5.4A Notes **Modeling and Optimization**

1. Two numbers add to 30, find the maximum product of the two numbers.

$x + y = 30$

$y = 30 - x$

$xy = P = x(30-x)$

$= -x^2 + 30x$

$P' = -2x + 30$

$-2x + 30 = 0$

$x = 15$

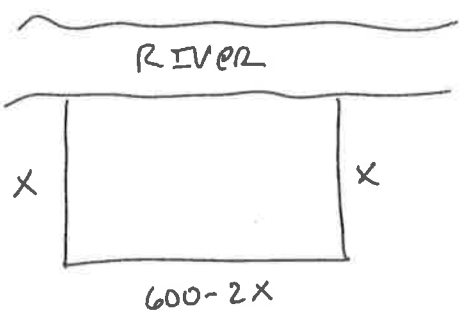
P': $+$ | $-$ at 15

$x = 15$, $y = 30 - 15 = 15$

$P_{MAX} = 15 \cdot 15 = 225$

2. A wire fence to keep horses contained is to border a river. No fencing is necessary along the river. If 600 m of fencing is available, write an equation for the area of the rectangle and find the dimensions of the rectangle that maximize the area.

$0 \le x \le 300$



A': $+$ | $-$ at 150, x

no need to check endpts. @ $x=0$ or $x = 300$

$A = x(600-2x) = -2x^2 + 600x$

$A' = -4x + 600$

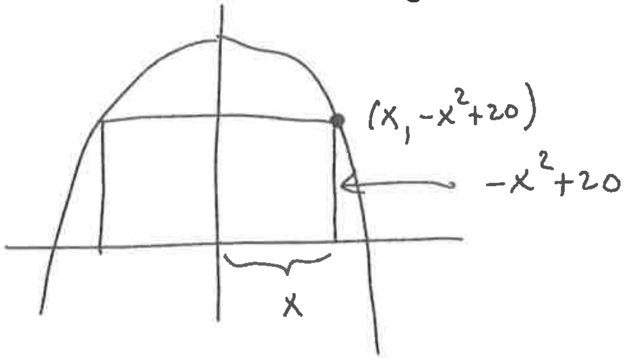
$-4x + 600 = 0$

$x = 150$

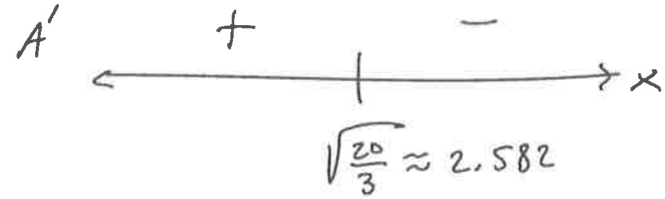
$x = 150\text{m}$, $600 - 2x = 300\text{m}$

$A_{MAX} = 150 \times 300 = 45{,}000 \text{ m}^2$

3. A rectangle has its base on the x-axis and its upper vertices on the parabola $y = -x^2 + 20$. Determine the dimensions of the rectangle with maximum area.



$0 < x < \sqrt{20}$



$A = 2x(-x^2+20) = -2x^3 + 40x$

$A' = -6x^2 + 40 = 0$

$x^2 = \frac{40}{6} = \frac{20}{3}$

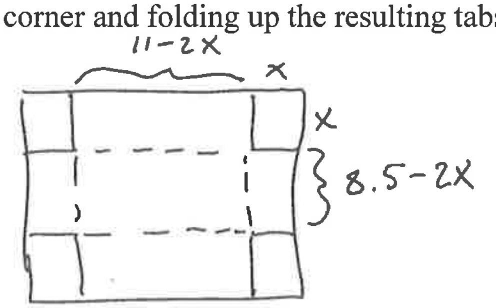
$x = \pm\sqrt{\frac{20}{3}}$

No need to check endpts.

