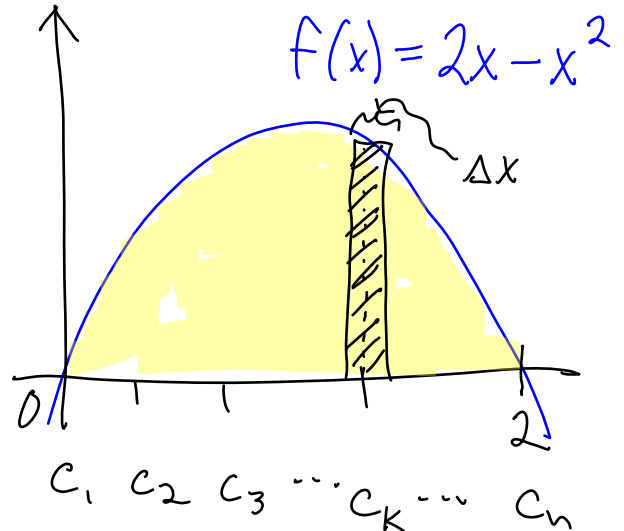
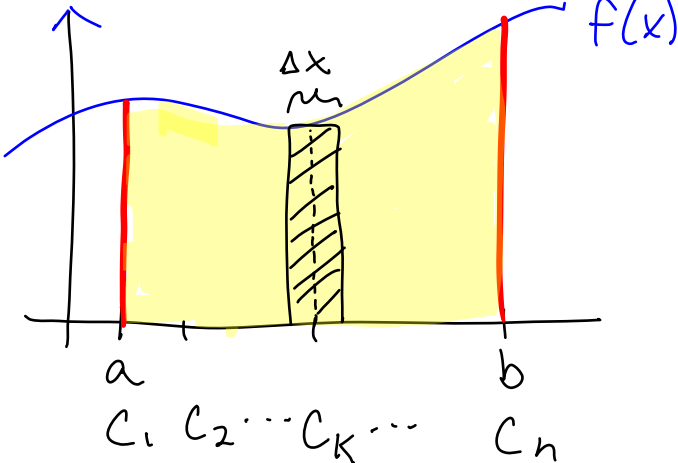


Let's sketch 1 generic function and 1 specific function:



What does each of these represent?

1. $f(c_k) \cdot \Delta x$
 = height · base
 = area of 1 rect.

4. $(2c_k - (c_k)^2) \cdot \Delta x$
 = height · base
 = area of 1 rect.

2. $\sum_{k=1}^n f(c_k) \cdot \Delta x$
 sum area of rect.
 = total area of all "n" rectangles

5. $\sum_{k=1}^n (2c_k - (c_k)^2) \cdot \Delta x$
 sum area of 1 rect.
 = total area of "n" rectangles

3. $\lim_{\Delta x \rightarrow 0} \sum_{k=1}^n f(c_k) \cdot \Delta x$

6. $\lim_{n \rightarrow \infty} \sum_{k=1}^n (2c_k - (c_k)^2) \cdot \Delta x$
 (see # 5)

- base gets infinitesimally thin
- $n \rightarrow \infty$ # of rects.
- gives exact area

$= \int_a^b f(x) dx$
 sum height · base

$= \int_0^2 (2x - x^2) dx$

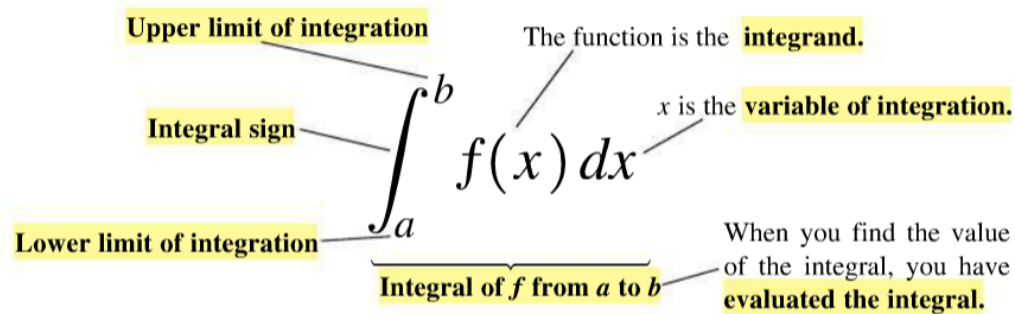
The Definite Integral of a Continuous Function on $[a, b]$

