^2}}}{-\frac{1}{x^2}}=\lim_{x\to 0}2x=0$

$\lim_{x\to 0}\left(\frac{1}{x^2}\right)^x=\lim_{x\to 0}e^{\ln f(x)}=e^0=1$

41. $\lim_{x\to\pm\infty}\frac{3x-5}{2x^2-x+2}=\lim_{x\to\pm\infty}\frac{3}{4x-1}=0$

42. $\lim_{x\to 0}\frac{\sin 7x}{\tan 11x}=\lim_{x\to 0}\frac{7\cos 7x}{11\sec^2 11x}=\frac{7}{11}$

43. The limit leads to the indeterminate form ∞^0.

Let $f(x) = (1+2x)^{1/(2\ln x)}$.

$$\ln(1+2x)^{1/(2\ln x)} = \frac{\ln(1+2x)}{2\ln x}$$

$$\lim_{x\to\infty}\frac{\ln(1+2x)}{2\ln x} = \lim_{x\to\infty}\frac{\frac{2}{1+2x}}{\frac{2}{x}} = \lim_{x\to\infty}\frac{x}{1+2x} = \lim_{x\to\infty}\frac{1}{2} = \frac{1}{2}$$

$$\lim_{x\to\infty}(1+2x)^{1/(2\ln x)} = \lim_{x\to\infty} e^{\ln f(x)} = e^{1/2} = \sqrt{e}$$

44. The limit leads to the indeterminate form 0^0.

Let $f(x) = (\cos x)^{\cos x}$.

$$\ln(\cos x)^{\cos x} = (\cos x)\ln(\cos x) = \frac{\ln(\cos x)}{\sec x}$$

$$\lim_{x\to\pi/2^-}\frac{\ln(\cos x)}{\sec x} = \lim_{x\to\pi/2^-}\frac{\frac{-\sin x}{\cos x}}{\sec x\tan x} = \lim_{x\to\pi/2^-}\frac{-\tan x}{\sec x\tan x} = \lim_{x\to\pi/2^-} -\cos x = 0$$

$$\lim_{x\to\frac{\pi}{2}^-} = (\cos x)^{\cos x} = \lim_{x=\frac{\pi}{2}^-} e^{\ln f(x)} = e^0 = 1$$

45. The limit leads to the indeterminate form 1^∞.

Let $f(x) = (1+x)^{1/x}$.

$$\ln(1+x)^{1/x} = \frac{\ln(1+x)}{x}$$

$$\lim_{x\to0^+}\frac{\ln(1+x)}{x} = \lim_{x\to0^+}\frac{\frac{1}{1+x}}{1} = 1$$

$$\lim_{x\to0^+}(1+x)^{1/x} = \lim_{x\to0^+} e^{\ln f(x)} = e^1 = e$$

46. The limit leads to the indeterminate form 0^0.

Let $f(x) = (\sin x)^{\tan x}$

$$\ln(\sin x)^{\tan x} = \tan x\ln(\sin x) = \frac{\ln(\sin x)}{\cot x}$$

$$\lim_{x\to0^+}\frac{\ln(\sin x)}{\cot x} = \lim_{x\to0^+}\frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to0^+}(-\sin x\cos x) = 0$$

$$\lim_{x\to0^+}(\sin x)^{\tan x} = \lim_{x\to0^+} e^{\ln f(x)} = e^0 = 1$$

47. The limit leads to the indeterminate form $1^{-\infty}$.

Let $f(x) = x^{1/(1-x)}$.

$$\ln x^{1/(1-x)} = \frac{\ln x}{1-x}$$

$$\lim_{x\to1^+}\frac{\ln x}{1-x} = \lim_{x\to1^+}\frac{\frac{1}{x}}{-1} = -1$$

$$\lim_{x\to1^+} x^{1/(1-x)} = \lim_{x\to1^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$$

48. $$\int_x^{2x}\frac{dt}{t} = \Big[\ln|t|\Big]_x^{2x} = \ln|2x| - \ln|x| = \ln\left|\frac{2x}{x}\right|$$

$$\lim_{x\to\infty}\int_x^{2x}\frac{dt}{t} = \lim_{x\to\infty}\ln\left|\frac{2x}{x}\right| = \lim_{x\to\infty}\ln 2 = \ln 2$$

49. $$\lim_{x\to1}\frac{x^3-1}{4x^3-x-3} = \lim_{x\to1}\frac{3x^2}{12x^2-1} = \frac{3}{11}$$

50. $$\lim_{x\to\infty}\frac{2x^2+3x}{x^3+x+1} = \lim_{x\to\infty}\frac{4x+3}{3x^2+1} = \lim_{x\to\infty}\frac{4}{6x} = 0$$

51. $$\lim_{x\to1}\frac{\int_1^x \cos t\,dt}{x^2-1} = \lim_{x\to1}\frac{\sin x - \sin 1}{x^2-1} = \lim_{x\to1}\frac{\cos x}{2x} = \frac{\cos 1}{2}$$