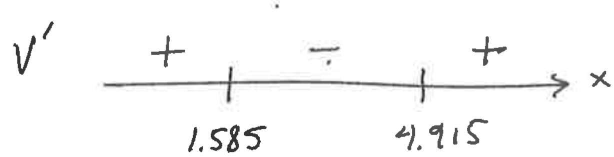
$x = \sqrt{\frac{20}{3}}$ $\quad y = -\left(\sqrt{\frac{20}{3}}\right)^2 + 20 = \frac{40}{3}$

$A_{MAX} = 2\sqrt{\frac{20}{3}} \cdot \frac{40}{3} \approx 68.853$

4. Given a sheet of cardboard, 8.5 inches by 11 inches. Create an open top box by cutting a square from each corner and folding up the resulting tabs. Find the dimensions of the box that maximize the volume.



$0 < x < 4.25$



Don't need to check endpts.

$V = x(11-2x)(8.5-2x)$

$V = (11x - 2x^2)(8.5 - 2x)$

$V = 4x^3 - 17x^2 - 22x^2 + 93.5x$

$V = 4x^3 - 39x^2 + 93.5x$

$V' = 12x^2 - 78x + 93.5$

$12x^2 - 78x + 93.5 = 0$

Solve using Calc.

$x \approx 1.585$ or 4.915

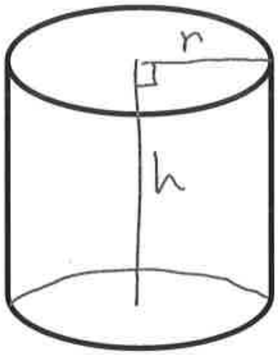
$x \approx 1.585$

$8.5 - 2x \approx 5.33$

$11 - 2x \approx 7.83$

$V_{MAX} \approx 66.148 \text{ IN}^3$

5. You have been asked to design a 1000 cubic centimeter cylindrical can. What dimensions of the can will require the least amount of material to make it? (Hint: What are we going to minimize?)



$1000 = \pi r^2 h$ $\qquad \dfrac{1000}{\pi r^2} = h$ $\qquad 0 < r < \infty$

$SA = 2\pi r h + 2\pi r^2$

$= 2\pi r \left(\dfrac{1000}{\pi r^2}\right) + 2\pi r^2$

$= \dfrac{2000}{r} + 2\pi r^2$

$SA' = \dfrac{-2000}{r^2} + 4\pi r = 0$

$4r\left(-\dfrac{500}{r^3} + \pi\right) = 0$

$r \neq 0$ or $r^3 = \dfrac{500}{\pi}$

$r = \sqrt[3]{500/\pi}$

≈ 5.419

SA' $\quad -$ $\quad +$

$\sqrt[3]{500/\pi}$

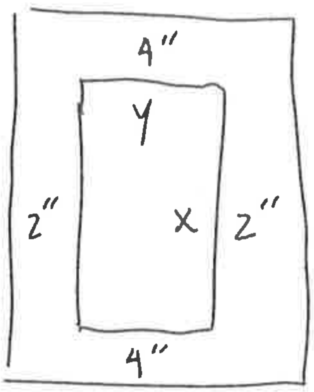
≈ 5.419

no endpts to check

$r \approx 5.419\text{cm}$ $\quad h = \dfrac{1000}{\pi r^2} \approx 10.839$ cm

5.4B Notes **More Optimization Problems!!!**

1. You are designing a rectangular poster to contain 50 square inches of printing with a 4-in margin at the top and bottom and a 2 in margin on each side. What overall dimensions will minimize the amount of paper used?