Let f be continuous on $[a, b]$, and let $[a, b]$ be partitioned into n subintervals of equal length $\Delta x = (b - a)/n$. Then the definite integral of f over $[a, b]$ is given by

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x,$$

where each c_k is chosen arbitrarily in the k^{th} subinterval.

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x = \int_a^b f(x) dx.$$



The value of the definite integral of a function over any particular interval depends on the function and not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x , we simply write the integral as

$$\int_a^b f(t) dt \quad \text{or} \quad \int_a^b f(u) du \quad \text{instead of} \quad \int_a^b f(x) dx.$$

No matter how we represent the integral, it is the same *number*, defined as a limit of Riemann sums. Since it does not matter what letter we use to run from a to b , the variable of integration is called a **dummy variable**.

A Definition of the Integral Based on RRAM

Name _____

To find $\int_a^b f(x)dx$, the "area" between the x-axis and a curve $f(x)$ on an interval $[a,b]$ we can first divide the area between the x-axis and the curve $f(x)$ into "rectangles", each of width $\Delta x = \frac{b-a}{n}$ and "height" $f(c_k)$, where $c_k = a + k\Delta x$. We can then find the area under the curve by multiplying the width $\Delta x = \frac{b-a}{n}$ of each rectangle by its height $f(c_k)$ and adding/summing the areas of all the rectangles as we let the number of rectangles approach ∞ .

In other words, $\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \cdot \Delta x$ where $\Delta x = \frac{b-a}{n}$ and $c_k = a + k\Delta x$

Examples: Write the following integrals using the limit definition above.

RRAM

$n = \text{number of intervals}$

1. $\int_0^5 \cos x dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \cos\left(\frac{5k}{n}\right) \cdot \frac{5}{n}$

$b-a = 5-0 = 5$

$\Delta x = \frac{5}{n}$

$c_k = 0 + k\left(\frac{5}{n}\right) = \frac{5k}{n}$

2. $\int_1^5 \cos x dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \cos\left(1 + \frac{4k}{n}\right) \cdot \frac{4}{n}$

$b-a = 4$

$\Delta x = \frac{4}{n}$

$c_k = 1 + k\left(\frac{4}{n}\right) = 1 + \frac{4k}{n}$

3. $\int_0^7 x^3 dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{7k}{n}\right)^3 \cdot \frac{7}{n}$

$b-a = 7$

$\Delta x = 7/n$

$c_k = 0 + 7\frac{k}{n} = \frac{7k}{n}$

4. $\int_1^6 x^3 dx = \lim_{n \rightarrow \infty} \left(1 + \frac{5k}{n}\right)^3 \cdot \frac{5}{n}$

$b-a = 5$

$\Delta x = \frac{5}{n}$

$c_k = 1 + \frac{5k}{n}$

5. $\int_{-4}^4 x^3 dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(-4 + \frac{8k}{n}\right)^3 \cdot \frac{8}{n}$

$b-a = 4 - (-4) = 8$

$\Delta x = 8/n$

$c_k = -4 + \frac{8k}{n}$

Examples: Write the limit as a definite integral.

$$1) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{5}{n} \ln \left(2 + \frac{5k}{n} \right) = \int_2^7 \ln x \, dx$$

$\Delta x = \frac{b-a}{n} = \frac{5}{n}$ $b-a=5$ $a=2$ $b=7$ $f(x) = \ln x$
 $b=5+a$

$$2) \lim_{n \rightarrow \infty} \frac{\pi}{3n} \sum_{k=1}^n \tan \left(\frac{k\pi}{3n} \right) = \int_0^{\pi/3} \tan x \, dx$$

