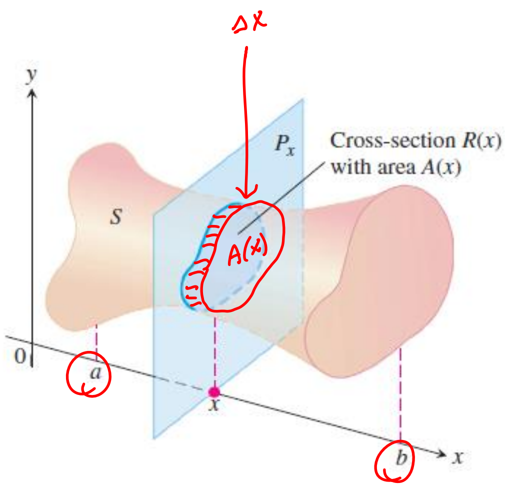
Volumes



$$V = Bh$$

$$= \sum A(x) \cdot \Delta x$$

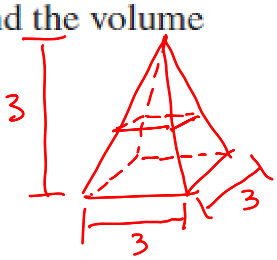
If $A(x)$ is changing with x, then

$$V = \lim_{\Delta x \to 0} \sum A(x) \cdot \Delta x$$

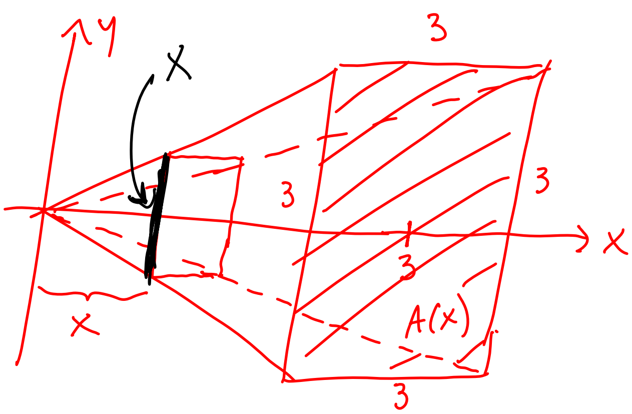
$$= \int_a^b A(x)\,dx$$

EXAMPLE 1 A Square-Based Pyramid

A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.



SOLUTION



So, $A(x) = x^2$

$$\therefore \quad V = \int_0^3 x^2\,dx$$

$$= \frac{1}{3}x^3 \Big|_0^3 = 9\,m^3$$

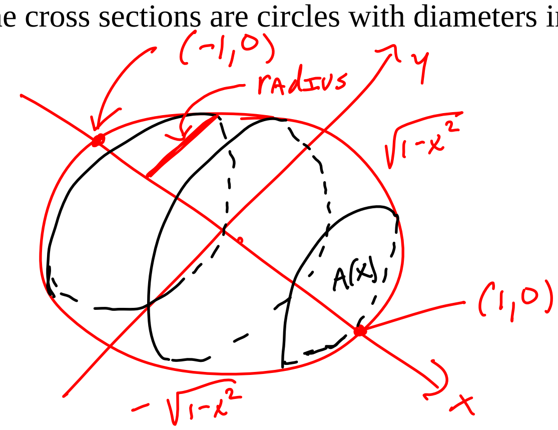
Note $B = x^2 \quad h = x$

$$\therefore \quad V = \frac{1}{3} \cdot x^2 \cdot x = \frac{1}{3}Bh$$

1. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$.

note: $y^2 = 1 - x^2$ or $x^2 + y^2 = 1$

a. The cross sections are circles with diameters in the xy-plane.



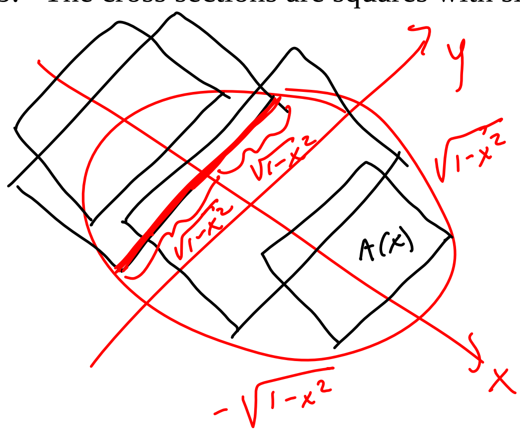
$$A(x) = \pi\left(\sqrt{1-x^2}\right)^2 = \pi(1-x^2)$$

$$V = \int_{-1}^{1} A(x)\,dx = 2\int_0^1 \pi(1-x^2)\,dx$$

$$= 2\pi\left[x - \tfrac{1}{3}x^3\right]_0^1 = 2\pi\left(\tfrac{2}{3}\right) = \tfrac{4}{3}\pi$$