$(0 < x < \infty)$

$50 = xy$

$\frac{50}{x} = y$

$A = (x+8)(y+4)$

$= (x+8)\left(\frac{50}{x}+4\right)$

$= 50 + \frac{400}{x} + 32 + 4x$

$A' = -400x^{-2} + 4 = 0$

$x^{-2} = \frac{-4}{-400} = \frac{1}{100}$

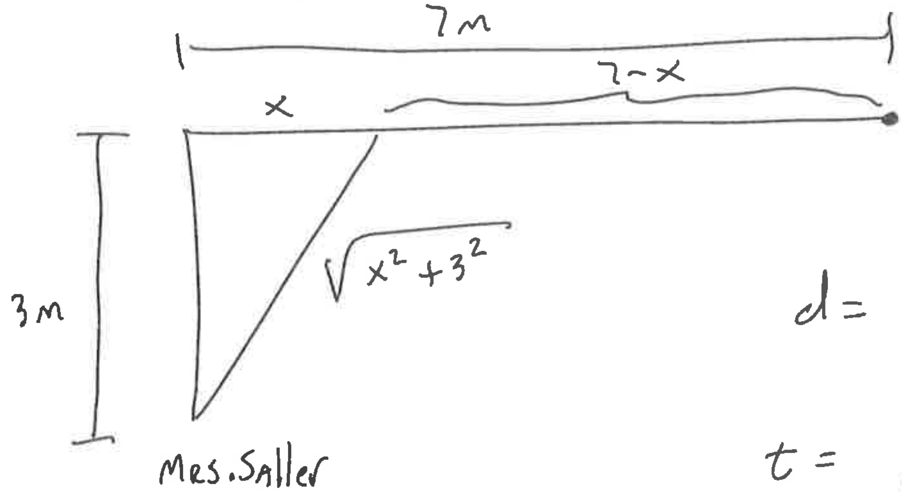
$x^2 = 100 \quad x = \pm 10$

A': $-$ | 10 | $+$

No endpts to check

$\therefore$ $\boxed{x+8 = 18''}$ $y = \frac{50}{10} = 5$ $\boxed{y+4 = 9''}$

2. Mrs. Saller is in a boat 3 miles offshore and wants to meet up with Mrs. Tyler and Ms. Orloff in a village 7 miles down a straight shoreline from the point nearest the boat. Mrs. can row 3 mph and can walk 4 mph. Where should she land the boat to reach her two colleagues in the least amount of time?



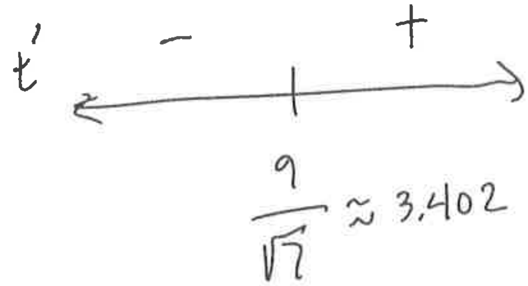
$$0 \le x \le 7$$

$$d = \sqrt{x^2+3^2} + 7 - x$$

$$t = \frac{\sqrt{x^2+3^2}}{3} + \frac{7-x}{4}$$

$$= \frac{1}{3}(x^2+9)^{1/2} + \frac{1}{4}(7-x)$$

$$t' = \frac{1}{3}\cdot\frac{1}{2}(x^2+9)^{-1/2}\cdot 2x - \frac{1}{4} = 0$$



$$\frac{x}{3\sqrt{x^2+9}} = \frac{1}{4}$$

No need to check end pts

$$3\sqrt{x^2+9} = 4x$$

$$9(x^2+9) = 16x^2$$

$$9x^2+81 = 16x^2$$

$$81 = 7x^2$$

$$\frac{81}{7} = x^2$$

$$x = \frac{9}{\sqrt{7}} \qquad \cancel{x = -\frac{9}{\sqrt{7}}}$$

$x = \frac{9}{\sqrt{7}} \approx 3.402$ miles

down shore

5.4D Notes Optimization Name *solutions*

Marginal Revenue – the rate of change of revenue

Marginal Cost – the rate of change of cost

Marginal Profit – The rate of change of profit where profit = revenue – cost

1. Suppose that Saller Inc. has a revenue given by $r(x) = 9x$ and has a cost given by $c(x) = x^3 - 6x^2 + 15x$ where x represents thousands of units of math facts produced (there are many math facts but they aren't that valuable). How many math facts should Saller Inc. produce to maximize profit?

P_{MAX}

Domain of x $[0, \infty)$

$r'(x) = 9$

$c'(x) = 3x^2 - 12x + 15$

$P'(x) = 9 - (3x^2 - 12x + 15) = 0$

$= -3x^2 + 12x - 6 = 0$

using a calculator

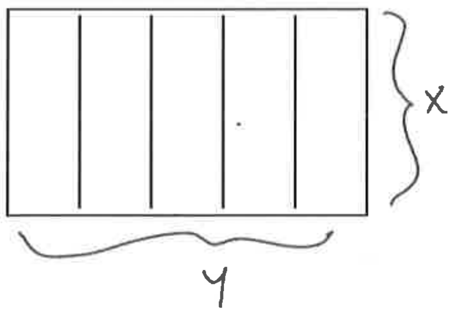
$x \approx 0.586$, $x \approx 3.414$

$P'(x)$: $-$ | $+$ | $-$ (0.586, 3.414)

3,414 math facts will maximize profit because that is when $P'(x)$ changes from + to −.