$\frac{\pi}{3n} = \frac{\pi/3}{n}$ $a + \frac{b-a}{n} = \frac{\pi/3}{n}$ $f(x) = \tan x$
 $\frac{b-a}{n} = \frac{\pi/3}{n}$ $a=0$ $b=\pi/3$

Homework

(1-2) Multiple Choice

$$1. \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \dots + \sqrt{\frac{n}{n}} \right] = \int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f \left(a + k \cdot \frac{b-a}{n} \right) \cdot \frac{b-a}{n}$$

(A) $\frac{1}{2} \int_0^1 \frac{1}{\sqrt{x}} \, dx$

(B) $\int_0^1 \sqrt{x} \, dx$

(C) $\int_0^1 x \, dx$

1. $\frac{b-a}{n} = \frac{1}{n}$

(D) $\int_1^2 x \, dx$

(E) $2 \int_1^2 x \sqrt{x} \, dx$

2. $a + \frac{b-a}{n} = \frac{1}{n}$

$a=0$ $b=1$

$f(x) = \sqrt{x}$

2. The expression $\frac{1}{50} \left(\sqrt{\frac{1}{50}} + \sqrt{\frac{2}{50}} + \sqrt{\frac{3}{50}} + \dots + \sqrt{\frac{50}{50}} \right)$ is a Riemann sum Approximation for

(A) $\int_0^1 \sqrt{\frac{x}{50}} \, dx$

(B) $\int_0^1 \sqrt{x} \, dx$

(C) $\frac{1}{50} \int_0^1 \sqrt{\frac{x}{50}} \, dx$

$\frac{b-a}{n} = \frac{1}{50}$

(D) $\frac{1}{50} \int_0^1 \sqrt{x} \, dx$

(E) $\frac{1}{50} \int_0^{50} \sqrt{x} \, dx$

$a + \frac{b-a}{n} = \frac{1}{50}$ $a=0$ $b=1$

c. $f(x) = \sqrt{x}$

(3-6) Rewrite the given limit as a definite integral.

$$3. \lim_{n \rightarrow \infty} \left(\frac{1}{n} \left(\left(9 + \frac{1}{n}\right)^2 + \left(9 + \frac{2}{n}\right)^2 + \left(9 + \frac{3}{n}\right)^2 + \left(9 + \frac{4}{n}\right)^2 + \dots + \left(9 + \frac{n}{n}\right)^2 \right) \right)$$

$$a) \quad \frac{b-a}{n} = \frac{1}{n}$$

$$b. \quad \frac{b-a}{n} = 9 + \frac{1}{n}$$

$$a = 9$$

$$b = 10$$

$$f(x) = x^2$$

$$= \int_9^{10} x^2 dx$$

$$4. \lim_{n \rightarrow \infty} \left(\frac{7}{n} \left(\sqrt[3]{-2 + \frac{7}{n}} + \sqrt[3]{-2 + \frac{14}{n}} + \sqrt[3]{-2 + \frac{21}{n}} + \sqrt[3]{-2 + \frac{28}{n}} + \dots + \sqrt[3]{-2 + \frac{7n}{n}} \right) \right)$$

$$\frac{b-a}{n} = \frac{7}{n}$$

$$f(x) = \sqrt[3]{x}$$

$$= \int_{-2}^5 \sqrt[3]{x} dx$$

$$b-a=7$$

$$a=-2 \quad b=5$$

$$5. \lim_{n \rightarrow \infty} \left(\frac{8}{n} \sum_{k=1}^n \left(4 + \frac{8k}{n} \right)^3 \right)$$

$$\frac{b-a}{n} = \frac{8}{n}$$

$$a=4$$

$$f(x) = x^3$$

$$b-a=8$$

$$b=12$$

$$= \int_4^{12} x^3 dx$$

$$6. \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{9}{n} \left(\frac{9k}{n} \right)^2 \right) = \int_0^9 x^2 dx$$

$$\Delta x = \frac{b-a}{n} = \frac{9}{n}$$

$$b-a=9 \quad f(x) = x^2$$

$$a=0 \quad b=9$$

