Honors Precalculus
Matrices 7.2

Matrix: is a rectangular array of numbers. We use them as an efficient way to solve systems of linear equations and collect data.

DEFINITION Matrix

Let m and n be positive integers. An **$m \times n$ matrix** (read "m by n matrix") is a rectangular array of m rows and n columns of real numbers.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We also use the shorthand notation $[a_{ij}]$ for this matrix.

Example: Identify the element specified for the following matrix. $\begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -7 \\ -3 & 4 & 8 \end{bmatrix}$

a. a_{32} = 4 b. a_{13} = 5 c. a_{21} = 0

The **order** of a matrix is denoted by m x n (rows x columns). If m=n then the matrix is said to be a **square** matrix.

Example: Determine the order of each matrix. Identify any square matrices.

a. $\begin{bmatrix} 1 & 4 & 6 \\ -2 & 5 & 8 \end{bmatrix}$

2 x 3

b. $\begin{bmatrix} -3 & 4 & 0 & 7 \\ 1 & 9 & -10 & 3 \\ 7 & 3 & -2 & -1 \\ 0 & 6 & 4 & 0 \end{bmatrix}$

4 x 4

c. $\begin{bmatrix} 8 & -1 \end{bmatrix}$

1 x 2

DEFINITION Matrix Addition and Matrix Subtraction

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of order $m \times n$.

1. The **sum $A + B$** is the $m \times n$ matrix

$$A + B = [a_{ij} + b_{ij}].$$

2. The **difference $A - B$** is the $m \times n$ matrix

$$A - B = [a_{ij} - b_{ij}].$$

Example : $[A] = \begin{bmatrix} 5 & 0 \\ -9 & 8 \\ 2 & 7 \end{bmatrix}$ $[B] = \begin{bmatrix} -3 & 1 \\ 0 & -4 \\ 2 & 6 \end{bmatrix}$ Find:

a. $[A]+[B]$

$$\begin{bmatrix} 2 & 1 \\ -9 & 4 \\ 4 & 13 \end{bmatrix}$$

b. $[B]-[A]$

$$\begin{bmatrix} -8 & 1 \\ 9 & -12 \\ 0 & -1 \end{bmatrix}$$

c. $[A]-2[B]$

$$\begin{bmatrix} 11 & -2 \\ -9 & 16 \\ -2 & -5 \end{bmatrix}$$

Matrices inherit many properties possessed by the real numbers. Let $A = [a_{ij}]$ be any $m \times n$ matrix. The $m \times n$ matrix $O = [0]$ consisting entirely of zeros is the **zero matrix** because $A + O = A$. In other words, O is the **additive identity** for the set of all $m \times n$ matrices. The $m \times n$ matrix $B = [-a_{ij}]$ consisting of the *additive inverses* of the entries of A is the **additive inverse of A** because $A + B = O$. We also write $B = -A$. Just as with real numbers,

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

DEFINITION **Matrix Multiplication**

Let $A = [a_{ij}]$ be an $m \times r$ matrix and $B = [b_{ij}]$ an $r \times n$ matrix. The **product** $AB = [c_{ij}]$ is the $m \times n$ matrix where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$.

The key to understanding how to form the product of any two matrices is to first consider the product of a $1 \times r$ matrix $A = [a_{1j}]$ with an $r \times 1$ matrix $B = [b_{j1}]$. According to the definition, $AB = [c_{11}]$ is the 1×1 matrix where $c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1r}b_{r1}$. For example, the product AB of the 1×3 matrix A and the 3×1 matrix B, where

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

is

$$A \cdot B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = [1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6] = [32].$$

Then, the ij-entry of the product AB of an $m \times r$ matrix with an $r \times n$ matrix is the product of the ith row of A, considered as a $1 \times r$ matrix, with the jth column of B, considered as a $r \times 1$ matrix, as illustrated in Example 4.

Example: Find the product, if possible, of the two given matrices.

a. $\begin{bmatrix} -1 & 6 \\ 8 & 2 \\ 3 & -2 \end{bmatrix} \bullet \begin{bmatrix} 9 & -1 \\ 2 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 3 & 2 \\ -8 & 0 & 2 \\ 4 & -5 & 6 \end{bmatrix} \bullet \begin{bmatrix} 3 & 4 \\ -5 & 6 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Question: Is matrix multiplication commutative?

The $n \times n$ matrix I_n with 1's on the main diagonal (upper left to lower right) and 0's elsewhere is the **identity matrix of order $n \times n$**

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

DEFINITION Inverse of a Square Matrix

Let $A = [a_{ij}]$ be an $n \times n$ matrix. If there is a matrix B such that

$$AB = BA = I_n,$$

then B is the **inverse** of A. We write $B = A^{-1}$ (read "A inverse").

We will see that not every square matrix (Example 7) has an inverse. If a square matrix A has an inverse, then A is **nonsingular**. If A has no inverse, then A is **singular**.

Example: (using your calculator) Verify the given matrices are inverses.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ -1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0.1 & 0.2 & -0.5 \\ 0.5 & 0 & 0.5 \\ 0.1 & 0.2 & 0.5 \end{bmatrix}$$

Inverse of a 2 × 2 Matrix

If $ad - bc \neq 0$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The number $ad - bc$ is the **determinant** of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and is denoted

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Example: (without a calculator) Find the inverse of the given matrix.

$$\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

Properties of Matrices

Let A, B, and C be matrices whose orders are such that the following sums, differences, and products are defined.

1. Commutative property

Addition:
$A + B = B + A$
Multiplication:
(Does not hold in general)

2. Associative property

Addition:
$(A + B) + C = A + (B + C)$
Multiplication:
$(AB)C = A(BC)$

3. Identity property

Addition: $A + O = A$
Multiplication: order of $A = n \times n$
$A \cdot I_n = I_n \cdot A = A$

4. Inverse property

Addition: $A + (-A) = O$
Multiplication: order of $A = n \times n$
$AA^{-1} = A^{-1}A = I_n, \quad |A| \neq 0$

5. Distributive property

Multiplication over addition
$A(B + C) = AB + AC$
$(A + B)C = AC + BC$

Multiplication over subtraction
$A(B - C) = AB - AC$
$(A - B)C = AC - BC$