$\boxed{\text{3,414 math facts}}$

2. A small 5 room motel is to be built as shown in the sketch with two long walls y feet each and 6 short walls x feet long each. The total length of the walls is to be 300 feet. Write an equation for the area taken up by the motel. What should the dimensions of the individual rooms be to maximize the area of each room.



$6x + 2y = 300 \Rightarrow y = \frac{300 - 6x}{2}$

$= 150 - 3x$

$A = xy$

$A = x(150 - 3x) = 150x - 3x^2$

DOMAIN OF x $(0, 50)$

$A' = 150 - 6x = 0$

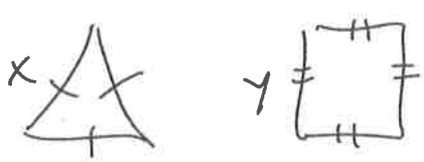
$x = \frac{150}{6} = 25$

A': $+$ | $-$ at 25

$x = 25$ MAXIMIZES The total Area of the motel because A' changes from $+$ to $-$ @ $x = 25$.

INDIVIDUAL Room DIMENSIONS ARE x by $\frac{y}{5}$ or 25 by 15

3. The sum of the perimeters of an equilateral triangle and a square is 10. Find the dimensions of the triangle and square that produce a minimum total area. Justify your answer.



$3x + 4y = 10 \Longrightarrow y = \frac{10 - 3x}{4}$

$A_{TOTAL} = \frac{x^2}{4}\sqrt{3} + y^2$

$= \frac{\sqrt{3}}{4}x^2 + \left(\frac{10 - 3x}{4}\right)^2 = \frac{\sqrt{3}}{4}x^2 + \frac{1}{16}(100 - 60x + 9x^2)$

$A'_{TOTAL} = 2\frac{\sqrt{3}}{4}x + \frac{1}{16}(-60 + 18x) = 0$

A'_{TOTAL}: $-$ | $+$ at 1.883

$x = 1.883$ MINIMIZES A_{TOTAL} because A'_{TOTAL} changes from $-$ to $+$ @ $x = 1.883$

$x \approx 1.883$ $\quad y = \frac{10 - 3x}{4} \approx 1.088$

Practice WS 5.4 **Calculator OK** Name Solutions

1. A number plus twice a second number is 108. Find the two numbers that give a maximum product. Justify your answer.

$$x + 2y = 108$$
$$y = \frac{108 - x}{2}$$
$$P = xy = x\left(\frac{108-x}{2}\right)$$
$$P = 54x - \frac{1}{2}x^2$$
$$P' = 54 - x = 0$$
$$x = 54$$

P': $+$ | $-$ at 54

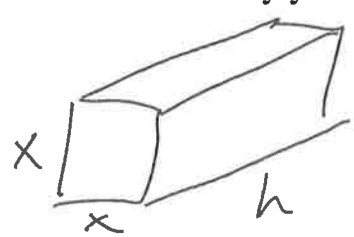
$x = 54$ IS A MAXIMUM because P' changes from $+$ to $-$ @ $x = 54$

$$y = \frac{108 - 54}{2} = 27$$

$$x = 54, \; y = 27$$

2. A rectangular solid with a square base has a surface area of 150 square inches. Find the maximum volume of the solid and its dimensions. Justify your answer.

