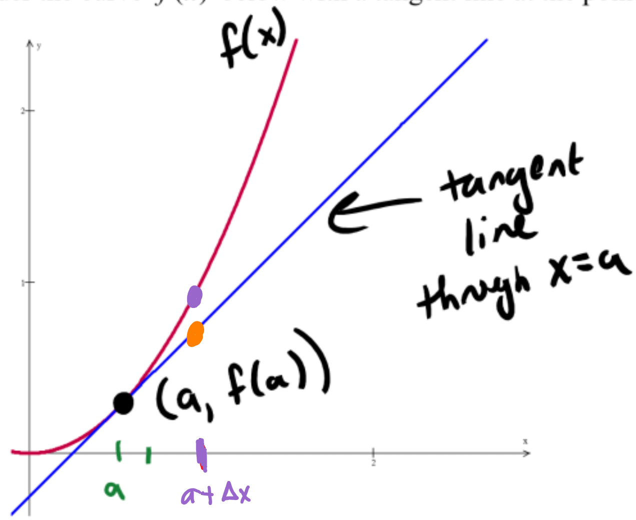
# 5.5 Day 1 (Friday 11/15)

Thursday, November 14, 2019 1:57 PM

## Section 5.5A Notes – Linearization

One of the neatest ideas that Newton developed when he came up with the idea of the Calculus is what you'll be finding out about today. Imagine a world without calculators, and how frustrating must it have been to find a root without a calculator. For instance, how hard was it to calculate $\sqrt[3]{124}$ which is not a perfect cube? What if you needed an approximation of that to compute a critical value for an optimization problem? There were of course formulas they used to estimate these values, but they were not easy to work with. Let's look at how we could use derivatives to make estimation easier.

Consider the curve $f(x)$ below with a tangent line at the point $(a, f(a))$:




$\Delta x = .01$

What if we wanted to estimate the value of f(x) at a point very close to x = a? Instead of plugging that decimal into a function and getting more yucky decimals, we could instead find the point close to x = a on the tangent line and use that to estimate the function value. So, in other words, we could use the tangent line to estimate values of the original function! This is called **linear approximation.**

**Definition:** If f(x) is differentiable at x = a, then the equation of the tangent line is:

$$y - f(a) = f'(a)(x - a)$$

$y - y_1 = m(x - x_1)$

This can be rewritten as:

$$L(x) = f'(a)(x - a) + f(a)$$

This is called the **linearization of f at a**, and can be used to estimate function values close to x = a on the curve!

Let's look at some examples of how this can be used.

Ex: Find the linearization of $f(x)=\sqrt{x}$ at x = 4. Use it to approximate $\sqrt{4.01}$ without a calculator.

$f(4)=\sqrt{4}=2$ $\quad$ T.o.t $(4,2)$

$m=f'(4)=\frac{1}{2}(4)^{-1/2}=\frac{1}{2}\cdot\frac{1}{2}=1/4$

$y-2=\frac{1}{4}(x-4)$

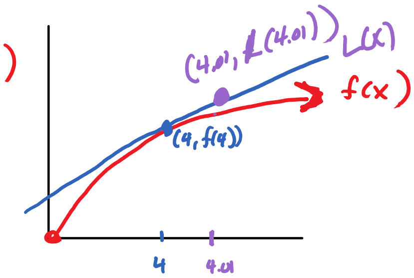
$L(x)=2+.25(x-4)$

$L(4.01)=2+.25(4.01-4)$

$=2+.25(.01)$

$=2.0025$

$f(4.01)\approx 2.0025$




Error: $|f(4.01)-L(4.01)|\approx 1.056\times10^{-6}$

Less than $10^{-5}$

Ex: Use a linearization to approximate $\sqrt{123}$.

$f(x)=\sqrt{x}$ $\quad$ $x=a=121$

$f(121)=11$ $\quad$ $(121,11)$

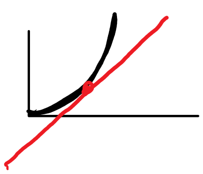
$f'(121)=\frac{1}{2}(121)^{-1/2}=\frac{1}{22}$

$L(x)=11+\frac{1}{22}(x-121)$ $\quad$ $L(123)=11+\frac{1}{22}(123-121)=11+\frac{1}{11}$ $\quad$ $11.\overline{09}$

Let's think about over and under estimates….

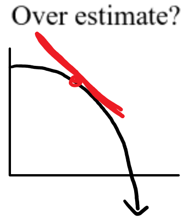
When is a linear approximation an:

Underestimate?



when $f''(a)>0$

Over estimate?



when $f''(a)<0$

Ex: Find the linearization of $f(x)=x^3-2x+3$ at x = 2. Use linear approximation to approximate $f(2.01)$. What's the error?

$f(2)=8-4+3=7$ $\quad$ $(2,7)$

$f'(2)=3(2)^2-2=10$

$y-7=10(x-2)$

$L(x)=7+10(x-2)$

$L(2.01)=7+10(2.01-2)=7.1$

error $|f(2.01)-7.1|=.0006$

Less than $10^{-3}$

$f''(x)=6x$

$f''(2)=>0$

★ 7.1 is an underestimate of $f(2.01)$ b/c $f''(2)>0$.

Would your estimate be an under or overestimate? How could we determine this?

Ex: For the function $f(x) = lnx$ @ $x=1$, approximate $f(0.9)$ using a tangent line approximation method. Is your estimate an underestimate or an over estimate. Justify your answer.

$f(1)=0$

$f'(x)=\frac{1}{x}$

$f'(1)=1$

$f(1)=\ln 1 = 0$

$y-0=1(x-1)$

$L(x)=x-1$

$L(.9)=.9-1=-.1$

$f''(x)=-\frac{1}{x^2}$

$f''(1)<0$

the estimate of $-0.1$ is an overestimate b/c $f''(1)<0$.

Ex: Let $f$ be the function given by $f(x) = sin(3x)$. What is the approximation for $f(\cancel{1.01})$ 0.35 found by using linearization of $f$ at $x = \cancel{\frac{\pi}{3}}$ $\frac{\pi}{9}$. Is this an under or overestimate? Justify your answer.

$f(\frac{\pi}{9}) = \sin(3(\frac{\pi}{9})) = \frac{\sqrt{3}}{2}$

$f'(x) = 3\cos(3x)$

$f'(\frac{\pi}{9}) = 3\cos(\frac{\pi}{3}) = 3/2$

$= 3\cos(3(\pi/9))$

$= 3\cos(\frac{\pi}{3})$

$3(1/2) = \frac{3}{2}$

$L(x) = \frac{\sqrt{3}}{2} + \frac{3}{2}(x - \frac{\pi}{9})$

$L(.35) = \frac{\sqrt{3}}{2} + \frac{3}{2}(.35 - \frac{\pi}{9}) \approx 0.867$

$f''(x) = -9\sin(3x)$

$f''(\pi/9) = -9\sin(\frac{\pi}{3}) = -\frac{9\sqrt{3}}{2} < 0$

the estimate of 0.867 is an overestimate b/c $f''(\pi/9) < 0$