note: $V = \frac{4}{3}\pi r^3$

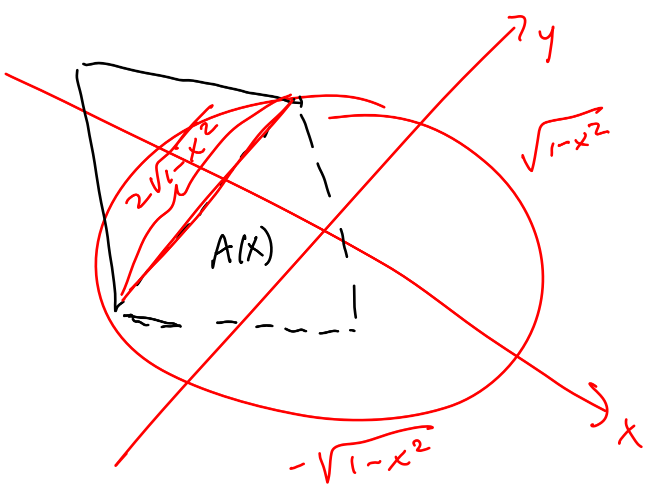
b. The cross sections are squares with sides in the xy-plane.



$$A(x) = \left(2\sqrt{1-x^2}\right)^2 = 4(1-x^2)$$

$$V = 2\int_0^1 4(1-x^2)\,dx = 8\left[x - \tfrac{1}{3}x^3\right]_0^1 = \frac{16}{3}$$

c. The cross sections are squares with diagonals in the xy-plane.



$$A(x) = \tfrac{1}{2}d_1 \cdot d_2 = \tfrac{1}{2}\left(2\sqrt{1-x^2}\right)^2$$

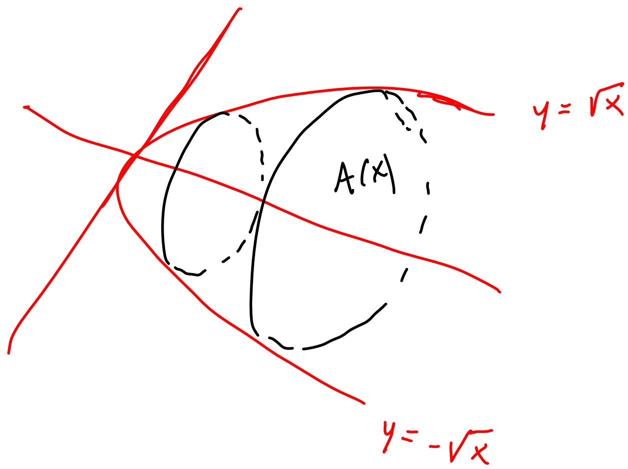
$$= \tfrac{1}{2}\cdot 4(1-x^2) = 2(1-x^2)$$

$$V = 2\int_0^1 2(1-x^2)\,dx = 4\left[x - \tfrac{1}{3}x^3\right]_0^1$$

$$= \frac{8}{3}$$

2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

a. The cross sections are circles with diameters in the xy-plane.

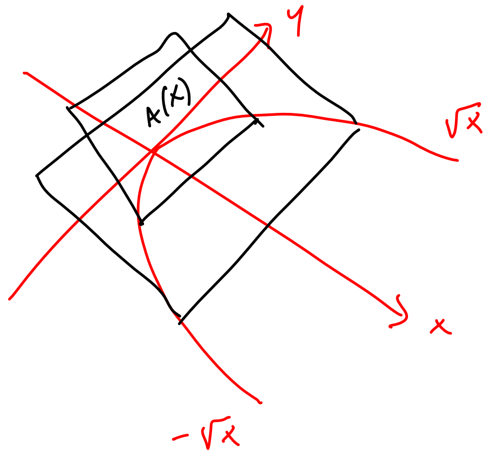


$A(x) = \pi \cdot \left(\sqrt{x}\right)^2 = \pi x^2$

$V = \int_0^4 \pi x^2 \, dx = \pi \left[\frac{1}{3}x^3\right]_0^4$

$=$

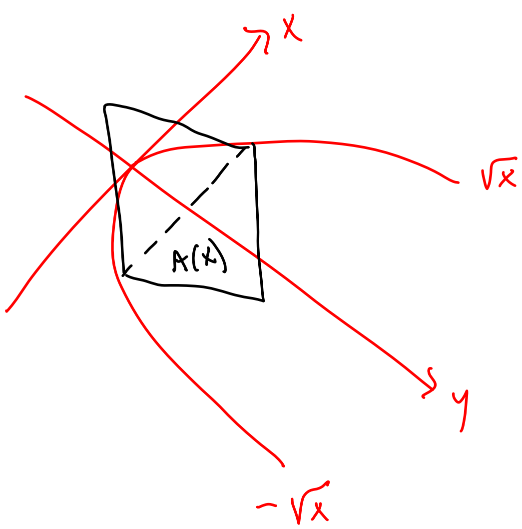
b. The cross sections are squares with sides in the xy-plane.



$A(x) = \left(2\sqrt{x}\right)^2 = 4x$

$V = \int_0^4 4x \, dx = 4\left[\frac{1}{2}x^2\right]_0^4 = 32$

c. The cross sections are squares with diagonals in the xy-plane.



$A(x) = \frac{1}{2}\left(2\sqrt{x}\right)^2 = 2x$

$V = \int_0^4 2x \, dx = 2\left[\frac{1}{2}x^2\right]_0^4 = 16$