$$SA = 2x^2 + 4xh = 150$$



$$h = \frac{150 - 2x^2}{4x}$$

$$V = x^2 \cdot h = x^2\left(\frac{150 - 2x^2}{4x}\right)$$

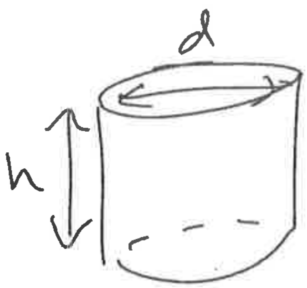
V': $+$ | $-$ at 5

$$= \frac{150}{4}x - \frac{2}{4}x^3$$
$$V' = \frac{150}{4} - \frac{6}{4}x^2 = 0$$
$$x^2 = \frac{150}{4} \cdot \frac{4}{6} = 25$$

$x = 5$ MAXIMIZES V because V' change from $+$ to $-$ @ $x = 5$. $h = 5$

$$V_{MAX} = x^2 \cdot h = 125 \text{ IN}^3$$

3. The diameter plus the height of a cylindrical package is equal to 108 inches. Find the dimensions of the package that gives you a maximum volume. Justify your answer.



$d + h = 108$

$h = 108 - d$

$V = \pi \left(\frac{d}{2}\right)^2 h$

$V = \pi \frac{d^2}{4}(108 - d)$

$V = \pi \cdot 27d^2 - \frac{\pi}{4} \cdot d^3$

$V' = 54\pi d - \frac{3}{4}\pi d^2 = 0$

$d\left(54\pi - \frac{3}{4}\pi d\right) = 0$

$d = 0$ or $d = \frac{54\pi}{\frac{3}{4}\pi} = 72$

V': $+$ on the left of 72, $-$ on the right of 72

72

$d = 72$ MAXIMIZES V because V' changes from $+$ to $-$ @ $d = 72$ in

$h = 108 - d = 36$ in

Practice WS 5.4 **Calculator OK** Name Alternate solutions

1. A number plus twice a second number is 108. Find the two numbers that give a maximum product. Justify your answer.

$x + 2y = 108$

$x = 108 - 2y$

$P = xy = (108 - 2y)y = 108y - 2y^2$

$P' = 108 - 4y = 0$

$y = \frac{108}{4} = 27$

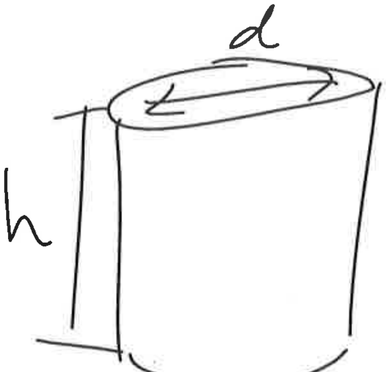
P' + | −
27

y = 27 IS A MAXIMUM because P' changes from + to − @ y = 27

$x = 108 - 2y = 54$

2. A rectangular solid that has a square base has a surface area of 150 square inches. Find the maximum volume of the solid and its dimensions. Justify your answer.

3. The diameter plus the height of a cylindrical package is equal to 108 inches. Find the dimensions of the package that gives you a maximum volume. Justify your answer.



$$d + h = 108$$

$$d = 108 - h$$

$$V = \pi \left(\frac{d}{2}\right)^2 \cdot h$$

$$V = \pi \left(\frac{108-h}{2}\right)^2 \cdot h = \frac{\pi}{4}\left(108^2 - 216h + h^2\right) \cdot h$$

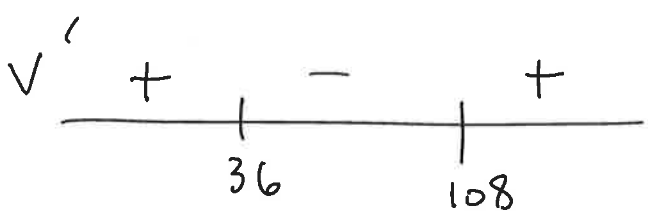
$$V = \frac{\pi}{4}\left(11{,}664h - 216h^2 + h^3\right)$$

$$V' = \frac{\pi}{4}\left(11{,}664 - 432h + 3h^2\right) = 0$$

$$= \frac{3\pi}{4}\left(3888 - 144h + h^2\right) = 0$$

$$= \frac{3\pi}{4}(36 - h)(108 - h) = 0$$

$$h = 36 \text{ or } h = 108$$



$h = 36$ MAXIMIZES V
because V' changes
from + to − @ $h = 36$
$d = 108 - h = 72$