6.6 Day 1 — DeMoivre's Theorem & n^{th} Roots

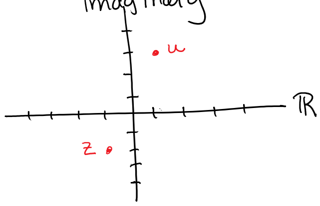
Tuesday, May 14, 2019 9:06 AM

Complex Numbers $= a+bi$

Reals — Imaginary

Complex plane

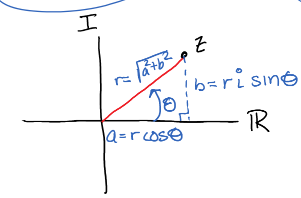
$u = 1+3i \qquad z = -1-2i$



Absolute value (modulus) of a complex number:

$z = a+bi$ (rectangular form = standard form)

θ = directional angle (arguement)



Trig form or Polar form

$z = r(\cos\theta + i\sin\theta)$

shorthand: $r\,\text{cis}\,\theta$

Ex: Find the trig form of a complex number where the arguement satisfies $0 \le \theta \le 2\pi$.

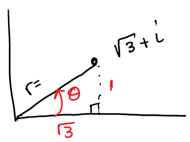
a. $\sqrt{3} + i \qquad a = \sqrt{3} \qquad b = 1$

$r = \sqrt{\sqrt{3}^2 + 1^2} = 2$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$

$\dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$ or $\dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$



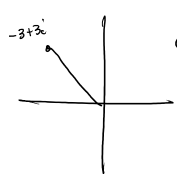
$2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$2\,\text{cis}\,\frac{\pi}{6}$

b. you try..... $-3+3i$

$r = 3\sqrt{2}$

$\theta = \frac{3\pi}{4}$



$\boxed{3\sqrt{2}\,\text{cis}\frac{3\pi}{4}}$

Product & Quotient of Complex Numbers:

Product and Quotient of Complex Numbers:

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ $\quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 \cdot z_2 = r_1 \cdot r_2 \left[(\cos\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) + (\cos\theta_1 \cdot i\sin\theta_2 + i\sin\theta_1\cos\theta_2) \right]$

$= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \right]$

$= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i(\sin\theta_1 + \theta_2) \right]$

$$\boxed{z_1 \cdot z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)}$$

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) \quad \text{where } z_2 \neq 0}$$

Example Find $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$

$z_1 = 1 - i$ $\quad z_2 = \sqrt{3} + i$

$z_1 = \sqrt{2}\operatorname{cis}\frac{7\pi}{4}$ $\quad z_2 = 2\operatorname{cis}\frac{\pi}{6}$

$z_1 \cdot z_2 = 2\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{4} + \frac{\pi}{6}\right) = 2\sqrt{2}\operatorname{cis}\frac{23\pi}{12}$ exact

standard form:

approx:

$\left(2\sqrt{2}\cos\frac{23\pi}{12}\right) + \left(2\sqrt{2}\sin\frac{23\pi}{12}\right)i$

$\boxed{2.73 - 0.73i}$

exact: standard form

$\underset{z_1}{(1-i)} \cdot \underset{z_2}{(\sqrt{3}+i)}$

$\sqrt{3} + i - i\sqrt{3} - i^2$

$\sqrt{3} + i - i\sqrt{3} + 1$

$\boxed{(\sqrt{3}+1) + i(1-\sqrt{3})}$

$\frac{z_1}{z_2} = \underbrace{\frac{\sqrt{2}}{2}\operatorname{cis}\frac{19\pi}{12}}_{\text{Trig Form}}$

standard

$0.18 + 0.68i$

Powers of complex numbers:

$z = r(\cos\theta + i\sin\theta)$ then $z^2 = z \cdot z$

$z^2 = r^2 \operatorname{cis}(\theta + \theta) = r^2 \operatorname{cis}(2\theta)$

In general - DeMoivre's Theorem:

$z = r(\cos\theta + i\sin\theta)$

$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) = \underline{r^n \operatorname{cis}(n\theta)}$

Simplify & express in rectangular form:

$$\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^5$$

$2^5 \operatorname{cis}\left(5 \cdot \frac{\pi}{3}\right) = \underline{32 \operatorname{cis} \frac{5\pi}{3}}$ (Trig or Polar Form)

$32\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$32\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right]$

$32\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$ (standard $a+bi$)

$\rightarrow 16 - 16i\sqrt{3}$