3.4a Velocity and Other Rates of Change

Average Rate of Change
$$\frac{f(x+h)-f(x)}{h}$$
over the interval *x* to *x* + *h*

Instantaneous Rate of Change of *f* w.r.t. *x* at *a* is the derivative:
$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
provided the limit exists

Note: When we say" rate of change", we mean "instantaneous rate of change".

Example 1:

Find the rate of change of the volume of the sphere with respect to the length of its radius. Then, find the rate of change when the radius is 3 inches.

Start with the formula for volume of a sphere: $V = \frac{4}{3}\pi r^3$

Find the first derivative of the volume w.r.t. its radius: $V' = 3\cdot\frac{4}{3}\pi r^2 = 4\pi r^2$

Evaluate at r = 3: $V'(3) = 4\pi\cdot 3^2 = 36\pi$

Supply the correct units: IN^2/IN

Example 2:

Find the rate of change of the area A of a circle with respect to its radius r. Find the rate of change of A at r = 5 and at r = 10 (use the appropriate units)

$A = \pi r^2$ $\quad A' = 2\pi r$ $\quad A'(5) = 10\pi\ IN^2/IN$ $\quad A'(10) = 20\pi\ IN^2/IN$

In the above problem, think of concentric circles where the radius is growing at a constant rate. Does this describe how trees grow?

yes!

Position, Velocity, and Acceleration
Moving along a Line

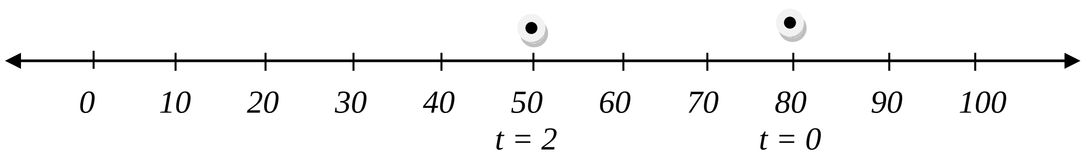
Suppose an object is moving along a linear path with distance measured in feet and time in seconds.

I. We use $s(t)$ to represent the **position** of the object at time *t*. $s(t)$ measures the position of the object compared to zero where s(t) = 0. (Note: at t = 0 the object may not be at position zero.)

Ex: *s(2)=50 means that at t = 2 seconds the object is 50 feet to the right (or above) position zero .*

- $s(b)-s(a)$ is the **displacement** (how much the position of the object has changed) during the time interval [a,b].

Ex: *s(2) – s(0) = -30 ft means that the object has traveled 30 ft to the left (or down) during the 2 second time period. (So in the example below, s(0) was 80 ft).*



- $\dfrac{s(b)-s(a)}{b-a}$ is **average velocity** (average rate of change of position) during the interval [a,b]. In the example above, $average\ velocity = \dfrac{s(2)-s(0)}{2-0} = \dfrac{50-80}{2-0} = -15\,ft/s$

II. $s'(t)=v(t)$ is the **(instantaneous) velocity** of the object at time t.

Ex: $s'(5)$ *= -20 ft/s means that at t = 5 seconds, the object is moving left (or down) at 20 ft/sec*

- $|v(t)|$ is the **speed** at time *t*. It's the same as velocity, except it ignores direction.

Ex: $|s'(5)|=|v(5)|=20\,ft/s$ *means the object was moving 20 ft/sec at t = 5 seconds without specifying the direction.*

III. $s''(t)=v'(t)=a(t)$ is the **acceleration** at time *t*. It measures how quickly your velocity changes.

Ex: *a(5) = -3ft/s^2 means that at t = 5 seconds the velocity of the object is decreasing by 3 ft/second^2 . Since the object's velocity is -20 ft/sec at that time and the velocity is decreasing makes the velocity more negative, which is actually increasing the speed of the object. The object is moving left (or down), and it is speeding up in that direction.*

Modeling Vertical Motion

Distance a body released from rest falls freely is proportional to the square of the amount of time it has fallen. This is expressed as:

$$s = \frac{1}{2}gt^2$$

where s is the distance, g is the acceleration due to Earth's gravity, and t is time.

$g = 32$ ft/sec^2 if s is measured in feet
$g = 9.8$ m/sec^2 if s is measured in meters

Example:

On the moon, a rock is thrown vertically upward from the surface at a velocity of 10 m/sec and reaches a height of $s = 10t - 0.8t^2$. (Equation used is: $s(t) = \frac{1}{2}gt^2 + v_0t + s_0$, g = 1.60 m/sec^2 on moon)

a. What is the velocity of the rock at 3 seconds?

$s'(t) = 10 - 1.6t$ $\quad s'(3) = 10 - 1.6(3) = 5.2$ m/s

b. What is the velocity after the rock has risen 25 meters?

$s(t) = 25$
$10t - .8t^2 = 25$
$t \approx 3.5$ s
$s'(3.5) \approx 4.472$ m/s

c. What is the maximum height of the rock? (Do not use max/min programs ☹)

$s'(t) = 0$
$10 - 1.6t = 0$
$t = 6.25$ s
$s(6.25) = 31.25$ m

d. When did the rock hit the ground?

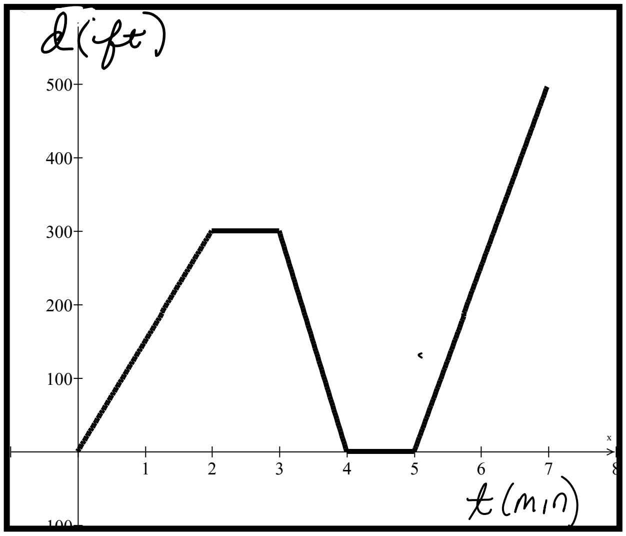
$s(t) = 0$
$10t - .8t^2 = 0$
$t(10 - .8t) = 0$
$t = 12.5$ s

e. What was the velocity of the rock when it hit the ground?

$s'(12.5) = -10$ m/s

Reading Position Graphs

Below is a graph of Janie's distance from home while walking to school.

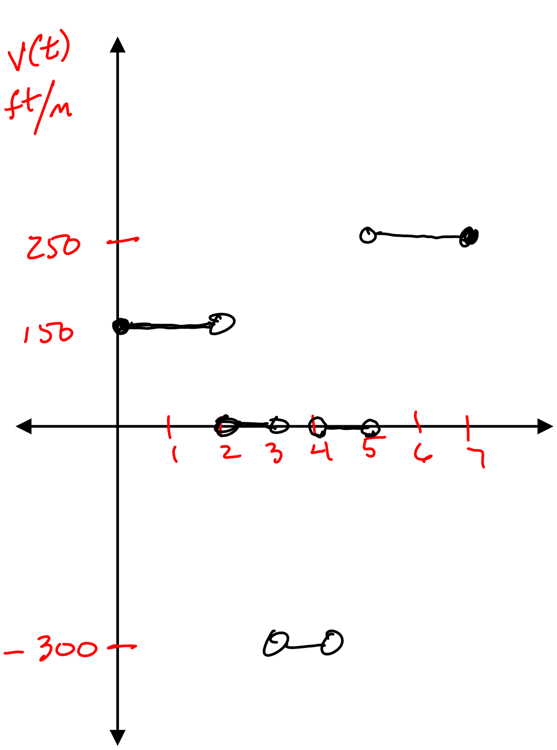


a. Describe her trip to school.

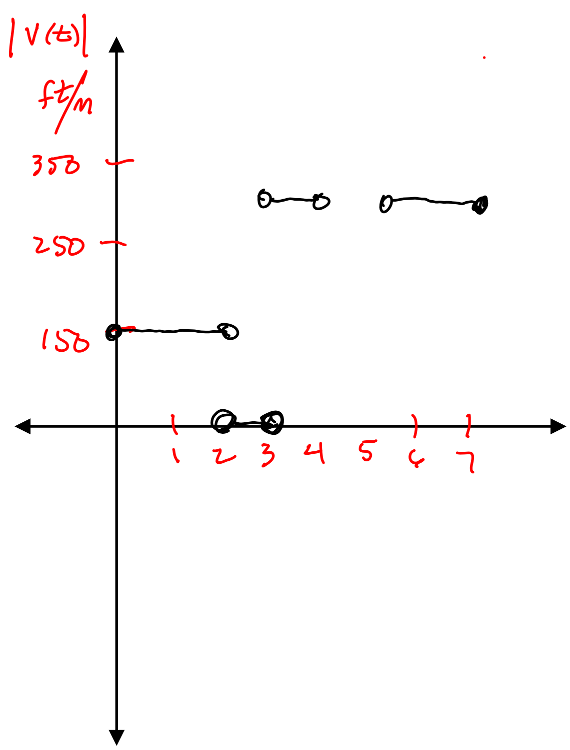
FROM [0, 2) JANIE's velocity IS 150 ft/MIN
" (2, 3) " " " 0 ft/MIN
" (3, 4) " " " −300 ft/MIN
" (4, 5] " " " 0 ft/MIN
" (5, 7) " " " 250 ft/MIN

b. Sketch a graph of her velocity and her speed.

Velocity graph:

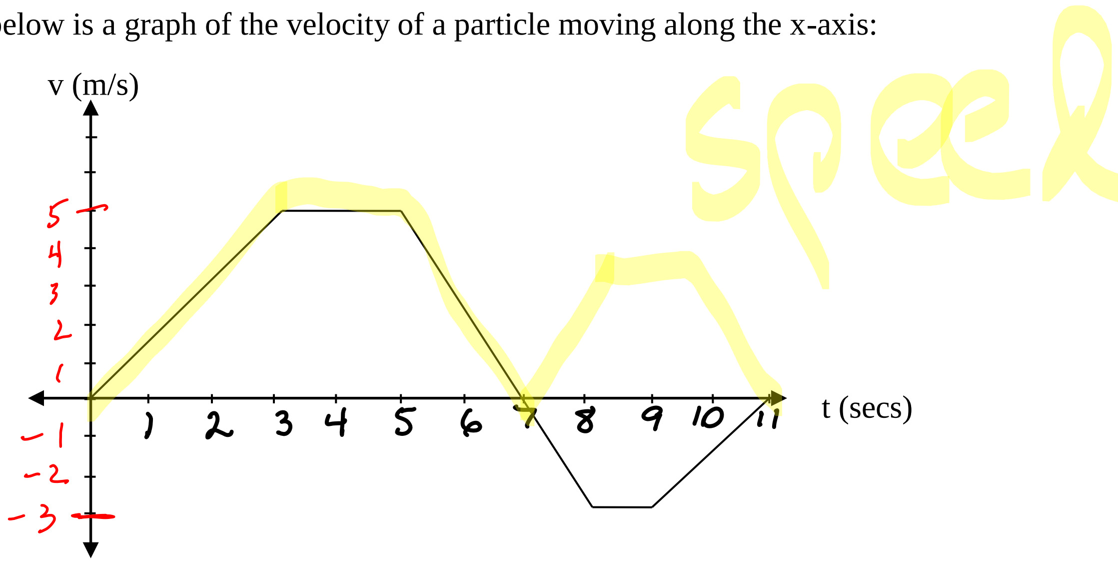


Speed graph:



Reading Velocity Graphs

The graph below is a graph of the velocity of a particle moving along the x-axis:

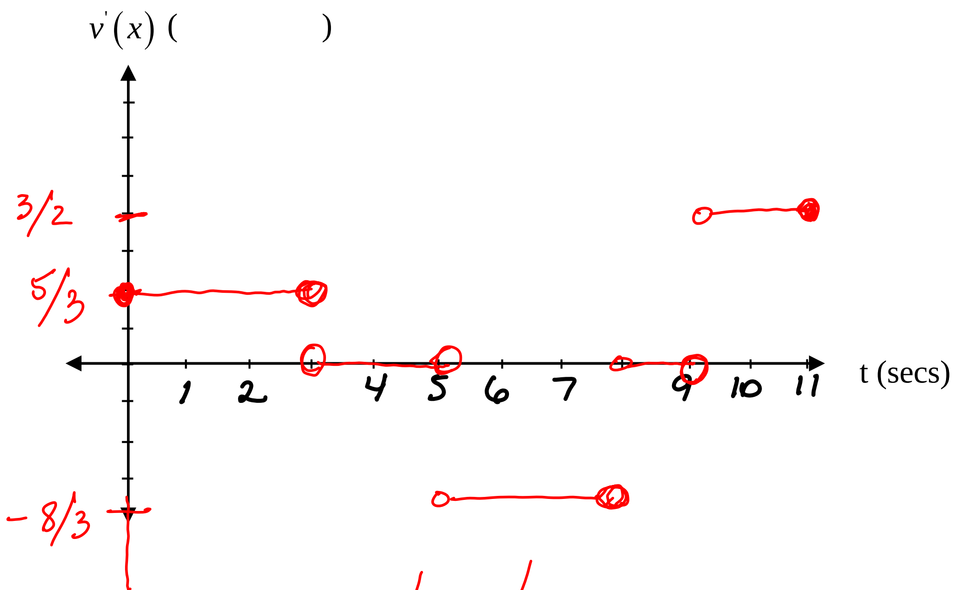


a. State the intervals that the velocity is positive? Negative?

Positive Velocity: (0, 7)   Zero Velocity: t = 1, 7, 11

Negative Velocity: (7, 11)

b. Draw a sketch of $v'(x)$ and state the intervals when it is positive? Negative?



c. What is $v'(x)$ called?   Acceleration

d. How can we determine when the particle is speeding up and slowing down?

If v(x) & a(x) have same sign the particle is speeding up. If v(x) & a(x) have opposite signs the particle is slowing down.